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A NOTE ON THE KORTEWEG-DE-VRIES EQUATION

The Korteweg-de-Vries equation

$$u_t + uu_x + \beta u_{xxx} = 0 \tag{1}$$

originally arose in the theory of surface waves on shallow water waves. Later it was encountered in numerous other problems. u is proportional to the horizontal component of the velocity which is constant over the channel depth. An analogous equation is also valid for the elevation of the surface over its undisturbed level. The KdV equation has a solution of traveling wave type, the so called soliton:

$$u = \frac{u_0}{\cosh^2\left(\sqrt{\frac{u_0}{12\beta}}(x - x_0 - st)\right)}, \quad \text{where} \quad u_0 = 3s. \tag{2}$$

This solution is very stable. Let us derive it. We will look for $u(x,t) = w(x - st) \equiv w(y)$, satisfying the boundary conditions $w(\pm\infty) = 0$ and $w'(\pm\infty) = 0$.

$$\begin{aligned}
-sw' + ww' + \beta w''' &= 0 \\
-sw' + \left(\frac{w^2}{2}\right)' + \beta w''' &= 0 \\
-sw + \frac{w^2}{2} + \beta w'' &= C \quad |\cdot 2w' \\
2\beta w''w' - 2sww' + w^2w' &= 2Cw' \\
\beta(w')^2 - sw^2 + \frac{w^3}{3} &= 2Cw + B
\end{aligned}$$

Since w and w' are zero at $\pm\infty$, $C = 0$ and $B = 0$. We continue.

$$\begin{aligned}
\beta(w')^2 &= sw^2 - \frac{w^3}{3} \\
w' &= \pm\sqrt{\frac{s}{\beta}w^2 - \frac{w^3}{3\beta}} \\
\int \frac{dw}{w\sqrt{s - \frac{w}{3}}} &= \pm\int \frac{dy}{\sqrt{\beta}}
\end{aligned}$$

The integral in the left-hand side can be taken using the substitution

$$\begin{aligned}
z &= \sqrt{s - \tfrac{w}{3}}, \quad dz = -\frac{dw}{6\sqrt{s - \frac{w}{3}}}, \\
z^2 &= s - \tfrac{w}{3}, \quad\ \ w = 3s - 3z^2.
\end{aligned}$$

Then we have

$$\begin{aligned}\int \frac{dw}{w\sqrt{s-\frac{w}{3}}} &= -6\int \frac{dz}{3s-3z^2}\\ =-2\int \frac{dz}{s-z^2} &= -\frac{2}{\sqrt{s}}\int \frac{dz}{1-\left(\frac{z}{\sqrt{s}}\right)^2}\\ =\frac{2}{\sqrt{s}}\int \frac{dp}{p^2-1} &= \frac{1}{\sqrt{s}}\log\left|\frac{p-1}{p+1}\right|,\end{aligned}$$

where

$$p=\frac{z}{\sqrt{s}}=\sqrt{1-\frac{w}{3s}}.$$

This equation shows that $p \le 1$. Now we continue.

$$\begin{aligned}\frac{1}{\sqrt{s}}\log\left|\frac{p-1}{p+1}\right| &= \pm\frac{y}{\sqrt{\beta}}+C\\ \left|\frac{p-1}{p+1}\right| &= e^{\pm(y-x_0)\sqrt{s/\beta}}.\end{aligned}$$

Let us introduce

$$A=\pm(y-x_0)\sqrt{s/\beta}.$$

Opening the absolute value we obtain

$$\begin{aligned}1-p &= pe^A+e^A\\ p(1+e^A) &= 1-e^A\\ p &= \frac{1-e^A}{1+e^A}.\end{aligned}$$

Now we return to the variable w.

$$w=3s-3z^2=3s(1-p^2)=3s\left(1-\left(\frac{1-e^A}{1+e^A}\right)^2\right).$$

Using the identity

$$1-\left(\frac{1-e^A}{1+e^A}\right)^2=\frac{1}{\cosh^2\frac{A}{2}}$$

we get

$$w=\frac{3s}{\cosh^2\left(\frac{y-x_0}{2}\sqrt{\frac{s}{\beta}}\right)}.$$

Introducing $u_0=3s$ and plugging in $y=x-st$ we obtain Eq. (2).

References

[1] G. I. Barenblatt, Scaling, self-similarity, and intermediate asymptotics, Cambridge University Press, 1996

[2] Walter A. Strauss, Partial Differential Equations: An Introduction, 2nd edition, John Wiley and Sons, 2008