This honors course will cover the basic ordinary differential equations material (Chapters 10-13), plus some additional, more challenging topics.
The final grade will be based on Homework (20%) three in-class exams (15% each), and a final exam (35%).

Make-up policy: There will be no make-ups for in-class exams. In the case of an absence due to illness, religious observance, participation in a University activity at the request of University authorities, or other compelling circumstances, your blank grade will be replaced by the average of your other in-class exams. Also, your 2 lowest homework grades will be dropped. This is meant to accommodate all excused absences, and no late homework will be accepted.

Homework 1, due Tuesday February 2:
Page 466: 7, 9, 24
Page 478: 1, 5, 18, 23, 28
(Variant of Example 7, p 476. Check your notes to see how this was solved in class). Assume \( \frac{d^2 r}{dt^2} + \sin r = 0 \). Find \( f(r) \) so that the "energy" \( \frac{1}{2} \left( \frac{dr}{dt} \right)^2 + f(r) \) is constant in \( t \) (or conserved), and justify your answer.
Find \( v_0 > 0 \), as small as possible, so that if \( r(0) = 0 \) and \( r'(0) > v_0 \) then \( r(t) \) increases without bound as \( t \to \infty \). (You can assume the equation has a solution for all \( t \)).

Homework 2, due Thursday, February 4:
Page 487: 5, 8, 9
Page 489: 20
Additional problems:
Problem 1:
Consider the equation \( y' = t^2 + y^2, y(0) = 1 \). Find the approximate value of \( y(0.2) \) in 3 different ways:
a) Euler method in one step
b) Euler method in 2 steps
c) Improved Euler method in one step

Problem 2: (Challenging) Linear differential inequalities. Throughout this problem assume \( t \geq 0 \).

(a) If \( y' - y = f \) prove \( y(t) = e^{t} \left( y(0) + \int_{0}^{t} e^{-s} f(s) ds \right) \).

(b) (Inequality version of (a)). Carefully repeat the previous argument to show if \( f_1(t) \leq y'(t) - y(t) \leq f_2(t) \) for all \( t \), then

\[
e^{t} \left( y(0) + \int_{0}^{t} e^{-s} f_1(s) ds \right) \leq y(t) \leq e^{t} \left( y(0) + \int_{0}^{t} e^{-s} f_2(s) ds \right)
\]

The fun part: Let \( y_n(t) = \left( 1 + \frac{t}{n} \right)^n - e^t \). You probably know already what \( \lim_{n \to \infty} y_n(t) \) is. In this part, you prove it using differential inequalities.

(c) Prove \( y_n' - y \leq 0 \) and \( y_n(0) = 0 \). What can you conclude about \( y_n \)?

(d) Prove \( y_n' - y_n \geq -\frac{t}{n} e^t \). What lower bound for \( y_n \) do you conclude from here?

(e) Use (c) and (d) to find (and justify!) \( \lim_{n \to \infty} y_n(t) \).

Homework 3 , due Tuesday, February 9:

Page 498: 7, 15, 16, 31
Page 506: 1, 12
Page 512: 1

Challenging problem. In this problem you will discover Hermite functions as an application of factoring second differential operators. This involves important ideas from Math and Quantum Mechanics, and no messy calculations.

Denote \( D_+ = \frac{d}{dx} + x \) and \( D_- = \frac{d}{dx} - x \). These act on functions, for instance \( (D_+ f)(x) \) is a new function given by \( (D_+ f)(x) = \frac{df}{dx}(x) + xf(x) \). They can also be composed. For instance \( D_+ D_- f = D_+ (D_- f) \) is a new function.

(a) Check \( D_- D_+ = \frac{d^2}{dx^2} - x^2 + 1 \) This means \( (D_- (D_+ f))(x) = \frac{df}{dx}(x) - x^2 f(x) + f(x) \).

(b) Check \( D_+ D_- = \frac{d^2}{dx^2} - x^2 - 1 \). Thus \( D_- D_+ - D_+ D_- = 2 \). Mathematicians denote \( D_- D_+ - D_+ D_- = [D_-, D_+] \) and call it a "commutator".

(c) Compute \( D_- D_+ e^{-\frac{x^2}{2}} \). (Hint: Do \( D_+ e^{-\frac{x^2}{2}} \) first.)

(d) Compute \( D_- D_+ D_- e^{-\frac{x^2}{2}} \) (Hint: The answer is a multiple of \( D_- e^{-\frac{x^2}{2}} \), and the calculation takes one line, using commutators).
(e) Generalize your previous calculation to find $D_-D_+(D_-)^n e^{-\frac{t^2}{2}}$.

(Hint: Again, this is a multiple of $(D_-)^n e^{-\frac{t^2}{2}}$)

(f) Let $h_n(x) = (D_-)^n e^{-\frac{x^2}{2}}$. Show that

$$\frac{d^2}{dx^2} h_n(x) - x^2 h_n(x) = (-2n - 1) h_n(x)$$

Note: $h_n$ are the Hermite functions. They are "eigenfunctions" of $\frac{d^2}{dx^2} - x^2$ corresponding to the eigenvalues $(-2n - 1)$.

Homework 4, due Tuesday, March 9
Page 520: 1, 5, 23
Page 529: 7, 15
Also: You are told $y_1 = t$ is a solution of

$$t^2 y'' - 2ty' + 2y = 0$$

Look for a second solution of this equation of the form $y_2 = uy_1$. What equation must $u$ satisfy? Derive that equation and then solve it.
If you get stuck, read Example 7, page 522.
Page 537: 26
Page 548: 17, 22

Recommended for Thursday, March 23: Read and understand section 1D, page 580.
Solve (but do not turn in) problem 5, page 584.
and also:
Assume

$$\frac{dx}{dt} = xy$$
$$\frac{dy}{dt} = -xy$$

Prove that if $x(0) > 0, y(0) > 0$ then $x(t) > 0$ and $y(t) > 0$ as long as the solutions exist.

Homework due Tuesday April 13:
Page 624: 7
Page 630: 7d
Page 637: 1, 2
Page 639: 2, 4, 8
Page 644: 2
Page 652: 1, 3, 4.
Stability theorems. Besides Theorem 4.2, page 655, we will use the following Theorem.

Definition Consider the system \( x'(t) = F(x(t)) \), where \( x : \mathbb{R} \to \mathbb{R}^n, F : \mathbb{R}^n \to \mathbb{R}^n \). Assume \( F(x_0) = 0 \). A function \( L \) defined on a neighborhood \( V \) of \( x_0 \), continuous on \( V \), differentiable for \( x \neq x_0 \), is a Liapunov function for the above system if 
\[
L(x_0) = 0, \quad L(x) > 0 \quad \text{for all} \quad x \in V, \quad x \neq x_0, \quad \text{and} \quad L' := \nabla L(x) \cdot F(x) \leq 0 \quad \text{for all} \quad x \in V, \quad x \neq x_0.
\] 
\( L \) is a strict Liapunov function if \( L' < 0 \) all \( x \in V, \ x \neq x_0 \).

Theorem 0.1. Consider the system of equations \( x'(t) = F(x(t)) \), where \( x : \mathbb{R} \to \mathbb{R}^n \). Assume \( x_0 \) is an equilibrium solution (\( F(x_0) = 0 \)). If there exists a Liapunov function \( L \), then \( x_0 \) is stable. If there exists a strict Liapunov function \( L \), then \( x_0 \) is asymptotically stable.

Problems due Tuesday April 20:
Page 657-658: 5, 6, 11
and also
Find a Liapunov function for
\[
\begin{align*}
x' &= 2yz \\
y' &= -xz \\
z' &= -z^3
\end{align*}
\]
for the equilibrium (0,0,0), and discuss its stability.

Problems due Tuesday, May 4
Page 713: 1, 5, 22
Page 720: 1, 5, 11, 16
Page 727: 1, 6, 16, 19

Exam 3: Thursday April 29.
Final exam: Tuesday, May 18 from 1:30 to 3:30.