Kaniel-Shinbrot Iteration and Global Solutions of the Cauchy Problem for the Boltzmann Equation

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Introduction

Every solution of the Cauchy problem for the Boltzmann equation over the spatial domain $\mathbb{R}^D$ has a unique global Maxwellian associated with it that is determined by the initial values of its formally conserved quantities.

Kaniel-Shinbrot iteration can be used to establish the existence of solutions to the Boltzmann equation over the spatial domain $\mathbb{R}^D$ for some initial data that is pointwise bounded above and below by a class of global Maxwellians larger than previously considered.

When these solutions are global in time, we use the bounds on them to obtain results on their large-time asymptotics.

Outline

1. Cauchy Problem for the Boltzmann Equation
2. Conserved Quantitites and Global Maxwellians
3. Kaniel-Shinbrot Iteration and Global Solutions
4. Long-Time Behavior of Global Solutions
5. Conclusion

1. Cauchy Problem for the Boltzmann Equation

We consider a kinetic density $F(v, x, t)$ governed by the Cauchy problem for the Boltzmann equation:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F)\,, \qquad F\Big|_{t=0} = F^{\mathsf{in}}\,. \tag{1}$$

We assume that $F^{\mathsf{in}}(v, x)$ is nonnegative and satisfies the bounds

$$0 < \iint_{\mathbb{R}^D \times \mathbb{R}^D} \left(1 + |v|^2 + |x|^2\right) F^{\mathsf{in}} \,\mathrm{d}v\,\mathrm{d}x < \infty\,. \tag{2}$$

The collision operator $\mathcal{B}(F, F)$ has the form

$$\mathcal{B}(F, F) = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} \left(F'_* F' - F_* F\right) \mathsf{b}\,\mathrm{d}\omega\,\mathrm{d}v_*\,, \tag{3}$$

where $\omega \in \mathbb{S}^{D-1}$, $\mathsf{b}(\omega, v - v_*)$ is the collision kernel, while F_*, F', and F'_* denote $F(\,\cdot\,, x, t)$ evaluated at v_*, $v' = v - \omega\,\omega \cdot (v - v_*)$, and $v'_* = v_* + \omega\,\omega \cdot (v - v_*)$ respectively.

Collision Kernels

We will assume that the collision kernel has the separable form

$$\mathsf{b}(\omega, v - v_*) = \widehat{\mathsf{b}}(\omega \cdot n)|v - v_*|^\beta \,, \qquad \text{where} \quad n = \frac{v - v_*}{|v - v_*|} \,, \tag{4}$$

for some $\beta \in (-D, 2)$ while $\widehat{\mathsf{b}}(\omega \cdot n)$ is positive almost everywhere and satisfies the weak small-deflection cutoff condition

$$\left\|\widehat{\mathsf{b}}\right\|_{L^1(\mathrm{d}\omega)} = \int_{\mathbb{S}^{D-1}} \widehat{\mathsf{b}}(\omega \cdot n)\, \mathrm{d}\omega < \infty \,. \tag{5}$$

The conditions $\beta > -D$ and (5) are required for $\mathsf{b}(\omega, v - v_*)$ given by (4) to be locally integrable with respect to $\mathrm{d}\omega\, \mathrm{d}v_*$.

Classical Collision Kernels

The form (4) arises from the classical scattering cross section calculation for identical hard spheres of mass m and diameter d_o, which yields

$$\mathsf{b}(\omega, v - v_*) = \frac{d_o^{D-1}}{2m} \left| \omega \cdot (v - v_*) \right| . \tag{6}$$

This corresponds to the case $\beta = 1$ and $\widehat{\mathsf{b}}(\omega \cdot n) = \frac{d_o^{D-1}}{2m} \left| \omega \cdot n \right|$ in (4).

The form (4) also arises from potentials proportional to r^{-k}, where r is the distance between the center of masses, one has

$$\beta = 1 - 2\,\frac{D-1}{k} \qquad \text{for } k > 2\,\frac{D-1}{D+1} . \tag{7}$$

The cases $\beta \in (-D, 0)$, $\beta = 0$, $\beta \in (0, 1]$, and $\beta \in (1, 2)$ are called respectively the “soft”, “Maxwell”, “hard”, and “super-hard” cases. The super-hard cases do not arise from an inverse-power interparticle potential.

Attenuation and Gain Operators

The weak small-deflection cutoff condition allows us to decompose the collision operator (3) as

$$\mathcal{B}(F,F) = \mathcal{G}(F,F) - \mathcal{A}(F)\,F\,, \tag{8}$$

where the attenuation operator $\mathcal{A}(F)$ and the gain operator $\mathcal{G}(F,F)$ are defined by

$$\begin{aligned} \mathcal{A}(F) &= \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} F_* \,\mathsf{b}\,\mathsf{d}\omega\,\mathsf{d}v_*\,, \\ \mathcal{G}(F,F) &= \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} F'_* F' \,\mathsf{b}\,\mathsf{d}\omega\,\mathsf{d}v_*\,. \end{aligned} \tag{9}$$

2. Consered Quantities and Global Maxwellians

Local Conservation Laws

Any solution F of the Boltzmann equation (1) formally satisfies the local conservation law

$$\partial_t \int_{\mathbb{R}^D} \xi \, F \, \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^D} v \, \xi \, F \, \mathrm{d}v = 0 \, ,$$

when $\xi(v, x, t)$ is any quantity that satisfies

$$\xi(\,\cdot\,, x, t) \in \operatorname{span}\{1, v_1, v_2, \cdots, v_D, |v|^2\} \, , \qquad \partial_t \xi + v \cdot \nabla_x \xi = 0 \, . \tag{10}$$

It has been known essentially since Boltzmann, who worked out the case $D = 3$, that the only such quantities ξ are linear combinations of the $4 + 2D + \frac{D(D-1)}{2}$ quantities

$$1 \, , \quad v \, , \quad x - vt \, , \quad \tfrac{1}{2}|v|^2 \, , \quad v \wedge x \quad v \cdot (x - vt) \, , \quad \tfrac{1}{2}|x - vt|^2 \, , \tag{11}$$

where $v \wedge x = v \, x^T - x \, v^T$ is the skew tensor product.

Globally Conserved Quantities

By integrating the coresponding local conservation laws over space and time, we formally obtain the globally conserved quantities

$$\iint_{\mathbb{R}^D \times \mathbb{R}^D} \begin{pmatrix} 1 \\ v \\ x - vt \\ \frac{1}{2}|v|^2 \\ v \wedge x \\ v \cdot (x - vt) \\ \frac{1}{2}|x - vt|^2 \end{pmatrix} F(v, x, t)\, \mathrm{d}v\, \mathrm{d}x \,, \tag{12}$$

where these quantities exist by the bounds (2). These are associated respectively with the conservation laws of mass, momentum, initial center of mass, energy, angular momentum, scalar momentum moment, and scalar inertial moment. The last two are not general physical laws, but are shared by the solutions of other kinetic equations.

Local Maxwellians

One consequence of Boltzmann's celebrated H-theorem is that if $f(v)$ is any nonnegative integrable function that has appropriate behavior as $|v| \to \infty$ then $\mathcal{B}(f, f) = 0$ if and only if

$$f = \frac{\rho}{(2\pi\theta)^{\frac{D}{2}}} \exp\left(- \frac{|v-u|^2}{2\theta} \right), \tag{13}$$

for some $(\rho, u, \theta) \in \mathbb{R}_+ \times \mathbb{R}^D \times \mathbb{R}_+$.

Functions of the form (13) where ρ, u, and θ functions of (x, t) are called *local Maxwellians*.

Global Maxwellians

Local Maxwellians that satisfy the Boltzmann equation are called *global Maxwellians*. The family of global Maxwellians over $\mathbb{R}^D$ with positive mass, zero net momentum, and center of mass at the origin has the form

$$\begin{aligned} \mathcal{M} &= \frac{m}{(2\pi)^D} \sqrt{\det(Q)}\, \exp\Big(- q(v,x,t)\Big)\,, \\ Q &= (ac - b^2)I + B^2\,, \\ q(v,x,t) &= \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix}\,, \end{aligned} \tag{14}$$

with $m > 0$ and $(a,b,c,B) \in \Omega$ where Ω is defined by

$$\Omega = \Big\{ (a,b,c,B) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{D\wedge D} \;:\; (ac - b^2)I + B^2 > 0 \Big\}\,.$$

Here $\mathbb{R}_+$ denotes the positive real numbers and $\mathbb{R}^{D\wedge D}$ denotes the skew-symmetric $D \times D$ real matrices.

This form can be derived from the fact that $\log(\mathcal{M})$ must satisfy

$$\log(\mathcal{M}) \in \operatorname{span}\{1, v_1, v_2, \cdots, v_D, |v|^2\}, \quad \Big(\partial_t + v \cdot \nabla_x\Big) \log(\mathcal{M}) = 0,$$

whereby $\log(\mathcal{M})$ must be a linear combination of the quantities

$$\begin{array}{cccc} & 1, & v, & x - vt, \\ \frac{1}{2}|v|^2, & v \wedge x & v \cdot (x - vt), & \frac{1}{2}|x - vt|^2. \end{array}$$

The form of $\mathcal{M}$ then comes from the requirements that it have finite mass, zero net momentum, and center of mass at the origin.

The larger family of global Maxwellians with positive mass is obtained from the form $\mathcal{M}$ by introducing translations in v and x, whereby it has a total of $4 + 2D + \frac{D(D-1)}{2}$ parameters.

We can bring $\mathcal{M}$ into the local Maxwellian form

$$\mathcal{M} = \frac{\rho(x,t)}{\big(2\pi\theta(t)\big)^{\frac{D}{2}}} \exp\left(-\frac{|v - u(x,t)|^2}{2\theta(t)}\right) ,$$

where the temperature $\theta(t)$, bulk velocity $u(x,t)$, and mass density $\rho(x,t)$ are given by

$$\begin{aligned} \theta(t) &= \frac{1}{at^2 - 2bt + c}, \qquad u(x,t) = \theta(t)\big(axt - bx - Bx\big) , \\ \rho(x,t) &= m \left(\frac{\theta(t)}{2\pi}\right)^{\frac{D}{2}} \sqrt{\det(Q)} \exp\left(-\frac{\theta(t)}{2} x^T Q x\right) . \end{aligned} \tag{15}$$

Because $a, c > 0$ and $ac > b^2$, we see that $at^2 - 2bt + c > 0$ for every t. Because $Q > 0$, we see that $\rho(x,t)$ is integrable over $\mathbb{R}^D$.

The second moments of $\mathcal{M}$ may be computed by evaluating the Gaussian integrals as

$$\begin{aligned}\iint_{\mathbb{R}^D\times\mathbb{R}^D}\begin{pmatrix} v \\ x-vt\end{pmatrix}\begin{pmatrix} v \\ x-vt\end{pmatrix}^T \mathcal{M}\,\mathrm{d}v\,\mathrm{d}x &= m\begin{pmatrix} cI & bI+B \\ bI-B & aI\end{pmatrix}^{-1} \\ &= m\begin{pmatrix} aQ^{-1} & -bQ^{-1}-Q^{-1}B \\ -bQ^{-1}+Q^{-1}B & cQ^{-1}\end{pmatrix}.\end{aligned}$$

In particular, the values of the quadratic conserved quantities are given by

$$\begin{aligned}\iint_{\mathbb{R}^D\times\mathbb{R}^D}|v|^2\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x &= m\,\mathrm{tr}\Big(Q^{-1}\Big)a\,,\\ \iint_{\mathbb{R}^D\times\mathbb{R}^D} v\cdot(x-vt)\,\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x &= -m\,\mathrm{tr}\Big(Q^{-1}\Big)b\,,\\ \iint_{\mathbb{R}^D\times\mathbb{R}^D}|x-vt|^2\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x &= m\,\mathrm{tr}\Big(Q^{-1}\Big)c\,,\\ \iint_{\mathbb{R}^D\times\mathbb{R}^D} v\wedge x\,\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x &= -2mQ^{-1}B\,.\end{aligned} \tag{16}$$

Conserved Quantities and Global Maxwellians

Every Cauchy problem (1) whose initial data $F^{\mathrm{in}}(v,x)$ satisfies the bounds

$$0 < \iint_{\mathbb{R}^D\times\mathbb{R}^D} \left(1 + |v|^2 + |x|^2\right) F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x < \infty\,, \tag{17}$$

can be associated with a unique global Maxwellian determined by the values of the conserved quantities computed from F^{in}. By choosing an appropriate rescaling and Galilean frame, we may assume without loss of generality that

$$\begin{gathered} \iint_{\mathbb{R}^D\times\mathbb{R}^D} F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = 1\,, \\ \iint_{\mathbb{R}^D\times\mathbb{R}^D} v\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{R}^D\times\mathbb{R}^D} x\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = 0\,. \end{gathered} \tag{18}$$

Theorem. 1 *Let $F^{\mathrm{in}}(v,x)$ be a nonnegative function that satisfies the bounds (2) and the normalizations (18). Let a_*, b_*, c_*, and B_* be given by*

$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} |v|^2 F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = a_*\,, \qquad \iint_{\mathbb{R}^D\times\mathbb{R}^D} v\cdot x\, F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = b_*\,,$$
$$\iint_{\mathbb{R}^D\times\mathbb{R}^D} |x|^2 F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = c_*\,, \qquad \iint_{\mathbb{R}^D\times\mathbb{R}^D} v\wedge x\, F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = B_*\,. \tag{19}$$

Then $(a_*, b_*, c_*, B_*) \in \Omega_*$, where Ω_* is defined by

$$\Omega_* = \left\{ (a_*, b_*, c_*, B_*) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{D\wedge D} \,:\, \tfrac{1}{2}\,\mathrm{tr}(|B_*|) < \sqrt{a_* c_* - b_*^2} \right\},$$

with $|B_*| = \sqrt{B_*^T B_*} = \sqrt{-B_*^2}$.

Conversely, if $(a_*, b_*, c_*, B_*) \in \Omega_*$ *then there exists a unique global Maxwellian* $\mathcal{M}$ *given by (14) with* $m = 1$ *and* $(a, b, c, B) \in \Omega$ *such that the quadratic converved quantities associated with* $\mathcal{M}$ *through (16) have values* (a_*, b_*, c_*, B_*) *— i.e. such that*

$$a_* = \operatorname{tr}\Big(Q^{-1}\Big)a\,, \quad b_* = -\operatorname{tr}\Big(Q^{-1}\Big)b\,, \quad c_* = \operatorname{tr}\Big(Q^{-1}\Big)c\,, \quad B_* = -2Q^{-1}B\,, \tag{20}$$

where $Q = (ac - b^2)I + B^2$ *and the set* Ω *was defined after (14).*

Remark. The first part of this theorem states that Ω_* contains the set of values that can be realized by the conserved quantities given by (16). The second part asserts that every point in Ω_* can be so realized. Therefore Ω_* characterizes all such values. This is a moment realizability result. The trace norm appearing in the characterization of Ω_* makes it unusual.

Ordering Global Maxwellians

Lemma. 2 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be global Maxwellians of the form (14) with parameters given by $(m_1, a_1, b_1, c_1, B_1) \in \mathbb{R}_+ \times \Omega$ and $(m_2, a_2, b_2, c_2, B_2) \in \mathbb{R}_+ \times \Omega$ respectively. Then $\mathcal{M}_1 \leq \mathcal{M}_2$ for every (v, x, t) if and only if*

$$\begin{pmatrix} c_2 I & b_2 I + B_2 \\ b_2 I - B_2 & a_2 I \end{pmatrix} \leq \begin{pmatrix} c_1 I & b_1 I + B_1 \\ b_1 I - B_1 & a_1 I \end{pmatrix}, \tag{21}$$

$$m_1 \sqrt{\det\Big(\big(a_1 c_1 - b_1^2\big) I + B_1^2\Big)} \leq m_2 \sqrt{\det\Big(\big(a_2 c_2 - b_2^2\big) I + B_2^2\Big)}. \tag{22}$$

Similarly, $\mathcal{M}_1 < \mathcal{M}_2$ for every (v, x, t) if and only if (21) holds and (22) is a strict inequality.

3. Kaniel-Shinbrot Iteration and Global Solutions

Kaniel-Shinbrot iteration can be used to prove the existence of solutions F to the Boltzmann initial-value problem posed over the spatial domain $\mathbb{R}^D$ with initial data $F^{\text{in}}(v,x)$ that satisfies certain pointwise bounds. More specifically, it is used to construct two sequences of approximate solutions, $\left\{F_j^L\right\}_{j\in\mathbb{N}}$ and $\left\{F_j^U\right\}_{j\in\mathbb{N}}$, the first of which converges monotonically to F from below, while the second converges monotonically to F from above. In other words, for some $T \in (0,\infty]$ these opposing monotone sequences should satisfy the order relationships

$$F_j^L \leq F_{j+1}^L \leq F_{j+1}^U \leq F_j^U \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D \times [0,T) \text{ for every } j \in \mathbb{N}\,. \tag{23}$$

The expectation is that these sequences will converge to $F(v,x,t)$ over $\mathbb{R}^D \times \mathbb{R}^D \times [0,T)$. When this construction can be carried out for $T = \infty$ then the solution F will be global in time.

Kaniel-Shinbrot iteration constructs sequences as follows. Given F^L_{j-1} and F^U_{j-1} for any $j \in \mathbb{Z}_+$ we define the Kaniel-Shinbrot iterates F^L_j and F^U_j to be the solution of the linear system

$$\begin{aligned} \partial_t F^U_j + v \cdot \nabla_x F^U_j + \mathcal{A}\Big(F^L_{j-1}\Big)\, F^U_j &= \mathcal{G}\Big(F^U_{j-1}, F^U_{j-1}\Big)\ , \\ \partial_t F^L_j + v \cdot \nabla_x F^L_j + \mathcal{A}\Big(F^U_{j-1}\Big)\, F^L_j &= \mathcal{G}\Big(F^L_{j-1}, F^L_{j-1}\Big)\ , \end{aligned} \tag{24a}$$

$$F^U_j\Big|_{t=0} = F^L_j\Big|_{t=0} = F^{\text{in}}\,. \tag{24b}$$

The existence of F^L_j and F^U_j can be inferred from the mild formulation of system (24a).

Notice that F^{in} enters the construction of F^L_j and F^U_j for $j \geq 1$ in (24b). The only iterates that generally do not satisfy (24b) are F^L_0 and F^U_0.

The form of the Kaniel-Shinbrot iteration (24) is motivated by the fact that $\mathcal{A}$ and $\mathcal{G}$ have the monotonicity properties

$$F \le G \quad \Longrightarrow \quad \mathcal{A}(F) \le \mathcal{A}(G) \quad \text{and} \quad \mathcal{G}(F,F) \le \mathcal{G}(G,G)\,. \tag{25}$$

The following lemma shows that this form insures that Kaniel-Shinbrot iterates preserve the order relationships that appear in (23).

Lemma. 3 (Order Preservation Lemma) *If for some* $T \in (0,\infty]$ *the Kaniel-Shinbrot iterates* F_{j-1}^L, F_{j-1}^U, F_j^L, *and* F_j^U *satisfy*

$$F_{j-1}^L \le F_j^L \le F_j^U \le F_{j-1}^U \qquad \textit{over } \mathbb{R}^D \times \mathbb{R}^D \times [0,T)\,, \tag{26}$$

then the Kaniel-Shinbrot iterates F_{j+1}^L *and* F_{j+1}^U *satisfy*

$$F_j^L \le F_{j+1}^L \le F_{j+1}^U \le F_j^U \qquad \textit{over } \mathbb{R}^D \times \mathbb{R}^D \times [0,T)\,. \tag{27}$$

Kaniel-Shinbrot Theorem

Induction, Lemma 3, the Lebesgue Monotone Convergence Theorem, and a stability bound can be used to prove the following.

Theorem. 4 (Kaniel-Shinbrot) *If for some* $T \in (0, \infty]$ *the Kaniel-Shinbrot iterates* F_0^L, F_0^U, F_1^L, *and* F_1^U *satisfy the so-called beginning condition*

$$F_0^L \leq F_1^L \leq F_1^U \leq F_0^U \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D \times [0, T)\,, \tag{28}$$

then the Kaniel-Shinbrot iteration yields opposing monotone sequences $\left\{F_j^L\right\}_{j \in \mathbb{N}}$ *and* $\left\{F_j^U\right\}_{j \in \mathbb{N}}$ *over* $\mathbb{R}^D \times \mathbb{R}^D \times [0, T)$ *— i.e. sequences that satisfy the order relationship (23). These sequences converge to a unique mild solution of the initial-value problem (1) for the Boltzmann equation.*

Beginning with Local Maxwellians

The hard part of applying the Kaniel-Shinbrot Theorem is showing the first two iterates satisfy the beginning condition (28). When the initial Kaniel-Shinbrot iterates are local Maxwellians then there is a simple criterion that insures the beginning condition is satisfied for some $T \in (0, \infty]$.

Proposition 5 (Local Maxwellian Beginning Lemma) *Suppose M^L and M^U are local Maxwellians that satisfy*

$$M^L\Big|_{t=0} \leq M^U\Big|_{t=0} \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D\,, \tag{29a}$$

and for some $T \in (0, \infty]$ satisfy

$$\begin{aligned} \partial_t M^U + v \cdot \nabla_x M^U &\geq \mathcal{A}(M^U - M^L) M^U \qquad &\text{over } \mathbb{R}^D \times \mathbb{R}^D \times [0, T)\,, \\ -\,\partial_t M^L - v \cdot \nabla_x M^L &\geq \mathcal{A}(M^U - M^L) M^L \qquad &\text{over } \mathbb{R}^D \times \mathbb{R}^D \times [0, T)\,, \end{aligned} \tag{29b}$$

Then for every initial data F^{in} such that

$$M^L\Big|_{t=0} \leq F^{\mathsf{in}} \leq M^U\Big|_{t=0} \qquad \textit{over } \mathbb{R}^D \times \mathbb{R}^D ,$$

the Kaniel-Shinbrot iterates obtained by setting $F_0^L = M^L$ and $F_0^U = M^U$ satisfy the beginning condition (28) over $[0, T)$. Moreover, the Kaniel-Shinbrot Theorem (4) yields the unique mild solution $F(v, x, t)$ of the initial-value problem (1) for the Boltzmann equation that satisfies the bounds

$$M^L(v, x, t) \leq F(v, x, t) \leq M^U(v, x, t) \qquad \textit{over } \mathbb{R}^D \times \mathbb{R}^D \times [0, T) . \tag{30}$$

Proposition 5 requires us to find local Maxwellians M^L and M^U that meet criterion (29) for some $T \in (0, \infty]$. When $M^L > 0$ this criterion may be recast as

$$\begin{aligned} \Big(\partial_t + v \cdot \nabla_x\Big) \log\Big(M^U\Big) &\geq \mathcal{A}\Big(M^U - M^L\Big) \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D \times [0, T)\,, \\ -\Big(\partial_t + v \cdot \nabla_x\Big) \log\Big(M^L\Big) &\geq \mathcal{A}\Big(M^U - M^L\Big) \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D \times [0, T)\,, \end{aligned} \tag{31a}$$

$$M^L\Big|_{t=0} \leq M^U\Big|_{t=0} \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D\,. \tag{31b}$$

When $M^L = 0$ criterion (29) reduces to simply

$$\Big(\partial_t + v \cdot \nabla_x\Big) \log(M^U) \geq \mathcal{A}\Big(M^U\Big) \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D \times [0, T)\,. \tag{32}$$

We will construct such local Maxwellians from the global Maxwellians given by (14). When we can do this with $T = \infty$, the Kaniel-Shinbrot theorem will yield global solutions.

Building Local Maxwellians from Global Ones

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be global Maxwellians in the form (14) that are given by parameters $(m_1, a_1, b_1, c_1, B_1) \in \mathbb{R}_+ \times \Omega$ and $(m_2, a_2, b_2, c_2, B_2) \in \mathbb{R}_+ \times \Omega$ respectively, and that satisfy $\mathcal{M}_1 \leq \mathcal{M}_2$ for every (v, x, t). Set

$$\begin{aligned} Q_1 &= (a_1 c_1 - b_1^2) I + B_1^2 \,, \qquad Q_2 = (a_2 c_2 - b_2^2) I + B_2^2 \,, \\ q_1(v,x,t) &= \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} c_1 I & b_1 I + B_1 \\ b_1 I - B_1 & a_1 I \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} . \\ q_2(v,x,t) &= \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} c_2 I & b_2 I + B_2 \\ b_2 I - B_2 & a_2 I \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} , \end{aligned} \tag{33}$$

so that

$$\mathcal{M}_i = \frac{m_i}{(2\pi)^D} \sqrt{\det(Q_i)} \, \exp\Big(- q_i(v,x,t) \Big) \, .$$

Because $\mathcal{M}_1 \leq \mathcal{M}_2$, we know that $q_1(v,x,t) \geq q_2(v,x,t)$.

We will construct local Maxwellians M^U and M^L in the form

$$M^U(v,x,t) = \gamma(t)\,\frac{m_2}{(2\pi)^D}\,\sqrt{\det(Q_2)}\ \exp\!\left(-\frac{q_2(v,x,t)}{\eta(t)}\right)\,, \tag{34a}$$

$$M^L(v,x,t) = \frac{1}{\gamma(t)}\,\frac{m_1}{(2\pi)^D}\,\sqrt{\det(Q_1)}\ \exp\!\left(-\left(2-\frac{1}{\eta(t)}\right)\,q_1(v,x,t)\right)\,, \tag{34b}$$

where the functions $\gamma(t)$ and $\eta(t)$ satisfy

$$\gamma(0) = \eta(0) = 1\,, \qquad \gamma'(t) > 0\,, \qquad \eta'(t) \geq 0\,.$$

Clearly, inequality (31b) is satisfied because

$$M^L\Big|_{t=0} = \mathcal{M}_1\Big|_{t=0} \le \mathcal{M}_2\Big|_{t=0} = M^U\Big|_{t=0}\,.$$

Because $(\partial_t + v\cdot\nabla_x)q_2(v,x,t) = (\partial_t + v\cdot\nabla_x)q_1(v,x,t) = 0$, a direct calculation from (34) yields

$$\begin{aligned}
\Big(\partial_t + v\cdot\nabla_x\Big)\log\Big(M^U\Big) &= \frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q_2(v,x,t)}{\eta(t)}\,,\\
-\Big(\partial_t + v\cdot\nabla_x\Big)\log\Big(M^L\Big) &= \frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q_1(v,x,t)}{\eta(t)}\,.
\end{aligned}$$

Because $q_1(v,x,t) \geq q_2(v,x,t)$, we see that

$$-\Big(\partial_t + v\cdot\nabla_x\Big)\log\Big(M^L\Big) \geq \Big(\partial_t + v\cdot\nabla_x\Big)\log\Big(M^U\Big)\ ,$$

therefore criterion (31) will be satisfied if

$$\frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q_2(v,x,t)}{\eta(t)} \geq \mathcal{A}\Big(M^U - M^L\Big)\ . \tag{35}$$

Similarly, if $M^L = 0$ while M^U is given by (34a) then criterion (32) will be satisfied if

$$\frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q_2(v,x,t)}{\eta(t)} \geq \mathcal{A}\Big(M^U\Big)\ . \tag{36}$$

This extends the basic inequality derived in Kaniel-Shinbrot (1978) to a more general class of local Maxwellians. We can treat criterion (36) as criterion (35) with $M^L = 0$, The right-hand sides of (35) and (36) contain expressions of the form $\mathcal{A}(M)$ where M is some local Maxwellian. If $\rho(x,t)$, $u(x,t)$, and $\theta(x,t)$ are the fluid variables associated with M then

$$\begin{aligned}
\mathcal{A}(M) &= \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} M_* \,\mathsf{b}(\omega, v-v_*)\,\mathrm{d}\omega\,\mathrm{d}v_* \\
&= \big\|\widehat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)} \int_{\mathbb{R}^D} |v-v_*|^\beta \frac{\rho}{(2\pi\theta)^{\frac{D}{2}}}\,\exp\!\left(-\frac{|v_*-u|^2}{2\theta}\right)\mathrm{d}v_* \\
&= \rho\,\theta^{\frac{\beta}{2}}\,\mathsf{a}\!\left(\frac{v-u}{\sqrt{\theta}}\right),
\end{aligned} \tag{37}$$

where the attenuation coefficient $\mathsf{a}(w)$ is defined by

$$\mathsf{a}(w) = \big\|\widehat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)} \frac{1}{(2\pi)^{\frac{D}{2}}} \int_{\mathbb{R}^D} |w-w_*|^\beta \exp\!\left(-\tfrac{1}{2}|w_*|^2\right)\mathrm{d}w_*\,. \tag{38}$$

By rotation invariance $\mathsf{a}(w)$ is a function of only $|w|$. It is bounded when $\beta\in(-D,0]$, and unbounded when $\beta\in(0,2]$. This fundamental difference requires a different analysis for each of these cases.

Soft and Maxwell Cases

We now complete the foregoing construction of for cases when the collision kernel has the separable form (4) with $\beta \in (-D, 0]$. In other words, for cases when the kernel arises from either soft or Maxwell potentials. Because (38) shows the right-hand sides of (35) and (36) are bounded functions of v when $\beta \in (-D, 0]$, we take $\eta(t) = 1$ in the forms of M^U and M^L given by (34). Criterion (35) then reduces to

$$\frac{\gamma'(t)}{\gamma(t)} \geq \mathcal{A}\Big(M^U - M^L\Big) \,, \qquad \gamma(0) = 1 \,. \tag{39}$$

Below we show how this criterion can be met in certain cases.

Near Vacuum Case

The easiest case to treat is when the lower bound is the vacuum.

Proposition 6 *Let $\mathcal{M}$ be the global Maxwellian in the form (14) given by $m > 0$ and $(a, b, c, B) \in \Omega$. Let $F^{\mathsf{in}}(v, x)$ be any initial data such that*

$$0 \leq F^{\mathsf{in}}(v, x) \leq \mathcal{M}(v, x, 0) \qquad \textit{over } \mathbb{R}^D \times \mathbb{R}^D \,. \tag{40}$$

Let $\beta \in (-D, 0]$. Let

$$N(t) = \|\mathsf{a}\|_{L^\infty(\mathsf{d}w)} \sqrt{\det\!\left(\tfrac{1}{2\pi}\, Q\right)} \int_0^t \theta(s)^{\frac{D+\beta}{2}}\, \mathsf{d}s \,, \tag{41}$$

where $Q = (ac - b^2)I + B^2$ and $\theta(t)$ is given by (15).

Then there exists a mild solution $F(v,x,t)$ to the Cauchy problem (1) for the Boltzmann equation over $[0,T)$ that satisfies the bounds

$$0 \le F(v,x,t) \le \gamma(t)\,\mathcal{M}(v,x,t) \qquad \textit{over } \mathbb{R}^D \times \mathbb{R}^D \times [0,T)\,, \tag{42}$$

where

$$\gamma(t) = \frac{1}{1 - mN(t)} \quad \textit{over } [0,T)\,, \qquad T = \sup\{t > 0 \,:\, mN(t) < 1\}\,. \tag{43}$$

In particular, this solution is global when $\beta \in (1-D, 0]$ and

$$m\,N_\infty \le 1\,, \qquad \textit{where } N_\infty = \lim_{t\to\infty} N(t)\,. \tag{44}$$

and is globally bounded by a global Maxwellian when strict inequality holds.

Remark. The local result above is in the spirit of Kaniel and Shinbrot (1978), while the global result is in the spirit of Illner and Shinbrot (1984). The bounds obtained here are different because of our use of the global Maxwellian family (14).

Near Global Maxwellian Case

The next easiest case to treat is when the lower and upper bounds are proportional to the same global Maxwellian.

Proposition 7 *Let* $\mathcal{M}(t)$ *be the global Maxwellian in the form (14) given by* $m = 1$ *and some* $(a, b, c, B) \in \Omega$. *Let* $m_2 > m_1 > 0$. *Let* $F^{\text{in}}(v, x)$ *be any initial data such that*

$$m_1 \mathcal{M}(v, x, 0) \leq F^{\text{in}}(v, x) \leq m_2 \mathcal{M}(v, x, 0) \qquad \textit{over } \mathbb{R}^D \times \mathbb{R}^D . \quad (45)$$

Let $\beta \in (-D, 0]$. *Let* $N(t)$ *be given by (41). Let* $\gamma_{\text{mn}} = \sqrt{m_1/m_2}$.

Then there exists a mild solution $F(v, x, t)$ to the Cauchy problem (1) for the Boltzmann equation over $[0, T)$ that satisfies the bounds

$$\frac{m_1}{\gamma(t)} \mathcal{M}(v, x, t) \leq F(v, x, t) \leq \gamma(t)\, m_2\, \mathcal{M}(v, x, t) \tag{46}$$

over $\mathbb{R}^D \times \mathbb{R}^D \times [0, T)$*, where*

$$\gamma(t) = \gamma_{\mathsf{mn}} \frac{1 + \gamma_{\mathsf{mn}} + (1 - \gamma_{\mathsf{mn}}) e^{2\gamma_{\mathsf{mn}} m_2 N(t)}}{1 + \gamma_{\mathsf{mn}} - (1 - \gamma_{\mathsf{mn}}) e^{2\gamma_{\mathsf{mn}} m_2 N(t)}} \qquad \text{over } [0, T)\,,$$
$$T = \sup\{t > 0 \,:\, (1 - \gamma_{\mathsf{mn}}) e^{2\gamma_{\mathsf{mn}} m_2 N(t)} < 1 + \gamma_{\mathsf{mn}}\}\,. \tag{47}$$

In particular, this solution is global when $\beta \in (1 - D, 0]$ and

$$m_2 \, N_\infty \le \frac{1}{2\gamma_{\mathrm{mn}}} \log\left(\frac{1 + \gamma_{\mathrm{mn}}}{1 - \gamma_{\mathrm{mn}}}\right) , \qquad \textit{where } N_\infty = \lim_{t \to \infty} N(t) \, , \quad (48)$$

and is globally bounded by a global Maxwellian when strict inequality holds in (48).

Remark. Condition (48) yields global solutions with larger mass by picking γ_{mn} closer to 1.

Remark. This result is in the spirit of Toscani (1988). The bounds obtained here are different because of our use of the global Maxwellian family (14). In particular, we can treat initial data with significant rotation that are excluded earlier.

Hard and Super-Hard Cases

We now complete the foregoing construction for cases when the collision kernel has the separable form (4) with $\beta \in (0,2]$. Such kernels arise from hard potentials when $\beta \in (0,1)$, and from hard spheres when $\beta = 1$. When $\beta \in (1,2]$ the kernel is said to be super-hard. The super-hard case has only theoretical interest. We will only treat the near vacuum case.

Proposition 8 *Let $\mathcal{M}$ be the global Maxwellian in the form (14) given by $m > 0$ and $(a, b, c, B) \in \Omega$, so that*

$$\begin{aligned} \mathcal{M}(v,x,t) &= \frac{m}{(2\pi)^D} \sqrt{\det(Q)}\, \exp\Big(-q(v,x,t)\Big)\,, \\ Q &= (ac - b^2)I + B^2\,, \\ q(v,x,t) &= \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} . \end{aligned} \tag{49}$$

Let $F^{\mathrm{in}}(v,x)$ be any initial data such that

$$0 \leq F^{\mathrm{in}}(v,x) \leq \mathcal{M}(v,x,0) \qquad \textit{over } \mathbb{R}^D \times \mathbb{R}^D\,. \tag{50}$$

Let $\beta \in (0,2]$ and

$$N(t) = m \sqrt{\det\Big(\tfrac{1}{2\pi} Q\Big)}\, \big\|\widehat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)} \int_0^t \left(\frac{1}{as^2 - 2bs + c}\right)^{\frac{D+\beta}{2}} \mathrm{d}s\,. \tag{51}$$

Then there exists a mild solution $F(v, x, t)$ to the Cauchy problem for the Boltzmann equation over $[0, T)$ that satisfies the bounds

$$0 \le F(v, x, t) \le \gamma(t) \, \frac{m}{(2\pi)^D} \sqrt{\det(Q)} \, \exp\left(- \frac{q(v, x, t)}{\eta(t)} \right) \tag{52}$$

over $\mathbb{R}^D \times \mathbb{R}^D \times [0, T)$, where

$$\gamma(t) = \left(1 - \frac{N(t)}{\tau} \right)^{-\mu} , \qquad \eta(t) = \left(1 - \frac{N(t)}{\tau} \right)^{-\nu} , \tag{53a}$$

$$\mu = 1 - \frac{\beta}{2} \frac{D + \beta}{2D + \beta} , \qquad \nu = \frac{\beta}{2D + \beta} , \qquad \tau = (2D + \beta)^{-\frac{\beta}{2}} , \tag{53b}$$

$$T = \sup\{ t > 0 \, : \, N(t) < \tau \} . \tag{53c}$$

In particular, this solution is global when

$$N_\infty \leq \tau \,, \qquad \textit{where } N_\infty = \lim_{t\to\infty} N(t) \,, \tag{54}$$

and is globally bounded by a global Maxwellian when strict inequality holds.

Remark. The local result above is in the spirit of Kaniel and Shinbrot (1978), while the global result is in the spirit of Illner and Shinbrot (1984). The bounds obtained here are different because of our use of the global Maxwellian family (14) and because we employ a sharper bound on the attenuation coefficient. If we set $\beta = 0$ into the results here they are identical to those of Proposition 6 for Maxwell case.

4. Long-Time Behavior of Global Solutions

We begin with a rough L^1 stability estimate on the collision operator.

Lemma. 9 *Let the collision kernel* $\mathfrak{b}$ *have the separated form (4) for some* $\beta \in (-D, 2]$. *Let* $\mathcal{M}$ *be the global Maxwellian given by (14) for some* $(m, a, b, c, B) \in \mathbb{R}_+ \times \Omega$. *Let* F *be any measurable function that satisfies the pointwise bounds*

$$0 \le F(v,x,t) \le \mathcal{M}(v,x,t) \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D \times [0,\infty) \text{ for some } s \in \mathbb{R}\,. \tag{55}$$

Then for every $[t_1, t_2] \subset [0, \infty)$ *one has the* L^1*-bound*

$$\int_{t_1}^{t_2} \iiiint \left| F_*' F' - F_* F \right| \mathfrak{b}\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}t \le C_1 \int_{t_1}^{t_2} \theta(t)^{\frac{\beta+D}{2}}\, \mathrm{d}t\,, \tag{56}$$

where C_1 *is a constant and* $\theta(t)$ *is given by (15). Here all four-fold integrals are understood to be taken over the domain* $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R}^D$.

An immediate consequence of the foregoing lemma is the following.

Proposition 10 *Let the collision kernel have the separated form (4) for some* $\beta \in (1-D, 2]$*. Let* $F(v,x,t)$ *satisfy the hypotheses of Lemma 9. Then*

$$\int_0^\infty \left\| \mathcal{B}(F,F) \right\|_{L^1(\mathrm{d}v\,\mathrm{d}x)} \mathrm{d}t < \infty \,. \tag{57}$$

Scattering for Boltzmann Solutions

The advection operator $\mathsf{A} = -v \cdot \nabla_x$ generates the group $e^{t\mathsf{A}}$ that acts on every function F^{in} that is defined almost everywhere by the formula

$$e^{t\mathsf{A}} F^{\text{in}}(v,x) = F^{\text{in}}(v, x - vt)\,.$$

When F^{in} is locally integrable then $F = e^{t\mathsf{A}} F^{\text{in}}$ is the unique distribution solution of the initial-value problem

$$\partial_t F + v \cdot \nabla_x F = 0\,, \qquad F\Big|_{t=0} = F^{\text{in}}\,. \tag{58}$$

The main result of this section states that certain global solutions of the Cauchy problem for the Boltzmann equation will behave like solutions of (58) as $t \to \infty$. This kind of large-time asymptotic result is often called a scattering result.

Theorem. 11 *Let the collision kernel* b *have the separated form (4) for some* $\beta \in (1-D, 2]$*. Let* $F(v,x,t)$ *be a global mild solution of the Cauchy problem (1) for the Boltzmann equation that also satisfies the hypotheses of Lemma 9. There exists a unique* $F^{\infty}(v,x)$ *such that*

$$\lim_{t\to\infty} \left\| F(t) - e^{t\mathsf{A}} F^{\infty} \right\|_{L^1(\mathrm{d}v\,\mathrm{d}x)} = 0\,, \tag{59}$$

and F^{∞} *satisfies the bounds*

$$0 \leq F^{\infty}(v,x) \leq \mathcal{M}(v,x,0) \qquad \text{almost everywhere over } \mathbb{R}^D \times \mathbb{R}^D\,. \tag{60}$$

Proof. The fact that F is a global mild solution of the Cauchy problem for the Boltzmann equation means that for every $t \in [0, \infty)$

$$F(t) = e^{t\mathsf{A}} F^{\mathrm{in}} + \int_0^t e^{(t-t')\mathsf{A}} \mathcal{B}(F(t'), F(t'))\, \mathrm{d}t' . \tag{61}$$

Because $\beta \in (1 - D, 2]$ while F satisfies the hypotheses of Lemma 9, Proposition 10 implies that

$$\begin{aligned} &\int_0^\infty \left\| e^{-t'\mathsf{A}} \mathcal{B}(F(t'), F(t')) \right\|_{L^1(\mathrm{d}v\,\mathrm{d}x)} \mathrm{d}t' \\ &\quad = \int_0^\infty \left\| \mathcal{B}(F(t'), F(t')) \right\|_{L^1(\mathrm{d}v\,\mathrm{d}x)} \mathrm{d}t' < \infty . \end{aligned} \tag{62}$$

This implies (by the Cauchy criterion for families) that

$$\lim_{t\to\infty} \int_0^t e^{-t'\mathsf{A}} \mathcal{B}(F(t'), F(t'))\, \mathrm{d}t' \qquad \text{exists in } L^1(\mathrm{d}v\,\mathrm{d}x) .$$

We then define $F^{\infty} \in L^1(\mathrm{d}v\,\mathrm{d}x)$ by

$$F^{\infty} = F^{\mathrm{in}} + \int_0^{\infty} e^{-t'\mathsf{A}} \mathcal{B}(F(t'), F(t'))\,\mathrm{d}t' \,. \tag{63}$$

Upon solving this relationship for F^{in} and placing the result into (61), we find that

$$F(t) = e^{t\mathsf{A}} F^{\infty} - \int_t^{\infty} e^{(t-t')\mathsf{A}} \mathcal{B}(F(t'), F(t'))\,\mathrm{d}t' \,. \tag{64}$$

Because $e^{(t-t')\mathsf{A}}$ is an isometry in $L^1(\mathrm{d}v\,\mathrm{d}x)$ we see that

$$\left\| F(t) - e^{t\mathsf{A}} F^{\infty} \right\|_{L^1(\mathrm{d}v\,\mathrm{d}x)} \leq \int_t^{\infty} \left\| \mathcal{B}(F(t'), F(t')) \right\|_{L^1(\mathrm{d}v\,\mathrm{d}x)} \mathrm{d}t' \,.$$

But the right-hand side above vanishes as $t \to \infty$ by (62), whereby limit (59) is established.

Because by hypothesis $0 \leq F(v, x, t) \leq \mathcal{M}(v, x, t)$, it follows that

$$0 \leq e^{-t\mathsf{A}} F(v, x, t) \leq e^{-t\mathsf{A}} \mathcal{M}(v, x, t) = \mathcal{M}(v, x, 0) \qquad \text{over } \mathbb{R}^D \times \mathbb{R}^D \,.$$

By passing to the limit as $t \to \infty$ in these inequalities, we obtain

$$0 \leq F^{\infty}(v, x) = \lim_{t \to \infty} e^{-t\mathsf{A}} F(v, x, t) \leq \mathcal{M}(v, x, 0)$$

almost everywhere over $\mathbb{R}^D \times \mathbb{R}^D$. This establishes bound (60), and thereby completes our proof. □

5. Conclusion

At this point we have been unable to show that F^{∞} is a global Maxwellian. If it were, and we could show that it has the same values for its formally conserved quantities as F^{in}, then F^{∞} would be uniquely determined by the values of those quantities.

This is on-going work.

Thank You!