MATH858T Maria Cameron

Homework 8. Due April 15

1. **(6pts)** The Langevin equation models the dynamics of heavy particles in the potential force field pushed around by light particles:

$$\begin{aligned} dq &= \frac{p}{m}dt \\ dp &= (-\nabla V(q) - \gamma p)dt + \sqrt{2\gamma m \beta^{-1}}dw. \end{aligned} \tag{1}$$

Here (q, p) are the positions and momenta of the heavy particles, γ is the friction coefficient, m is the mass of the heavy particles, and $-\nabla V(q)$ is the potential force acting on the heavy particles. Eq. (1) can be written in the form

$$X_t = b(X_t)dt + \sigma(X_t)dw$$

by introducing

$$X_t = \begin{bmatrix} q \\ p \end{bmatrix}, \quad b(x) = \begin{bmatrix} p/m \\ -\nabla V(q) - \gamma p \end{bmatrix}, \quad \sigma = \sqrt{2\gamma m \beta^{-1}} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

(a) Show that the infinitesimal generator for Eq. (1) is given by

$$L = \frac{p}{m} \cdot \nabla_q - \nabla_q V \cdot \nabla_p + \gamma \left(-p\nabla_p + m\beta^{-1}\Delta_p\right).$$

(b) Derive the expression for the adjoint generator

$$L^* g = -\frac{p}{m} \cdot \nabla_q g + \nabla_q V \cdot \nabla_p g + \gamma \left(\nabla_p \cdot (pg) + m\beta^{-1}\Delta_p g\right).$$

(c) Solve the stationary Fokker-Planck equation and show that the invariant pdf is given by

$$\mu(q, p) = \frac{1}{Z}e^{-\beta H(q,p)}, \quad \text{where} \quad H(q, p) = \frac{|p|^2}{2m} + V(q).$$

2. **(6pts)** Apply the Euler-Maruyama and Milstein's methods to the geometric Brownian motion

$$dX_t = \lambda X_t dt + \mu X_t dw, \quad X_0 = 1, \quad t \in [0, 1].$$

This equation is solvable analytically. Let X_j be its analytic solution at the mesh points, while Y_j be its numerical solution, $0 \le j \le n$.

(a) Generate a Brownian random walk $w = \{w_j\}_{0 \le j \le 1024}$ on the interval $[0, 1]$. Out of it, create coarser Brownian random walks with 64 and 256 steps. Plot the exact solution X_t on the interval $[0, 1]$ and the numerical solutions by the Euler-Maruyama and Milstein's methods with $n = 64$ and 256 points. Plot all these graphs in the same figure. Write a summary of your observations regarding the accuracy of these methods.

(b) Determine experimentally the weak and strong orders of convergence of the Euler-Maruyama and Milstein's methods. Use time steps $h = 2^{-n}$ for $n = 5, 6, 7, 8, 9, 10$. Repeat calculations $M = 1000$ times for each time step. Plot the graphs of the weak error

$$\max_{0 \le j \le n} |E[Y_j] - E[X_j]|$$

and the strong error

$$\max_{0 \le j \le n} E[|Y_j - X_j|]$$

as functions of h in the log-log scale. Determine the slopes of these graphs. These slopes will be the weak and the strong orders of convergence respectively. Use can use the matlab function `polyfit` for determination of the slopes.

Hint: you might find the paper [1] very helpful. It contains a number of programs, in particular, for finding strong and weak orders of converge.

3. **(6pts)** Consider the stochastic cubic oscillator

$$dX_t = -X_t^3 dt + \sqrt{2\beta^{-1}} dw_t, \quad X_0 = 4. \tag{2}$$

Note that the function $b(x) = -x^3$ does not satisfy the global Lipschitz condition. Set $\beta = 1$.

(a) Show that the invariant pdf is given by

$$\pi(x) = \frac{1}{Z} e^{-\beta x^4/4}.$$

(b) Generate a numerical solution on the time interval $[0, 10]$ using Euler-Maruyama method with a very small time step $h = 10^{-4}$. Keep the generated Brownian random walk w. Plot the graph of the computed solution. You will treat this solution as the "true solution".

(c) Create coarser random walks out of w with time step 0.3125. Try to compute solutions using the Euler-Maruyama. It is likely that these solutions will quickly blow up. Report your observations.

(d) Using the same coarser random walks and time step 0.3125, compute numerical solutions using MALA. Plot them on the same figure as the "true solution".

Hint: You should obtain a figure similar to Fig. 3.1 in [2].

References

[1] Desmond J. Higham, An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations, SIAM Review, **43**, 3, (2001) 525-546

[2] N. Bou-Rabee, E. Vanden-Eijnden, Pathwise Accuracy and Ergodicity of Metropolized Integrators for SDEs, Commun Pure Appl Math, 63, 655-696, 2010