TOPICS IN ANALYSIS, READING COURSE

The following are suggested topics. I hope that the participants will do all the lecturing.

- Littlewood-Paley decomposition, the Stein-Tomas restriction theorem, Strichartz inequalities, bilinear estimates for the wave equation, following Sergiu Klainerman's note
https://web.math.princeton.edu/ seri/homepage/courses/Analysis2011.pdf
- $X^{s,b}$ spaces, pseudoconformal identity, Morawetz and interaction Morawetz estimates for the Schrödinger equation following chapters 2, 3 from Tao's book "Nonlinear dispersive equations, local and global analysis". See also
Viriel, Morawetz, and interaction Morawetz inequalities under "short stories" towards the end of the web page
http://www.math.ucla.edu/ tao/preprints/pde.html
- Refined spaces and harmonic analysis estimates adapted to Wave Maps, following Tataru's paper On global existence and scattering for the wave maps equation,
http://math.berkeley.edu/ tataru/papers/wm2b.pdf
- The Schrödinger representation of the Heisenberg group, the metaplectic representation of the symplectic group following Folland's book "Harmonic analysis in phase space". Aplications to the lens transform for the Schrödinger equation. The infinite dimensional analogues of these, and application to Bose-Einstein condensation.

Topics from Tao's book

- Blow-up for focusing NLS based on the pseudoconformal transformation (formula (3.15)) and viriel identity (formula (3.72)).
- Local existence for energy-subcritical NLS in 3 dimensions (Proposition 3.19), and global existence in the defocusing case.
- Scattering for NLS (Proposition 3.28)
- The I method (Proposition 3.28)

Topics from Folland's book, centered around the operator $\frac{1}{i}\frac{\partial}{\partial t} - \Delta + |x|^2$.

- Chapter 1 The Heisenberg group and its Lie algebra (formulas (1.15), (1.20); Automorphisms, particularly symplectic ones; The Schrödinger representation (sometimes called Stone-von Neumann) (1.25), and the Stone-von Neumann teorem (1.50); The Bargmann transform and Fock space (section 6), also (1.79), (1.80), and the conjugation formulas (1.73)-(1.79). At this point we can see that the second order Hermite operator $= \sum X_j^2 + D_j^2$ is (essentially) the first order operator $\sum z_j \frac{\partial}{\partial z_j}$ on the Fock side, so we can solve the Schrödinger equation for the Hermite operator there and invert back to $L^2(\mathbb{R}^n)$.
- Chapter 4 The metaplectic representation. The abstract definition (section 2). The infinitesimal representation (section 3), formula (4.44). Solving the Hermite-Schrödinger equation: Corollary (4.55), and also the section **Some applications** of section 4

1. Optional exercises

1) Recall that if $\hat{f} \in L^2(\mathbb{R})$ and $F(t,x) = \int e^{ix\xi} e^{it|\xi|^2} \hat{f}(\xi) d\xi$, then F is a solution of the homogeneous Schrödinger equation, and the space-time Fourier transform $\tilde{F}(\tau,\xi) = c\delta(\tau + |\xi|^2)\hat{f}(\xi)$. The restriction theorem and the Strichartz estimate state

$$\|F\|_{L^6(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R})} = c\|\hat{f}\|_{L^2(\mathbb{R})}$$

A version of the restriction conjecture (which is a theorem only in two dimensions) states that if $\hat{f} \in L^4$ supported in $(-1,1)$, then for any $p > 4$ there exists C_p such that

$$\|F\|^4_{L^p(\mathbb{R}^2)} \leq C_p \|\hat{f}\|^4_{L^4}$$

One usually denotes $p = 4+$.

Try to prove this.

HINT:

$$\|F\|^4_{L^{4+}(\mathbb{R}^2)} = \| \int e^{ix(\xi+\eta)} e^{it(|\xi|^2+|\eta|^2} \hat{f}(\xi)\hat{f}(\eta) d\xi d\eta\|^2_{L^{2+}(\mathbb{R}^2)}$$

Change variables $u = \xi + \eta, v = |\xi|^2 + |\eta|^2$ (this is a 2-1 change of variables), $dudv = 2|\xi - \eta| d\xi d\eta$, so the above equals

$$\| \int e^{ixu} e^{itv} \frac{\hat{f}(\xi)\hat{f}(\eta)}{|\xi - \eta|} dudv\|^2_{L^{2+}(\mathbb{R}^2)}$$

Now apply Hausdorff-Young (which is almost Plancherel in this case), and then undo the change of varialles.

2) Let $\hat{f} \in L^2(\mathbb{R})$, supported in $(-2,-1)$ and define $F(t,x) = \int e^{ix\xi} e^{it|\xi|^2} \hat{f}(\xi) d\xi$. Similarly, let $\hat{g} \in L^2(\mathbb{R})$, supported in $(1,2)$ and define $G(t,x) = \int e^{ix\xi} e^{it|\xi|^2} \hat{g}(\xi) d\xi$.

The restriction theorem and the Strichartz estimate state

$$\|FG\|_{L^3(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}$$

But in this case one can do much better:

$$\|FG\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2} \|g\|_{L^2}$$

Try to prove this using the same strategy as in the previous problem.