Math 130 – Spring 2015 – Final Exam – Solutions

- NO CALCULATORS OR ELECTRONIC DEVICES ALLOWED.
- Use a separate answer sheet for each question.
- Give your pledge on page 1 only, covering the whole test.
- Draw a box around a final answer to a problem.

1.
(a) You are given the following table of values:

x	1	2	3	4
$f(x)$	2	4	1	3
$f'(x)$	-6	-7	-8	-9
$g(x)$	2	3	4	1
$g'(x)$	2/7	3/7	4/7	5/7

Let $h(x) = g(f(x))$. What is the linear approximation to $h(3.02)$?
(This is also called the differential or tangent line approximation.)
Solution. $h'(3) = (g'(f(3))f'(3) = g'(1)f'(3) = (2/7)(-8) = -16/7$. The linear approximation to $h(3.02)$ is then $h(3) + h'(3)(.02) = 2 + (-16/7)(.02)$

(b) Suppose for a certain kind of rectangular frame, the cost of the frame is \$10 per foot for the vertical sides and \$20 per foot for the horizontal sides. What is the largest area the frame could enclose at a cost of \$160 ?
Solution.
Let x be the length of a horizontal size and y the length of a vertical side, in feet. The cost C in dollars is then $C = 20(2x) + 10(2y)$. The area A is $A = xy$. Then

$$\begin{aligned} 160 &= 40x + 20y \\ 20y &= 160 - 40x \\ y &= 8 - 2x\ . \end{aligned}$$

So,

$$\begin{aligned} A = xy = x(8 - 2x) &= 8x - 2x^2 \\ A'(x) &= 8 - 4x\ . \end{aligned}$$

Then $A'(x) = 0$ at $x = 2$. A is maximum at $x = 2$ because $A'(x) > 0$ for $x < 2$ and $A/(x) < 0$ for $x > 2$. When $x = 2$, we have $y = 8 - 2x = 4$. So, the maximum area xy is 8 square feet.

2.

(a) Compute the following limits. Possible correct answers are a number, ∞, $-\infty$ or DNE ("does not exist").

$$(i)\ \lim_{x\to\infty} \frac{x^2\sin(x)}{e^x} = \mathbf{0} \qquad (ii)\ \lim_{x\to 2^+} \frac{x^2-2x}{x^2-4} = \frac{\mathbf{1}}{\mathbf{2}} \qquad (iii)\ \lim_{x\to\infty} \frac{x\ln(x)}{x^2+1} = \mathbf{0}$$

(b) Find the equation of the tangent line to the curve $y = (\sin x)/x$ at the point where $x = \pi$.

Solution.

By the quotient rule,

$$y' = \frac{x\cos(x) - \sin(x)(1)}{x^2}$$

so at $x = \pi$ we have $y' = -1/\pi$. At the point $(\pi.0)$ on the graph, one equation for the tangent line is

$$y = (-1/\pi)(x - \pi)\ .$$

3. Evaluate the following definite integrals.

$$
\begin{aligned}
(a)\ \int_1^2 \frac{x^2+\sqrt{x}}{x}\,dx &= \int_1^2 \frac{x^2}{x} + \frac{\sqrt{x}}{x}\,dx \\
&= \int_1^2 x + x^{-1/2}\,dx \\
&= \left[\frac{x^2}{2} + 2\sqrt{x}\right]_{x=1}^2 \\
&= (2+2\sqrt{2}) - (\frac{1}{2}+2) \\
&= 2\sqrt{2} - \frac{1}{2}\ .
\end{aligned}
$$

(b) We use the substitution $u(x) = u = -1/x$. Then $du/dx = 1/x^2$, and

$$
\begin{aligned}
\int_1^4 \frac{e^{-1/x}}{x^2}\,dx &= \int_{u(1)}^{u(4)} e^u\,du = \int_{-1}^{-1/4} e^u\,du \\
&= \Big[e^u\Big]_{u=-1}^{-1/4} = e^{-1} - e^{-1/4}
\end{aligned}
$$

(c) We use the substitution $u(x) = u = 5-\sqrt{2}\sin x$. Then $du/dx = -\sqrt{2}\cos x$ and $(-1/\sqrt{2})du = \cos x\,dx$. So,

$$
\begin{aligned}
\int_{x=0}^{\pi/4} \frac{\cos x}{8-\sqrt{2}\sin x}\,dx &= \int_{u=u(0)}^{u(\pi/4)} \frac{1}{u}\,(-1/\sqrt{2})\,du \\
&= (-1/\sqrt{2})\int_{u=8}^{4} \frac{1}{u}\,du \\
&= (-1/\sqrt{2})\Big[\ln(u)\Big]_{u=8}^{4} \\
&= (-1/\sqrt{2})(\ln(4)-\ln(8)) \\
&= \frac{\ln(8)-\ln(4)}{\sqrt{2}} = \frac{\ln(8/4)}{\sqrt{2}} \\
&= \frac{\ln(2)}{\sqrt{2}}
\end{aligned}
$$

4.

(a) Suppose the rate of infection of a certain disease (in units of people per month) is modeled over a period of six months by the function $f(t) = 400(6t - t^2)$, where t is the time (in months) after the disease breaks out. In this model,

(i) What is the total number of people who are infected with the disease over the first six months after it breaks out?

Solution.

$$\int_{t=0}^{6} 400(6t - t^2)\,dt = 400\Big[3t^2 - (1/3)t^3\Big]_{t=0}^{6} = 400\big(3(6)^2 - (1/3)6^3\big)$$
$$= 400(3(36) - (6/3)(36)) = 400(36) = 14,400\ .$$

The answer: 14,400 people.

(ii) At what time t is the rate of infection greatest?

Solution.

The function $f(t)$ is maximum at $t = 3$ months.

(b) A certain number Q is approximated by sums of the form

$$\sum_{i=1}^{n} \sqrt{5 + i(3/n)}\,(3/n)$$

and these sums converge to Q as $n \to \infty$. Write a definite integral which is equal to Q. Do not compute the integral.

Solution.

There is more than one possible solution. Two natural correct solutions are

$$\int_{x=0}^{3} \sqrt{5 + x}\,dx \quad \text{and} \quad \int_{x=5}^{8} \sqrt{x}\,dx\ .$$

For the first solution, we interpret the sum to be the right endpoint Riemann sum $\sum_{i=1}^{n} f(x_i)\Delta x$ for $f(x) = \sqrt{5 + x}$ on $[0, 3]$ with $\Delta x = (3 - 0)/n$. For the second, we use instead a function $f(x) = \sqrt{x}$ on $[5, 8]$, with $\Delta x = (8 - 5)/n$.

5.

(a) A spherical shaped balloon is inflating at a rate of 2 cubic inches per second. How fast is the radius of the balloon increasing when the radius is 3 inches?

Solution.

The balloon volume at radius R is $V = (4/3)\pi R^3$. So,

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dR}\frac{dR}{dt} \\ 2 &= (4\pi R^2)\frac{dR}{dt} \end{aligned}$$

When $R = 2$,

$$2 = (4\pi 3^2)\frac{dR}{dt} = (36\pi)\frac{dR}{dt}$$

and $dR/dt = 1/(18\pi)$ inches per second .

(b) (8 pts) A radioactive substance is decaying exponentially. At noon there is 90 grams. One hour later, there is 30 grams.

At what time t hours after noon will there be 5 grams of the substance?

Solution.

Let $y(t)$ be the amount in grams at time t hours after noon. For some number $k > 0$, $y(t) = y_0 e^{-kt} = 90e^{-kt}$. Substituting at $t = 1$, we get $30 = 90e^{-kt}$; solving, $y = 90e^{-(\ln 3)t}$. Now solve the following equation for t:

$$\begin{aligned} 5 &= 90e^{-(\ln 3)t} \\ (1/18) &= e^{-(\ln 3)t} \\ \ln(1/18) &= -(\ln 3)t \\ -\ln(18) &= -(\ln 3)t \\ t &= (\ln 18)/(\ln 3) \ . \end{aligned}$$

The amount is 5 grams at $t = (\ln 18)/(\ln 3)$ hours after noon.

6. The function $f(x) = (\ln(x))/x$ satisfies

$$f'(x) = \frac{1 - \ln(x)}{x^2} \quad \text{and} \quad f''(x) = \frac{-3 + 2\ln(x)}{x^3} \ .$$

(i) What is the domain of f?
$0 < x < \infty$.

(ii) Find all asymptotes of f.
There is a vertical asymptote $x = 0$ and a horizontal asymptote $y = 0$.

(iii) On which intervals is f increasing, and on which intervals is f decreasing?
f is increasing on $(0, e)$ and decreasing on (e, ∞), because
f' is positive on $(0, e)$ and negative on (e, ∞).

(iv) Determine the intervals on which the graph of f is concave up or down.
The graph is concave down on $(0, e^{3/2})$ and concave up on $(e^{3/2}, \infty)$, because
f'' is negative on $(0, e^{3/2})$ and positive on $(e^{3/2}, \infty)$.

(v) Graph f.
The graph is not included for technical reasons.