Homework 5. Due March 25

1. **(5 pts)** The Metropolis-Hastings algorithm is a modification of the Metropolis algorithm in which the transition matrix Q generating proposed moves does not need to be symmetric (see the last paragraph in Section 3.1 in markov_chains.pdf). Prove that the frequencies of visits of states in the Metropolis-Hastings algorithm approach the invariant distribution.

2. **(5 pts)** A random variable $\eta : \Omega \to [0, \infty]$ has an exponential distribution if

$$\mathbb{P}(\eta > t) = e^{-\lambda t} \quad \text{for all} \quad t \geq 0,$$

where $\lambda \geq 0$ is a parameter.

Prove that a random variable $\eta : \Omega \to (0, \infty]$ has an exponential distribution if and only if it has the following memoryless property:

$$\mathbb{P}(\eta > t + s \mid \eta > s) = \mathbb{P}(\eta > t) \quad \text{for all} \quad s, t \geq 0. \tag{1}$$

Hint: To deduce an exponential distribution from Eq. (1), you may proceed as follows. Introduce

$$F(t) := \mathbb{P}(\eta > t) \quad \textit{and define} \quad \lambda := -\log F(1).$$

Then write $F(1) = F\left(\frac{1}{n} + \ldots + \frac{1}{n}\right)$ *and apply the memoryless property. Establish the exponential form for* F *for all rational* t*. Then use the fact that any real number can be approximated by rational numbers. Etc.*

3. **(5 pts)** Consider a continuous-time Markov chain with a generator matrix L with all diagonal entries being equal to $-\lambda$ where $\lambda > 0$. Prove that the number of jumps occurring in the corresponding Markov jump process in a time interval $[0, t]$ is distributed according to the Poisson distribution with parameter λt, i.e.,

$$\mathbb{P}(J_k \leq t, J_{k+1} > t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

Hint: First show that for a small time interval $[0, h]$, $\mathbb{P}(J_1 \leq h, J_2 > h) = \lambda h + o(h)$ *and* $\mathbb{P}(J_2 \leq h) = o(h)$ *where* $o(h)$ *denotes any function* $f(h)$ *such that* $\lim_{h \to 0} h^{-1} f(h) = 0$*. Then partition the interval* $[0, t]$ *into* n *subintervals where* n *is large. Show that* $\mathbb{P}(J_k \leq t, J_{k+1} > t) = \mathbb{P}(A) + \mathbb{P}(B)$ *where* A *is the event that exactly* k *subintervals contain one jump and the rest contain no jumps, and* B *is the event that at least one of the subintervals contains more than one jump and there are* k *jumps in total. Let* n *tend to* ∞ *and prove that* $\mathbb{P}(B) \to 0$ *while* $\mathbb{P}(A) \to e^{-\lambda t}(\lambda t)^k (k!)^{-1}$.