STUDY GUIDE, EXAM 2, MATH 463

The exam covers Chapters 4,5,6 of Brown and Churchill (Vol.9).
Some exam problems are inspired by a review of the assigned homework.

When I say f is analytic, I mean there is a domain D in $\mathbb{C}$ such that $f\colon D \to \mathbb{C}$ and $f'(z)$ exists everywhere. (It follows as described in Chapter 4 that $f(z)$ then has derivatives of all orders.) Below, D denotes a domain (connected open subset of $\mathbb{C}$).

Chapter 4.
There are a slew of definitions, facts and theorems you should know:

- Definitions of contour integral, domain, simply connected domain.
- If f is continuious on a contour C of length L, and M is a positive number such that $|f(z)| \leq M$ for all z on the contour, then $|\int_C f(z)\,dz| \leq ML$.
- For the contour $z(t) = Re^{it}$, $0 \leq t \leq 2\pi$, if n is an integer not equal to -1, then $\int_C z^n\,dz = 0$; and, $\int_C 1/z\,dz = 2\pi i$.
- For a continuous function $f : D \to \mathbb{C}$, TFAE.
  (1) Path independence for all contour integrals $\int_C f(z)\,dz$.
  (2) Integrates to zero on all closed contours.
  (3) f has an antiderivatve on D.
- Cauchy-Goursat Theorem.
- If f is analytic on a simply connected D, then for every closed contour C, $\int_C f(z)\,dz = 0$.
- Cauchy integral formula (for $f(z)$)
- Generalized Cauchy integral formula (for $f^{(n)}(z)$, the nth derivative)
- Principle of Deformation of Paths
- If $f\colon D \to \mathbb{C}$ and $f'(z)$ exists for all z in D, then $f^{(n)}$ exists for all $n > 0$, for all z in D.
- Morera's Theorem
- Cauchy's inequality ($|f^{(n)}| \leq (n!M_R)/R^n)$)
- Liouville's Theorem
- Fundamental Theorem of Algebra (proved using Liouville's Theorem)
- Maximum Modulus Principle
- Gauss Mean Value Theorem

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Chapter 5

For a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ with complex coefficients, there is a ρ in $[0, +\infty]$ (called the radius of convergence of the series) such that

- $|z - z_0| < \rho \implies$ the series converges ,
- $|z - z_0| > \rho \implies$ the series diverges .

Now suppose $\rho > 0$ and f is defined on the open disc $B_\rho(z_0)$ by setting $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Then

- f is analytic on $B_\rho(z_0)$.

- For every z in $B_\rho(z_0)$, the series converges absolutely.
- If $0 < s < \rho$, then the series converges uniformly on $\{z\colon |z - z_0| \leq s$.
- On $B_\rho(z_0)$, f has has derivatives of all orders, and these are given by series which can be computed by differentiating or antidifferentiating the series termwise.
- Likewise a series for an antiderivative of f on $B_\rho(z_0)$ can be computed by termwise antiderivation.
- The series can only be the Taylor series for $f(z)$ at z_0: for each n, $a_n = (1/n!)f^{(n)}(z_0)$.

If f is analytic on a domain D and g is analytic on a domain E containing D and $f = g$ on D, then g is called an *analytic continuation* of f. For example, define $f(z) = \sum_{n=0}^{\infty} z^n$ on $D = B_1(0)$ and define $g(z) = 1/(1-z)$ on $E = \{z \in \mathbb{C} : z \neq 1\}$.

The radius of convergence ρ is the smallest R such that there is a point w such that $|w - z_0| = R$ and there is no analytic continuation of f to a domain containing w. This can be used to say for example what the radius of convergence of a Taylor series of $f(z) = e^z/(z(z-3i))$ is at, say, the point $2+5i$ is – it is simply the distance to the closer of the numbers 0 and $3i$.

The radius of convergence ρ can also be described in terms of the coefficients a_n as follows. Let A be the largest number in $[0, +\infty]$ which is the limit of some subsequence of the sequence of numbers $|a_n|^{1/n}$. Then $\rho = 1/A$. (Here, $1/0$ means ∞ and $1/\infty$ means 0.) (For an example, consider $\sum_{n=0}^{\infty} 2^n z^n$.) (This characterization won't be used on Exam 2.)

Chapter 6

- Laurent series $f(z) = \sum_{n=-\infty}^{+\infty} c_n(z - z_0)^n$, valid for z in $D = \{z : r < |z - z_0| < R\}$ if f is analytic on D. Then

$$c_n = \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

  if (for example) C is the contour $z(t) = se^{it}$, $0 \leq t \leq 2\pi$ and $r < s < R$.
- The three types of isolated singularities. Definitions via Laurent series.
- Behavior of the three types in a small deleted neighborhood of the singularity.
- Casorati-Weierstrass Theorem, proved with Riemann's Theorem.
- Picard's Theorem
- Computing integrals with residues. With the residue at infinity.
- Computing the residue at a pole.
- Order of a zero. Order of a pole.
- If $f(z) = 1/g(z)$,then a zero of order m for g is a pole of order m for f.
- For f analytic on D, if $f(z_0) = 0$ and f is not identically zero on a neighbrhood of z_0, then f is nonzero on some deleted neighborhood of z_0.
- Coincidence Principle.
- Suppose f, g are analytic at z_0, and are not identically zero. (Suppose $z_0 = 0$ for convenience.) Then there are nonnegative integers j, k and analytic functions F, G not zero at 0 such that $f(z) = z^j F(z)$ and $g(z) = z^k G(z)$. Then on a deleted neighborhood of 0, $f(z)/g(z) = z^{j-k}F(z)/G(z)$ and

$F(z)/G(z)$ is analytic with some Taylor series $\sum_{n=0}^{\infty} a_n z^n$. Suppose you know the coefficients for the Taylor series of F and G. Then you can solve for the a_n:

$$F(z) = (a_0 + a_1 z + a_2 z^2 + \cdots)G(z)$$

$$(b_0 + b_1 z + b_2 z^2 + \cdots) = (a_0 + a_1 z + a_2 z^2 + \cdots)(d_0 + d_1 z + d_2 z^2 + \cdots)\ .$$

To solve. Note $b_0 = F(0) \neq 0 \neq G(0) = d_0$.
First $b_0 = a_0 d_0$ determines a_0.
Then $b_1 = a_0 d_1 + a_1 d_0$ gives you a_1. Then $b_2 = a_0 d_2 + a_1 d_1 + a_2 d_0$ gives you a_2.
Etc.
If say $j - k = -4$, then the Laurent series for f/g is $(1/z^4)(a_0 + a_1 z + a_2 z^2 + \cdots$, and the residue for f/g is a_3.

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Often the easiest way to find coefficients for a Taylor or Laurent series is through known series. Another useful trick is to use

$$\frac{1}{1 - f(z)} = 1 + f(z) + [f(z)]^2 + \ldots$$

when $|f(z)| < 1$.