II. Automorphisms of the Shift

\[ \text{Aut}(S_A) = \text{group of homeomorphisms } U \text{ such that } U S_A = S_A U \]

Investigate \( S_A \) via \( \text{Aut}(S_A) \). Suppose \( A \) is irreducible, \( \lambda > 1 \).

As an abstract group:

1. \( \text{Aut}(S_A) \) is countably infinite

2. " " residually finite

( homomorphisms into finite groups separate points )

3. \( \text{Center of Aut}(S_A) = \{ (S_A)^n : n \in \mathbb{Z} \} \) [Ryan]

4. \( \text{Aut}(S_A) \) contains
   - many complicated subgroups
   - no fin gen. subgroup with unsolvable word problem

Ryan's Theorem (3) is our only (!) tool for distinguishing different \( \text{Aut}(S_A) \).

**Ex.** The groups \( \text{Aut}(S_A) \) are pairwise not isomorphic for \( A = [2], [4], [8] \).

**Open.** \( \text{Aut}(S_{[2]}) \noneq \text{Aut}(S_{[3]}) \)?

\[ \text{Aut}(S_{[2]}) \noneq \text{Aut}(S_{[20]}) ? \]

We know only two ways to construct useful representations of \( \text{Aut} S_A \):

- via dimension module
- periodic points
Suppose $A = UV$, $B = VU$ over $\mathbb{Z}_+$. 

\[
\begin{array}{ccc}
Q^m & \xrightarrow{A} & Q^m \\
\downarrow u & & \downarrow v \\
Q^n & \xrightarrow{B} & Q^n
\end{array}
\]

$Au = uVu = UB$. 

Can check: 

\[ G_A \xrightarrow{\pi} G_B \]

Define $\hat{u} = u|_{G_A}$ 

\[ \hat{u} : (G_A, \hat{A}) \xrightarrow{\pi} (G_B, \hat{B}) \]

Given conjugation $\varphi : S_A \to S_B$, $\varphi$ is a composition $c(R_k, S_e) \cdots c(R_k, S_e) \cdot (\cdot)$. 

Define $\hat{\varphi} : (G_A, \hat{A}) \to (G_B, \hat{B})$ as 

\[ \hat{\varphi} = \hat{R}_k \cdots \hat{R}_k \]

Nontrivial: indeed $\hat{\varphi}$ is independent of choice of SSE for decomposition. ($\star$)

One proof: Kac's.

Define the dimension representation of $\text{Aut}(S_n)$ as the map 

\[ u_n : \text{Aut}(S_n) \to \text{Aut}(\hat{A}) \]

$q \mapsto \hat{q}$
\[ \text{Aut}_+ (\hat{A}) = \text{the automorphisms of } (G_A, G_+^\times) \text{ which commute with } \hat{A}. \]

\[ \text{Aut} (\hat{A}) \cong \text{Aut}_+ (\hat{A}) \oplus \mathbb{Z}/2 \]

Ex. \( A = [27] \)

\[ u \in \text{Aut} (\hat{A}) \text{ is restriction of a vector space map } \mathbb{Q} \rightarrow \mathbb{Q} \]

\[ a \rightarrow m x \]

\[ G_A = \text{dyadic rationals} \]

\[ m = \pm 2^{-k}, \ k \in \mathbb{Z}. \]

\[ \text{Aut}(\hat{A}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \]

\[ \text{Aut}_+ (\hat{A}) \cong \mathbb{Z} \]

Ex. \( A = [\boxed{24}] \)

\[ \text{Aut}_+ (\hat{A}) \cong \mathbb{Z} \]

Ex. \( A = [10] \)

\[ \text{Aut}_+ (\hat{A}) \cong \mathbb{Z} \]

Always \( \text{Aut}_+ (\hat{A}) = \text{Aut}_+ (\hat{A}), \ k \in \mathbb{Z}. \)
Periodic Points

\[ S = S_A \]
\[ P_n^o(S_A) = \text{points of least } S_A \text{-period } n \]
\[ |P_n^o(S_A)| < \infty \]

Define \( U_n = U \mid P_n^o(S_A) \)

\[ \text{Aut}(S) \rightarrow \text{Aut}(S_n) \]
\[ U \mapsto U_n \]

is a group homomorphism.

Given \( n \) and \( U \):

Let \( \Theta_1, \ldots, \Theta_k \) be the \( S \)-orbits of size \( n \).
Pick \( x_i \in \Theta_i, \quad 1 \leq i \leq k \)

\[ U_n : x_i \mapsto S^{m_i} x_i, \quad (m_i \text{ a permutation on } \{1, \ldots, kS\}) \]

Define

\[ \text{Sign}_n(U) = \begin{cases} 0 & \text{if } T \text{ is even} \\ 1 \exp \frac{i\pi}{2} & \text{if } T \text{ is odd} \end{cases} \]

\[ g_y_n(U) = \sum_{i=1}^{k} m_i x_i \in \mathbb{Z}/n \]

("\( g_y \) for "gyration").

Then \( \text{Sign}_n \) and \( g_y_n \) are

\[ \text{Sign}_n : \text{Aut}(S) \rightarrow \mathbb{Z}/2 \]
\[ g_y_n : \text{Aut}(S) \rightarrow \mathbb{Z}/n \]

are homomorphisms (easy to check).
Remake: a longer exercise:

if \( n \cdot \prod_{i=1}^{n} \text{Aut}(S^n) \to A \) is a homomorphism, then \( h \) factors through \( \prod_{i=1}^{n} (g_{i,n} \times \text{sign}_n) \).

Define \( g_{i,n} = \prod_{k=1}^{n} \text{sign}_k \).

Definition (sign-rotation-compatibility condition homomorphism)

\[ \text{SGCC}_n : \text{Aut}(S) \to \mathbb{Z}/n \]

\[ : U \mapsto g_{36}(U) + \sum_{1 \leq k < n, \frac{n}{k} \in \mathbb{Z}^3} \text{sign}_k(U) \]

Example: \( \text{SGCC}_{36} = g_{36} + 18(\text{sign}_{18} + \text{sign}_9) \in \mathbb{Z}/36 \)

Example: \( S = S_{527} \) the shift on \( 30,15^2 \)

\( S \)-orbits

\begin{align*}
\text{size } 1 & : 0^\infty 1^\infty \\
\text{size } 2 & : (01)^\infty \\
\text{size } 3 & : (001)^\infty (011)^\infty \\
\text{size } 4 & : (011)^\infty (0001)^\infty (0011)^\infty \\
\end{align*}

\[ \text{SGCC}_2(S) = 1 \in \mathbb{Z}/2 \]

For flip \( U \) \((Ux)_i = x_{i+1}\):

\[ \text{SGCC}_n(U) = 0 \quad \text{for } n = 1,2,3,4,\ldots \]
Now a miraculous connection of periodic point and dimension representations:

Fact: For $A$ irreducible over $\mathbb{Z}^+$ and $U \in \text{Aut}(S_A)$:

- $U^p(U) = 0 \Rightarrow SGCC_n(U) = 0 \quad \forall n$

Moreover there is an explicit (nasty) formula for $sgc_n$ (more later).

I. For example: if $U \in \text{Aut}(S_A)$ and $Ux = x$ for $x \in A$.

II. Where did SGCC come from?

II. Of what use is this?

I. What is action of $\text{Aut}(S_A)$ on finite subsystems $F(S_A) = \bigcup_{N \in N_S} P^0_m(S_A)$?

(Can SE $\mathbb{Z}^+$ mixing $S_A$ have nonisomorphic actions?) Essentially Williams 80
I

[B-Krieger 85]: A collection $I$ of involutions of full shifts $S$ such that $VN$, for $V \in \text{Aut}(S \, | \, F_N)$ TFAE.

1. $V = U |_{F_N}$ for some $U \in \langle J \rangle$

2. $SGCC_n(V) = 0$, $n \leq N$.

For an involution $J$ of a full shift $S$:
- $U \circ J$ is top. conjugate to an SFT
- $|P^n_0(U \circ J)| = |P^n_0(J)| \quad \forall n \quad (*)$

Exploring the limits of action of $\langle J \rangle$ and the constraint $(*)$ leads to $SGCC$.

II

THM (Kim-Raush-Wagoner, following Bk, BLB, Biebling, B-Friedgut): For irreducible SET $SA$ and $V \in \text{Aut}(S \, | \, F_N \, | \, S_A)$ ($N$ sufficiently large), TFAE.

1. $SGCC_n(V) = 0$, $n \leq N$

2. $V = U |_{F_N}$ for some $U \in \text{Ker}(\rho)$.

THM (B-Krieger): Likewise $SGCC = 0$ is the only constraint on extending automorphisms of subshifts of $S_A$ to automorphisms of $S_A$. 
To complete our understanding of the action of $\text{Aut}(S_A)$ on subsystems, we need to solve (and for other reasons)

Open Problem (RD) Given $A$ a primitive over $\mathbb{Z}_+$, what is the image of the dimension representation of $S_A$ in $\text{Aut}_+(\hat{A})$?

Some facts:

(1) The classification of SFT's up to topological conjugacy reduces to the solution of (RD) and the solution of the conjugacy problem for mixing SFT's. (Kim - Rash)

(2) $\hat{\phi}_A : \text{Aut}(S_A) \to \text{Aut}_+(\hat{A})$ can be non-surjective. (Later.)

(3) Modulo multiplications by $\hat{A}$

(a) Given $A$ primitive over $\mathbb{Z}_+$ and $\phi \in \text{Aut}_+(\hat{A})$

(b) Let $R$ be a rank $r$ matrix over $\mathbb{Q}$ such that $G_A^m \to G_A$ defines $\hat{R} \in \text{Aut}(\hat{A}^*)$. Likewise, let $G_A \to G_A$ have $\hat{S} = \hat{R}^{-1}$.

(c) Let $\text{rank } S = \text{rank } R = \dim V_A$.

Then for all large $n$, $A^{m+n} = (RA^m)(SA^n)$

$= (SA^n)(RA^m)$
with $RA^m$, $SA^n$ over $\mathbb{Z}^+$. So,

$\Phi \in \text{Aut}(A)$

$\Phi \in \text{image}(\hat{P}_A)$ for all large $l$.

(3) If char. poly. $(A)$ has no repeated nonzero root, then $\text{Aut}(A)$ is finitely generated (and abelian), so by (2), $\hat{P}_A$ is surjective for all large $l$.

(4) (N,Long) Problem (RD) is "not elementary".

- if $A$ is primitive with $A = UV = VU$ over $\mathbb{Z}^+$ and $\lambda_A = \text{prime } p \in \mathbb{N}$ and $A$ has no repeated nonzero root then $\exists k \in \mathbb{N}$

$\hat{U}^k = \text{id}$ or $\hat{A}^{-1} \hat{U}^k = \text{id}$. (*)

- Examples with $\hat{P}_A$ surjective, but not using only such (*).
For a primitive and $\lambda_A > 1$, understand $\text{Aut}(S_A)$ via:

- **Kernel ($p_A$):**
  - big, complicated, combinatorial
  - known actions are rather homogeneous, modulo low-order periodic points

- **Image ($\pi_A$):**
  - typically finitely generated abelian
  - fine structure, algebraic

But: $\text{Aut}(S_A) \rightarrow \text{Aut}(S_{\text{on}})$ if $n \geq 1$

Open Problem

For a primitive and $\lambda_B > 1$ and every $A$, does $\text{Aut}(S_A) \rightarrow \text{Aut}(S_B)$?