Numerical Estimates of Hausdorff Dimension

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Introduction/Motivation

Imagine you're a cartographer attempting to make the most accurate map of Britain, and you run into the problem of how to measure the coastline. You notice that as you increase the precision of your measurement the length increases indefinitely!

Some sets can behave pathologically under the Lebesgue measure, so we instead use the Hausdorff measure to quantify them. The Fractal, or box counting, dimension is another way but it has been shown to be very poor for sets that are not self-similar.

Hausdorff Measure & Hausdorff Dimension

Definition

Let (X, d) be a metric space, $\alpha \geq 0, \delta > 0$, we define the following function:

$$\mathcal{H}^{\alpha}_{\delta}(A) := \inf \left\{ \sum_{n=1}^{\infty} (\text{diam}(U_n))^{\alpha} : (U_n)_n \text{ are open, } \text{diam}(U_n) < \delta, A \subseteq \bigcup_{n=1}^{\infty} U_n \right\} \quad (1)$$

Remark

The function in (1) is non-increasing in δ and we define the *Hausdorff Outer Measure* as follows:

$$\mathcal{H}^{\alpha}(A) := \lim_{\delta \to 0} \mathcal{H}^{\alpha}_{\delta}(A) = \inf_{\delta > 0} \mathcal{H}^{\alpha}_{\delta}(A).$$

It can be shown that $\mathcal{H}^{\alpha}$ can be restricted to a σ-algebra, the σ-algebra of the Hausdorff measurable sets and there becomes a measure.

Definition

Let $A \subseteq \mathbb{R}^d$ be a set, we define the *Hausdorff dimension of A* as follows:

$$\dim_H(A) := \inf\{\alpha \geq 0 : \mathcal{H}^{\alpha}(A) = 0\} = \sup\{\alpha \geq 0 : \mathcal{H}^{\alpha}(A) = \infty\}.$$

Remark

Note that for all $\varepsilon > 0$, there are $\alpha_{\varepsilon}, \beta_{\varepsilon}$ such that

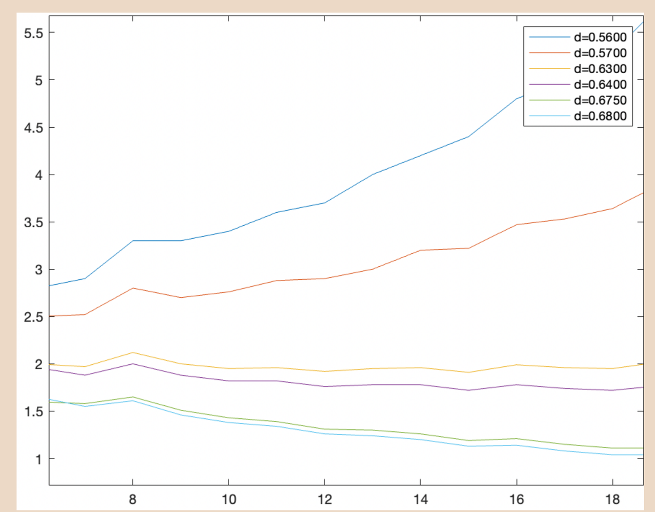
$$\dim_H(A) \leq \alpha_e < \dim_H(A) + \varepsilon, \quad \dim_H(A) - \varepsilon < \beta_{\varepsilon} \leq \dim_H(A)$$

and also implying that

$$\mathcal{H}^{\dim_H(A)+\varepsilon} = 0, \quad \mathcal{H}^{\dim_H(A)-\varepsilon} = \infty.$$

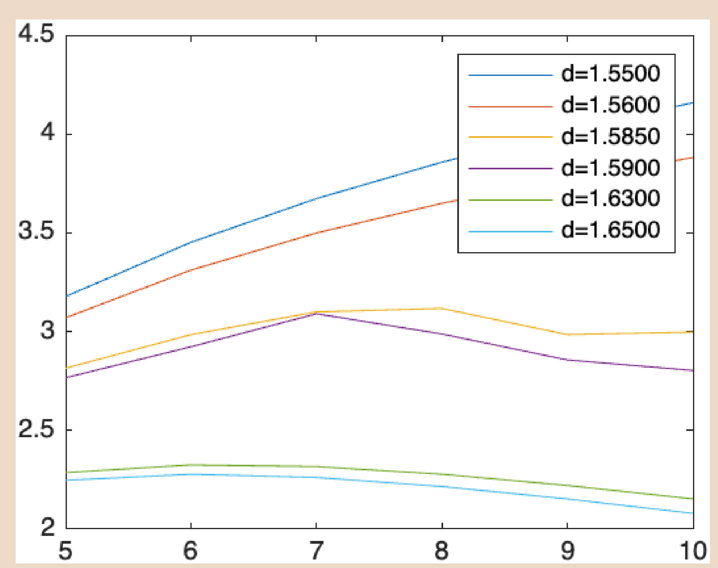
Results in One dimension

By initializing a tree of dyadic intervals with diameter 2^{-k} at level k, then intersecting the tree with the set we want to measure we can find the infimum covering of the set. Using this we can estimate the Hausdorff measure by varying the maximum level k over many d as the true value of d will have constant measure over varying k. I was able to measure the Hausdorff dimension of the Cantor set and matched the widely accepted value of 0.63.



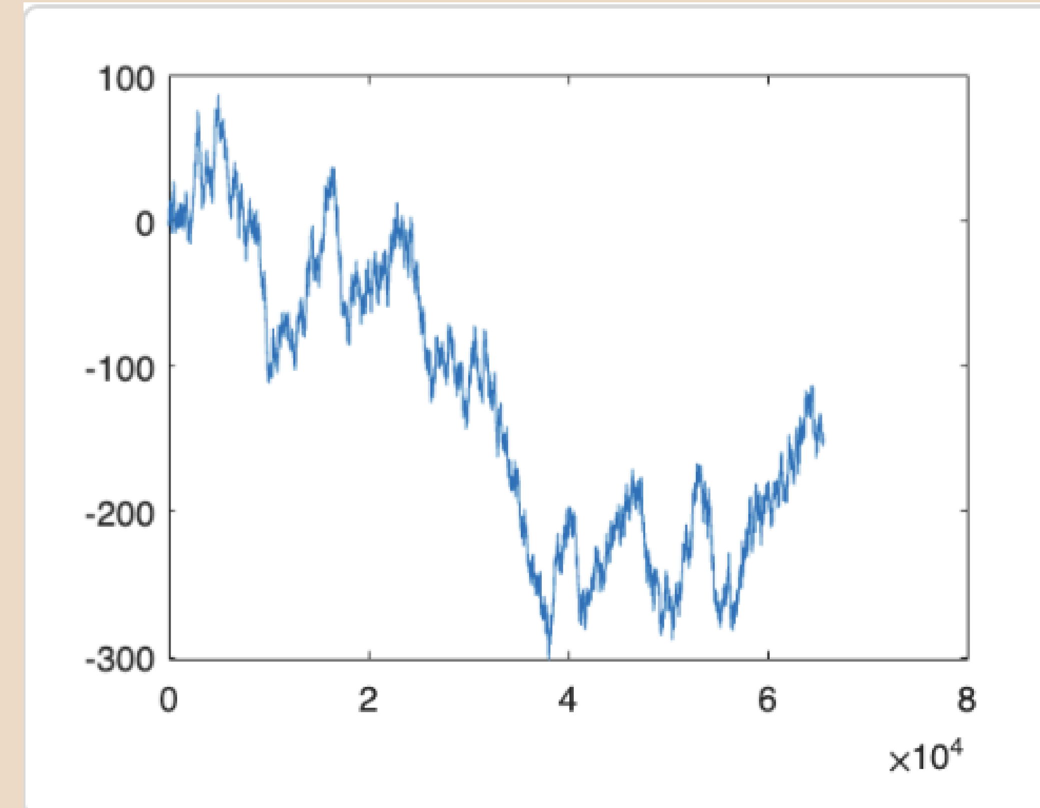
Results in Two Dimensions

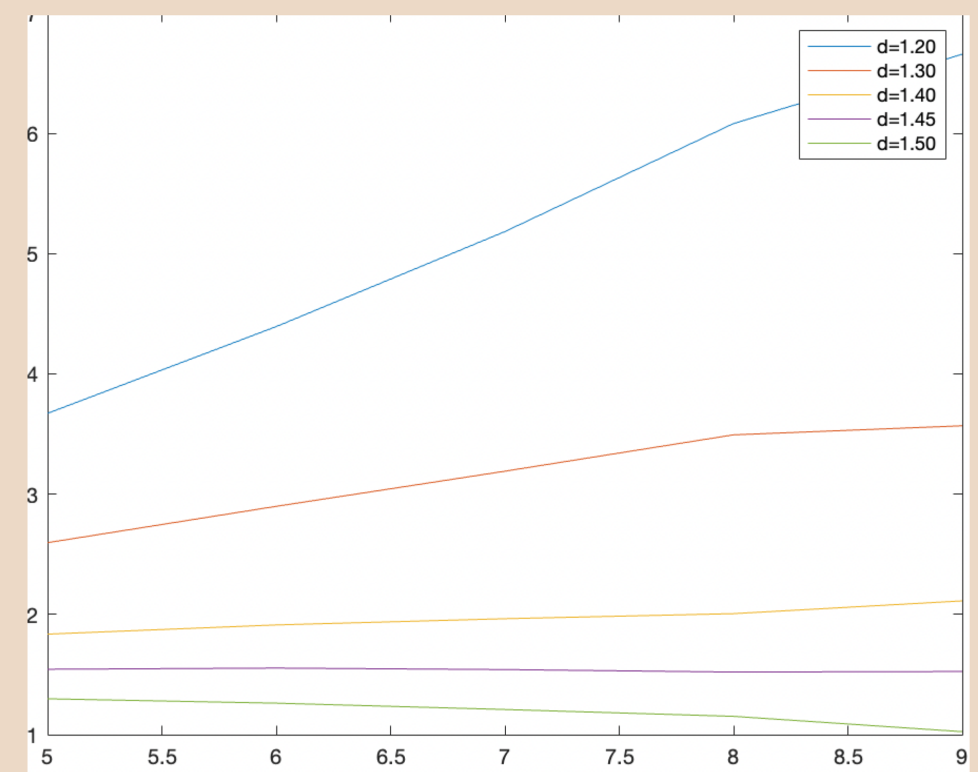
By modifying the tree from dyadic intervals to dyadic squares we can measure sets embedded in $\mathbb{R}^2$, My results for the Sierpinsky triangle below match the widely accepted value of 1.585.



Two Dimensional Results Contd.

The following figure shows the algorithms value of dimension for the pictured Brownian motion, while the fractal dimension gives a value from 1.55-1.6, this algorithm predicts a value of 1.45.





Conclusions/Future Work

While the Hausdorff dimension agrees with the fractal dimension for self-similar fractals, fractal dimension is a very poor estimate for sets which are not self-similar. This algorithm gives us a way to measure the Hausdorff dimension when it differs from the fractal dimension, while still agreeing with the fractal dimension for self-similar fractals. In the future it would be possible to expand this to three dimensions and generalize an efficient algorithm for the intersection between the tree and set, as the bulk of the computing was devoted to this process.

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