The Space of Cluster Polylogarithms

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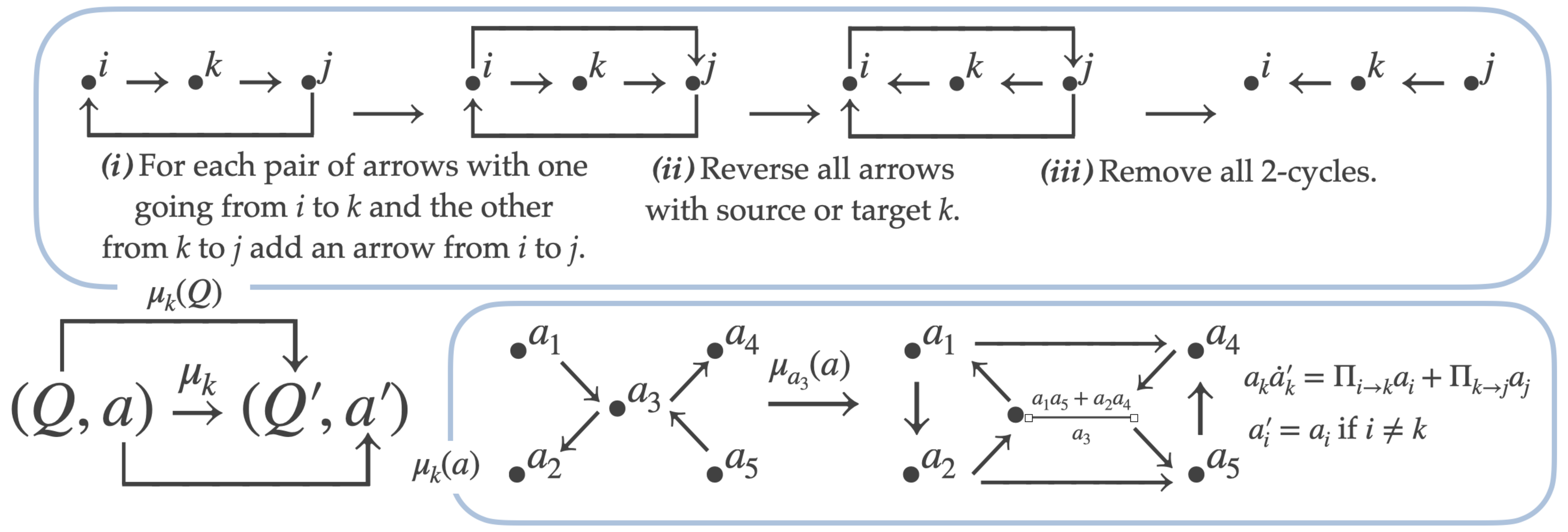
Cluster Algebra

Quiver: A directed graph with no one or two cycles with vertices labeled $1, 2 ... n$.

Seed: A pair (Q, a) for a quiver, Q and a tuple $a = (a_1, a_2, ... a_n)$ called a *cluster* labeling the vertices of Q respectively. The a_i's are called *A-coordinates*.

For each vertex, k, of a seed, a ***mutation*** at vertex k produces a new seed with a new quiver and a new list of variables in the following manner:



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The cluster algebra generated by a seed Q is the $\mathbf{Q}$−Algebra generated by all A-coordinates appearing in any seed obtained by mutating the original seed.

The Grassmannian

The **Grassmannian** over a field F, denoted $Gr(p, n)$ is the space of p-dimensional sub-spaces of an n-dimensional vector space over F.

Fact: An element of $Gr(p, n)$ can be identified with a $p \times n$ matrix which can in turn be identified by the determinants of p−column minors, called **Plucker Coordinates** (modulo some left action from SL_p).

The **coordinate ring** of the Grassmannian is the space of **Plucker Coordinates** modulo relations between them.

Theorem: The coordinate ring of $Gr(p, n)$ has a cluster algebra structure.

Polylogarithm Relation

Classical polylogarithms appeared in the 18th and 19th centuries under different guises in the works of Leibniz, Euler, Spence, Abel, Kummer, Lobachevsky, and many others.

- **Classical Polylogarithm** $Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
- **Multiple Polylogarithms** $Li_{n_1,...,n_d}(x_1, ..., x_d) = \sum_{k_1 < ... < k_d} \frac{x_1^{k_1} ... x_d^{k_d}}{k_1^{n_1} ... k_d^{n_d}}$

The *weight* of a polylogarithm is $n_1 + n_2 ... + n_d$ and the *depth* is d, the number of variables.

It was noticed early on that polylogarithms satisfy functional equations: a linear combination of polylogarithms of some weight are equal to lower weight terms. Here is the famous ***five-term relation*** for the dilogarithm obtained by Abel:

$$Li_2(-x) - Li_2(-y) + Li_2(-\frac{1+y}{x}) - Li_2(-\frac{1+x+y}{xy}) + Li_2(-\frac{1+x}{y}) = \textit{lower weight terms}$$

which holds when $0 < x < y < 1$. Similar relations for Li_3, Li_4, and Li_5 were found by Kummer.

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The relations proved very important to understanding polylogarithms. The discoveries by Abel, Kummer and others led to the following conjecture:

Conjecture: There exists a relation among the polylogarithms of each weight.

Cluster Polylogarithms

Observation: The five-term relation is a sum of dilogarithms evaluated at *A-coordinates* of a cluster algebra! This motivated the theory of **Cluster Polylogarithms**, which are polylogarithms evaluated at *A-coordinates*.

The space $CL_n(S)$ is the space of cluster polylogarithms of weight n associated to the seed S.

More formally, it is all of the following expressions of the form

$$\sum_{I=(i_1,...,i_n)} k_I \int_{\gamma} d\log(a_{i_1})...d\log(a_{i_n})$$

that satisfy two conditions:

- **Cluster adjacency** For each I, $a_{i_1} ... a_{i_n}$ all lie in a single cluster.
- **Cluster integrability** The iterated integral only depends on the homotopy class of γ

Symbols of Polylogarithms

The **symbol** of a polylogarithm is an algebraic invariant assigned to each cluster polylogarithm in the following way:

$$\sum_{I=(i_1,...,i_n)} k_I \int_{\gamma} d\log(a_{i_1})...d\log(a_{i_n}) \quad \rightarrow \quad \sum_{I=(i_1,...,i_n)} k_I \left(a_{i_1} \otimes ... \otimes a_{i_n} \right)$$

Fact: If a linear combination of polylogarithms forms a polylogarithm relation, this can be recovered from only the symbols.

Thus we identify polylogarithms and their symbols. Using polylogarithm symbols, Matveiakin and Rudenko gave a purely algebraic and combinatorial description of $CL_n(S)$.

Theorem: A polylogarithm P is in $CL_2(S)$ if the symbol $Sym(P)$ lies in the space generated by

$$\left\{ \frac{M_1}{aa'} \wedge \frac{M_2}{aa'} \,\middle|\, a \text{ an A-coordinate of } S \right\}$$

where M_1 and M_2 denote product in and out respectively, and a, a' denote the variable at a vertex before and after mutation at that vertex respectively.

A polylogarithm P is in $CL_n(S)$ if for all $1 \le j \le n-1$, we have

$$\sum_{I=(i_1,...i_n)} k_I a_{i1} \otimes ... a_{ij} \wedge a_{ij+1} \otimes a_{ij+2} ... \otimes a_{in} \quad \in \quad A \otimes ... CL2(S) \otimes ... A$$

Theorem by Andrei Matveiakin and Daniil Rudenko:

Dimension of the space $CL_n(Gr(2, m))$ of weight $n \ge 2$, $m \ge 4$ equals to

$$\binom{m-1}{3} + \binom{m-1}{4} + ... + \binom{m-1}{n+1}$$

What about for $Gr(3, 6), Gr(3, 7), Gr(3, 8)$, etc.?

The project: Calculation of dimension

$$\wedge^2(\mathbb{Z}[A]) \quad > \quad CL_2$$

generated by $a \wedge b$ generated by $\frac{M_1}{aa'} \wedge \frac{M_2}{aa'} = X \wedge (X+1)$

Wedge Product:
$a \wedge b = -b \wedge a$
$a \wedge a = 0$
$a \wedge bc = a \wedge b + a \wedge c$
$a \wedge \frac{b}{c} = a \wedge b - a \wedge c$

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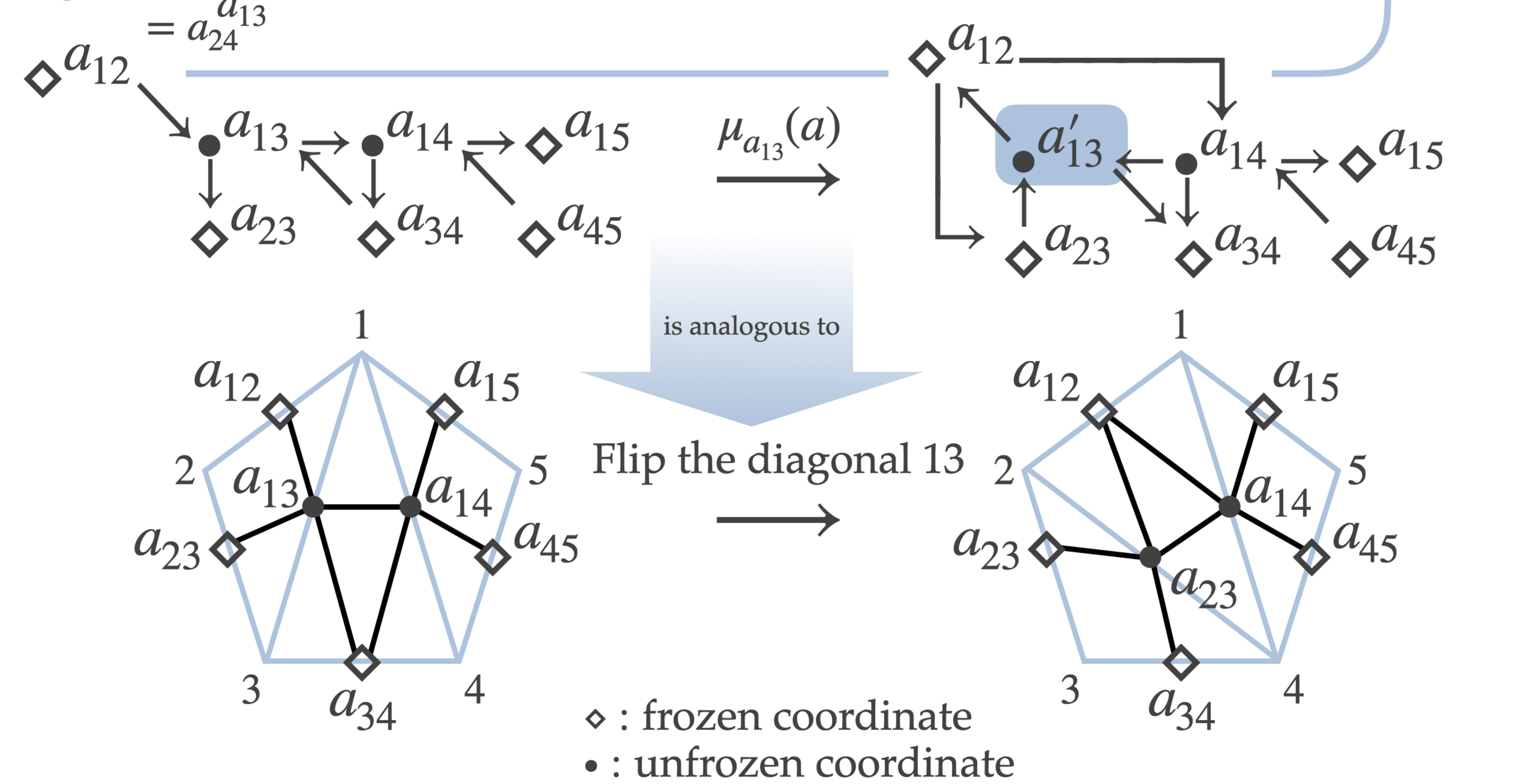
Goal: Find the dimension and basis for $CL_n(Gr(3, m))$ for $m = 6, 7, 8$.

Method: Create computational algorithms to compute the clusters and coordinates for the Grassmannians. From this, use linear algebra to construct a basis for $CL2$. Then, implement the linear map from symbols to tensor and wedge products, and use linear algebra to find the preimage of the necessary subspace to construct a basis for CL_n.

Example for $Gr(2, 5)$

$$a_{13}' = \frac{a_{12}a_{34} + a_{14}a_{23}}{a_{13}} \text{ by defination of the mutation on A-coordinates}$$
$$= \frac{a_{13}a_{24}}{a_{13}} \text{ by the Plucker relation}$$
$$= a_{24}$$



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$$\frac{M_1}{a_{13}a_{13}'} \wedge \frac{M_2}{aa'} = \frac{a_{12}a_{34}}{a_{13}a_{24}} \wedge \frac{a_{14}a_{23}}{a_{13}a_{24}}$$

$= a_{12}a_{34} \wedge a_{14}a_{23} - a_{12}a_{34} \wedge a_{13}a_{24} - a_{13}a_{24} \wedge a_{14}a_{23}$
$= a_{12} \wedge a_{14} + a_{12} \wedge a_{23} + a_{14} \wedge a_{12} + a_{14} \wedge a_{23} - a_{12} \wedge a_{13} - a_{12} \wedge a_{24} - a_{34} \wedge a_{13} - a_{34} \wedge a_{24} - a_{13} \wedge a_{14} - a_{13} \wedge a_{23} - a_{24} \wedge a_{14} - a_{24} \wedge a_{23}$

$$\frac{a_{12}a_{34}}{a_{13}a_{24}} \wedge \frac{a_{14}a_{23}}{a_{13}a_{24}} - \frac{a_{23}a_{45}}{a_{24}a_{35}} \wedge \frac{a_{25}a_{34}}{a_{24}a_{35}} + \frac{a_{34}a_{15}}{a_{35}a_{14}} \wedge \frac{a_{13}a_{45}}{a_{35}a_{14}} - \frac{a_{45}a_{12}}{a_{14}a_{25}} \wedge \frac{a_{24}a_{15}}{a_{14}a_{25}} + \frac{a_{15}a_{23}}{a_{25}a_{13}} \wedge \frac{a_{35}a_{12}}{a_{25}a_{13}} = 0$$

$$dim(CL_2(Gr(2, 5)) = 4$$

References

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[2] D. Rudenko, "On the goncharov depth conjecture and a formula for the volumes of orthoschemes," *arXiv:2012.05599*, Dec 2020.

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