1. Notes on compact sets

This is similar to ideas you learned in Math 410, except ”open sets” had not yet been defined.

**Definition 1.1.** $K \subseteq \mathbb{R}^n$ is compact if for every covering of $K$ by open sets $V_\alpha$ there exists a finite subcover $V_{\alpha_1}, \ldots V_{\alpha_m}$ (that is, if $V_\alpha$ are open sets so that $K \subseteq \bigcup_\alpha V_\alpha$ then you know for sure there exist finitely many $V_\alpha$ such that $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_m}$)

The following will be checked in detail in class:

- If $K$ is compact, then $K$ is bounded. (Use $V_m = B_m(0)$).
- If $K$ is compact, then $K$ is closed. (Proof: Let $u_i$ a sequence in $K$ converging to $u$. If $u$ is not in $K$, then the sets
  \[\{1/m < ||x - u|| < m\}\]
  are an open cover of $K$. This can’t have a finite subcover (why?).

Conversely, we prove that $K$ closed and bounded implies $K$ compact.

**Proposition 1.2.** If $K$ is a closed subset of a compact set $L$, then $K$ is compact.

*Proof.* Let $\{V_\alpha\}$ be an open cover of $K$. Then $\{V_\alpha\}$ together with the open set $K^c$ cover $L$. There exists a finite subcover of $L$ consisting of some $V_\alpha$s and possibly $K^c$. That finite subcover is also a cover of $K$, and you can remove $K^c$ from it, since it has no points in common with $K$. □

Now let $K$ be closed and bounded. It is contained in some closed cube $C$, so it suffices to show a closed cube is compact.

**Proposition 1.3.** Let $C$ be a closed cube in $\mathbb{R}^n$. Then $C$ is compact.

*Proof.* Let $\{V_\alpha\}$ be an open cover of $C$. Assume, by contradiction, it has no finite subcover. Divide $C$ into $2^n$ subcubes (of side half the side of $C$). At least one of them say $C_1$, has no finite subcover of $V_\alpha$s. Divide it again into $2^n$ subcubes. At least one of those cubes, called $C_2$ has no finite subcover. Repeat the procedure to get a nested sequence ...

...$C_3 \subseteq C_2 \subseteq C_1 \subseteq C$ of cubes with sides converging to 0, and the property that none can be covered by finitely many $V_\alpha$s. By the nested
cubes theorem (exercise) there exists a unique $u \in C_i$ for all $i$, and that $u$ belongs to at least one $V_{a_0}$. But then for all $i$ sufficiently large, $C_i$ is covered by just one $V_{a_0}$, contradiction. \qed

2. Notes on the inverse function theorem

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$, assume that $DF(x^*)$ is invertible. We will show that there exists $\delta_0 > 0$, such that for every $0 < \delta < \delta_0$ and every $y$ satisfying $\|F(x^*) - y\| < \frac{\delta}{2\|DF(x^*)^{-1}\|}$, there exists $x$ with $\|x - x^*\| \leq \delta$ such that $F(x) = y$.

The solution $x$ can be obtained as the limit of the sequence $x_{k+1} = x_k - (DF)(x^*)^{-1}(F(x_k) - y)$, where $x_1$ is chosen so that $\|x - x_1\| \leq \delta_0$. Notice some similarity with the Newton method for solving $f(x) - y = 0$. Actually, we will obtain the limit as a fixed point, $x = x - (DF)(x^*)^{-1}(F(x) - y)$.

The main step, which will be checked in class, is that there exists $\delta_0 > 0$ such that

$$\|x - z - (DF(x^*))^{-1}(F(x) - F(z))\| \leq \frac{1}{2}\|x - z\|$$

for all $x, z \in B_{\delta_0}(x^*)$ (closed ball). This follows by noticing that the derivative matrix of $G(x) = x - (DF(x^*))^{-1}F(x)$ evaluated at $x^*$ is 0. Once that is established, consider

$$T : \overline{B}_\delta(x^*) \to \mathbb{R}^n$$

given by $T(x) = x - (DF)(x^*)^{-1}(F(x) - y)$. We will check in class that

$$T : \overline{B}_\delta(x^*) \to \overline{B}_\delta(x^*)$$

is a contraction with Lipschitz constant $\frac{1}{2}$, hence it has a fixed point, which solves $F(x) = y$.

3. Convolutions

Definition 3.1. Let $g : \mathbb{R}^n \to \mathbb{R}$. The support of $g$ is the closure of the set $\{x \in \mathbb{R}^n | g(x) \neq 0\}$. Recall the closure of a set is the union of that set and its boundary, and is always a closed set.

Definition 3.2. Let $f, g : \mathbb{R} \to \mathbb{R}$ continuous, assume $g$ has compact support. The convolution of $f$ and $g$ is defined by

$$f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy = \int_{\mathbb{R}} f(x-y)g(y)dy$$
Notice that if \( g = 0 \) outside some \([-R, R]\), then \( f(y)g(x-y) = 0 \) for \( y \) outside \([x-R, x+R]\), so the integrals are in fact over bounded intervals, \([-R, R]\) and \([x-R, x+R]\).

**Theorem 3.3.** Let \( f, g : \mathbb{R} \to \mathbb{R} \) uniformly continuous, assume \( g \) has compact support contained in \([-1, 1]\) and \( \int \mathbb{R} g(x) \, dx = 1 \). Define \( g_\delta(x) = \frac{1}{\delta} g\left(\frac{x}{\delta}\right) \). Then

\[
f \ast g_\delta(x) \to f(x)
\]

uniformly as \( \delta \to 0 \).

**Proof.**

\[
f \ast g_\delta(x) = \int \mathbb{R} f(y)g_\delta(x-y) \, dy = \int \mathbb{R} f(x-y)g_\delta(y) \, dy
\]
and

\[
f \ast g_\delta(x) - f(x) = \int \mathbb{R} (f(x-\delta y) - f(x)) g(y) \, dy
\]

\[
\int_{-1}^{1} (f(x-\delta y) - f(x)) g(y) \, dy \to 0
\]
uniformly in \( x \). (Let \( M \) such that \( |g(x)| \leq M \) for all \( x \). Let \( \epsilon > 0 \).

Let \( \delta > 0 \) such that \( |f(x-y) - f(x)| \leq \frac{\epsilon}{2M} \) for all \( |y| \leq \delta \). Then

\[
\int_{-1}^{1} |f(x-\delta y) - f(x)| g(y) |dy| \leq \epsilon \text{ for all } x.
\]

\( \square \)

**4. Partitions of unity**

**Theorem 4.1.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function supported in a finite union of \( k \) balls \( \bigcup_{i=1}^{k} B_{r_i}(x_i) \). Then there exist \( C^1 \) functions \( f_i \) supported in \( B_{r_i}(x_i) \) such that \( f = \sum f_i \).

**Proof.** First we argue that there are \( s_i < r_i \) such that the support of \( f \) is covered by \( \bigcup B_{s_i}(x_i) \). Indeed, the infinite union \( \bigcup_{s_i < r_i} B_{s_i}(x_i) \) cover the compact support of \( f \), so finitely many also do.

Let \( \chi_i \) be \( C^1 \) functions supported in \( B_{r_i}(x_i) \), \( f_i = 1 \) on \( B_{s_i}(x_i) \). (It is easy to construct such functions). Write

\[
0 = f(x)(1 - \chi_1(x))(1 - \chi_2(x)) \cdots (1 - \chi_k(x))
\]
so

\[
f(x) = f(x)\chi_1(x) + f(x)(1 - \chi_1(x))\chi_2(x)
+f(x)(1 - \chi_1(x))(1 - \chi_2(x))\chi_3(x)
+ \cdots f(x)(1 - \chi_1(x))(1 - \chi_2(x)) \cdots \chi_k(x)
\]
The divergence theorem

**Definition 5.1.** $U$ is a $C^1$ open set if there exists a $C^1$ function $k$ with $\nabla k(x) \neq 0$ if $k(x) = 0$ such that $U = \{x \in \mathbb{R}^n | k(x) > 0\}$.

By the implicit function theorem, every point in $\partial U = \{x \in \mathbb{R}^n | k(x) = 0\}$ has a neighborhood $V$ where (after re-labeling coordinates) $U$ agrees with $\{x_n > r(x_1, \cdots, x_{n-1})\}$ for some $C^1$ function $r$. The outward pointing normal is $\tilde{N} = (\nabla r, -1)$, the unit outward pointing normal is $N(x) = \frac{\tilde{N}(x)}{\|\tilde{N}(x)\|}$.

If $f$ is a compactly supported continuous function supported in $V$, then, by definition,

$$\int_{\partial U} f dS = \int_{\mathbb{R}^{n-1}} f(x_1, \cdots, x_{n-1}, r(x_1, \cdots, x_{n-1})) \|\tilde{N}\| dx_1 \cdots x_{n-1} \tag{1}$$

If $f$ is a general compactly supported continuous function, and $U$ is a $C^1$ open, bounded set, then for every point $x$ in the support of $f$ there exists an open ball $B_r(x)$ such that either $B_r(x)$ does not intersect $\partial U$ or else $U$ agrees (after re-labeling coordinates) with $\{x_n > r(x_1, \cdots, x_{n-1})\}$ for some $C^1$ function $r$. Cover the support of $f$ with finitely many such balls, and apply theorem (4.1) to write $f = f_0 + f_1 + \cdots + f_k$ where all $f_i$ are compactly supported continuous functions and the support of $f_0$ does not meet $\partial U$ and $\partial U$ agrees with the graph of a $C^1$ function on the support of $f_1, \cdots, f_k$. Each $\int_{\partial U} f_i dS$ is well defined by (1) and we define

$$\int_{\partial U} f dS = \sum_{i=1}^{k} \int_{\partial U} f_i dS$$

One can show that the definition is independent of the choices made. This also follows from the divergence theorem once we prove it.

**Lemma 5.2.** Let $U = \{x_n > r(x_1, \cdots, x_{n-1})\}$ and $f$ a compactly supported $C^1$ function. Let $H : \mathbb{R} \to \mathbb{R}$ be the Heaviside function ($H(x) = 0 \text{ if } x \leq 0, H(x) = 1 \text{ in } x > 0$) and let $h : \mathbb{R} \to \mathbb{R}$ be a $C^1$ smoothed out Heaviside function ($h(x) = 0 \text{ if } x \leq 0, h(x) = 1 \text{ in } x > 1$, and $C^1$ in-between). Then

$$\int_U f(x) dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}^{n}} h\left(\frac{x_n - h(x_1, \cdots, x_{n-1})}{\epsilon}\right) f(x) dx$$
Proof.
\[
\int_U f(x)\,dx = \int_{\mathbb{R}^{n-1}} \left( \int_{r(x_1,\ldots,x_{n-1})}^{\infty} f(x)\,dx_n \right) dx_1,\ldots, dx_{n-1}
\]
\[= \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} H(x_n - h(x_1,\ldots,x_{n-1}) f(x)\,dx_n \right) dx_1,\ldots, dx_{n-1} \]
Now we notice that
\[
\int h\left(\frac{x_n - r(x_1,\ldots,x_{n-1})}{\epsilon}\right) f(x)\,dx_n
\]
\[
\rightarrow \int H(x_n - r(x_1,\ldots,x_{n-1}) f(x)\,dx_n
\]
uniformly in \(x_1,\ldots,x_{n-1}\). This uses the fact that \(f\) is bounded. Now we can conclude that
\[
\int_{\mathbb{R}^{n-1}} \left( \int_{\epsilon \mathbb{R}} H\left(\frac{x_n - r(x_1,\ldots,x_{n-1})}{\epsilon}\right) f(x)\,dx_n \right) dx_1,\ldots, dx_{n-1}
\]
\[
\rightarrow \int_{\mathbb{R}^{n-1}} \left( \int H(x_n - r(x_1,\ldots,x_{n-1}) f(x)\,dx_n \right) dx_1,\ldots, dx_{n-1} \]
\]
\[
\square
\]

\textbf{Theorem 5.3.} (The divergence theorem.) Let \(U\) be a bounded open set in \(\mathbb{R}^n\) with \(C^1\) boundary.
Let \(V_1,\ldots, V_n\) be \(C^1\) functions supported in a neighborhood of \(\bar{U}\). Then
\[
\int_U \left( \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n} \right) \,dx = \int_{\partial U} \langle V, N \rangle \,dS
\]
where \(N\) is the outward pointing unit normal to \(\partial U\).

Proof. We will prove the equivalent form
\[
\int_U \nabla f \,dx = \int_{\partial U} f N \,dS
\]
for each of the compactly supported \(C^1\) functions \(f_i\) of Theorem (4.1).
There is nothing to prove for \(f_0\). Let \(f\) be one of \(f_1,\ldots,f_k\), and write \(\partial U\) as the graph of \(r\) on the support of \(f\). We have
\[
\int_U \nabla f \,dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \frac{h(x_n - r(x_1,\ldots,x_{n-1})}{\epsilon} \nabla f(x) \,dx
\]
\[
= - \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \nabla \left( \frac{h(x_n - r(x_1,\ldots,x_{n-1})}{\epsilon} \right) f(x) \,dx
\]
\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^{n-1}} \left( \frac{1}{\epsilon} \frac{h(x_n - r(x_1,\ldots,x_{n-1})}{\epsilon} f(x)\,dx_n \right) dx_1,\ldots, dx_{n-1} := I_\epsilon
\]
The final remark is that by Theorem (3.3) (and its proof)
\[
\int \frac{1}{\epsilon} h'(\frac{x_n - r(x_1, \cdots x_{n-1})}{\epsilon}) f(x) \tilde{N}(x) dx_n \to \left(f \tilde{N}\right)(x_1, \cdots, x_{n-1}, r(x_1, \cdots x_{n-1}))
\]
uniformly in \(x_1, \cdots x_{n-1}\) and thus
\[
I_\epsilon \to \int_{\mathbb{R}^{n-1}} \left(f \tilde{N}\right)(x_1, \cdots, x_{n-1}, r(x_1, \cdots x_{n-1})) dx_1 \cdots dx_{n-1}
\]
\[
= \int_{\partial U} f N dS
\]