Symbolic extensions of intermediate smoothness

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This talk primarily reports


and also refers to the following


I. Background: symbolic extensions and entropy.

- All spaces are compact metrizable.

- \((X, T)\) denotes a homeomorphism, 
  \(T : X \rightarrow X\), with \(h_{\text{top}}(T) < \infty\).

- \(\mathcal{M}_T\) is the space of \(T\)-invariant Borel probabilities.

- A subshift \((Y, S)\) is the restriction of the full shift on a finite alphabet to a closed invariant subsystem.

- A \textit{symbolic extension} of \((X, T)\) is a subshift \((Y, S)\) with a continuous surjection \(\varphi : Y \rightarrow X\) such that \(T\varphi = \varphi S\).
Definition.
The (topological) residual entropy of $T$ is

$$h_{\text{res}}(T) = \inf \{ h_{\text{top}}(S) \} - h_{\text{top}}(T)$$

where the inf is over the symbolic extensions of $T$.

Theorem. [BFF, D1]
Given $0 < \alpha < \infty$ and $0 \leq \beta \leq \infty$, there exists $T$ with $h_{\text{top}}(T) = \alpha$, $h_{\text{res}}(T) = \beta$.

The intuition: $h_{\text{res}}(T) > 0$ reflects nonuniform emergence of entropy on refining scales.

To understand this it is essential to consider symbolic extensions in terms of invariant measures.
**Extension entropy.** Consider a homeomorphism $T$ of a compact metric space $X$. Given a symbolic extension $\varphi : (Y, S) \to (X, T)$ define its extension entropy function

$$h_{\text{ext}}^{\varphi} : \mathcal{M}_T \to [0, \infty)$$

$$\mu \mapsto \max\{h(S, \nu) : \varphi \nu = \mu\} .$$

**Symbolic extension entropy.** Given $(X, T)$, we define its symbolic extension entropy function to be the function $h_{\text{sex}}^T : \mathcal{M}_T \to [0, \infty]$ which is the infimum of all $h_{\text{ext}}^{\varphi}$ arising from symbolic extensions $\varphi$ of $(X, T)$.

($h_{\text{sex}}^T \equiv \infty$ if no symbolic extension exists.)

Abbreviate:

symbolic extension entropy $= \text{sex entropy}.$

When some symbolic extension exists, $h_{\text{sex}}^T$ is a bounded function, and $h_{\text{sex}}^T(\mu)$ gives a quantitative measure of the emergence of complexity on finer scales “near” the support of $\mu.$
**Entropy structure.** An entropy structure for \((X, T)\) is an allowed nondecreasing sequence of nonnegative functions \(h_n\) on \(\mathcal{M}_T\), converging to the entropy function \(h\).

**Example of an entropy structure.**
Suppose the system \((X, T)\) admits a refining sequence of partitions \(P_n\) with small boundaries (the boundary of the closure of each partition element has \(\mu\)-measure zero for every \(\mu\) in \(\mathcal{M}_T\)), and with the maximum diameter of elements of \(P_n\) going to zero as \(n \to \infty\). Define \(h_n(\mu) = h(\mu, P_n)\). The sequence \((h_n)\) is an entropy structure for \((X, T)\).

- \((h_n)\) reflects emergency of complexity on refining scales.

- The meaning of “allowed” is part of a deeper theory of entropy [D2].

- Every system has an entropy structure [BD1].
Superenvelopes. Below: \((h_n)\) is an entropy structure with \(h_0 \equiv 0\) and all \(h_n - h_{n-1}\) u.s.c. A bounded function \(E\) on \(\mathcal{M}_T\) such that every \(E - h_n\) is nonnegative u.s.c. is called a superenvelope of the entropy structure. (Also allow the constant function \(E \equiv \infty\) as a superenvelope.)

Sex Entropy Theorem [BD1].
Let \(E\) be a bounded function on \(\mathcal{M}_T\). T.F.A.E.

1. \(E\) is the extension entropy function of a symbolic extension of \((X,T)\).

2. \(E\) is affine and a superenvelope of the entropy structure.

(The statement does not depend on the choice of entropy structure.)

Functional analytic characterization of \(h_{\text{sex}}\).
\(h_{\text{sex}}\) is the minimum superenvelope of the entropy structure \((h_n)\).
Inductive Characterization of $h_{sex}$.
Let $\tilde{g}$ denote the u.s.c. envelope of a function $g$ (the inf of the continuous functions larger than $g$). Convention: $\tilde{g} \equiv \infty$ if $\sup g = \infty$.

Let $\mathcal{H} = (h_n)$ be an entropy structure, $h_n \rightarrow h$. Begin with the tail sequence $\tau_n = (h - h_n)$, which decreases to zero. We will define by transfinite induction a transfinite sequence $u_\mathcal{H}$ of functions $u_\alpha$ on $\mathcal{M}_T$. Set
- $u_0 \equiv 0$
- $u_{\alpha+1} = \lim_k (u_\alpha + \tau_k)$
- $u_\beta = \text{the u.s.c. envelope of } \sup \{u_\alpha : \alpha < \beta\}$, if $\beta$ is a limit ordinal.

**THEOREM** $u_\alpha = u_{\alpha+1} \iff u_\alpha + h = h_{sex}$, and such an $\alpha$ exists among countable ordinals (even if $h_{sex} \equiv \infty$).

The convergence above can be transfinite, and this indicates the subtlety of the emergence of complexity on ever smaller scales.
Sex entropy and smoothness

If \((X, T)\) is \(C^\infty\), then [Buzzi following Yomdin] \(T\) is asymptotically \(h\)-expansive, and [BFF] therefore \(h_{\text{sex}} = h\).

**Theorem** [DN] A generic \(C^1\) non-hyperbolic (i.e. non-Anosov) area preserving diffeomorphism of a compact surface has no symbolic extension (i.e. residual entropy \(= \infty\)).

**Theorem** [DN] For \(r > 1\) and any compact Riemannian manifold of dimension \(> 1\), there is a \(C^r\)-open set of \(C^r\) diffeomorphisms in which the diffeomorphisms with positive topological residual entropy are a residual set.

**Theorem** [A] For a smooth compact manifold \(M\) with \(\dim(M) \geq 3\), there is an open subset of \(\text{Diff}^1(M)\) in which generic diffeomorphisms have no symbolic extension.
The DN/A proofs involve complicated iterated constructions using genericity arguments and persistent homoclinic tangencies. We’ll give concrete $C^r$ examples ($1 \leq r < \infty$) a little later.

**The main open problem.** For a $C^r$ diffeomorphism $T$, $1 < r < \infty$, is it possible that $T$ has infinite residual entropy?

**Conjecture [DN].** Suppose $2 \leq r < \infty$ and $T$ is a $C^r$ diffeomorphism. Then

$$h_{sex}(T) \leq \left[ R(f) \dim(X) \right] \frac{r}{r-1},$$

where $R(f) := \lim_{n} (1/n) \log \max \| (T^n)' \|$. 
II. Functoriality of sex entropy. [BD2]

**Powers.** For $0 \neq n \in \mathbb{Z}$,

1. The restriction of $h_{\text{sex}}^{T^n}$ to $\mathcal{M}_T$ equals $|n|h_{\text{sex}}^T$.
2. $h_{\text{sex}}(T^n) = |n|h_{\text{sex}}^T$.

**Flows.** For $T$ a flow and $a, b$ nonzero in $\mathbb{R}$,

1. $h_{\text{sex}}(T^a, \mu) = |a/b|h_{\text{sex}}(T^b, \mu),$
   \quad for all $\mu \in \mathcal{M}_{T^a} \cap \mathcal{M}_{T^b}$.
2. $h_{\text{sex}}(T^a) = |a/b|h_{\text{sex}}(T^b)$.

**Products.** Suppose $(X, T)$ is the product of finitely or countably many systems $(X_k, T_k)$ such that $\sum_k h_{\text{sex}}(T_k) < \infty$, and $\mu \in \mathcal{M}_T$. Let $\mu_k$ be the coordinate projection of $\mu$. Then

1. $h_{\text{sex}}(T, \mu) \leq \sum_k h_{\text{sex}}(T, \mu_k)$.
2. If $\mu$ is the product measure $\prod_k \mu_k$, then
   \quad $h_{\text{sex}}(T, \mu) = \sum_k h_{\text{sex}}(T, \mu_k)$.
3. $h_{\text{sex}}(T) = \sum_k h_{\text{sex}}(T_k)$.
Fiber Products. Let \((X, T)\) be the fiber product of \((X', T')\) and \((X'', T''')\) over their common factor \((X, T''')\). Then

\[
(1) \quad h_{\text{sex}}(T, \mu) \leq h_{\text{sex}}(T', \mu') + h_{\text{sex}}(T'', \mu'') - h(T''', \mu''')
\]

where \(\mu \in \mathcal{M}_T\) and the other measures are its projections.

(2) If above \(\mu\) is the relatively independent joining of \(\mu'\) and \(\mu''\), and \(T''\) is asymptotically \(h\)-expansive, then

\[
\begin{align*}
& h_{\text{sex}}(T, \mu) \geq h_{\text{sex}}(T', \mu') + h_{\text{sex}}(T'', \mu'') - h_{\text{sex}}(T''', \mu''') \\
& \text{(3) If above } h(T''') = 0 \text{ and } T'' \text{ is asymptotically } h\text{-expansive, then} \\
& h_{\text{sex}}(T, \mu) = h_{\text{sex}}(T', \mu') + h_{\text{sex}}(T'', \mu'').
\end{align*}
\]

We need (3) for our explicit examples.

The proofs for products and fiber products use the (transfinite) inductive characterization and also the Downarowicz entropy structure defined from continuous functions [D2].
III. Examples.

Given $1 \leq r < \infty$, Misiurewicz (1973) manipulated several vector fields to construct a $C^r$ system $D : V \times S^1 \rightarrow V \times S^1$ with no measure of maximal entropy (the first smooth examples with no such measure). ($\dim(V)=3$.) Features of the example, given $r$: 

- Each $V \times \{t\}$ is $D$-invariant. Let $V_t = V \times \{t\}$, $D_t = D|V_t$, $S^1 = (-1/2, 1/2]$.

- $h_{\text{top}}(D_0) = 0$.

- Restriction of $D$ to $\bigcup_{t \geq \epsilon} V_t$ is $C^\infty$ with entropy $< h(D)$.

- $\limsup_{t \to 0} h(D_t) = h(D) > 0$. 

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It turns out that the sex entropy function $h^D_{\text{sex}}$ is simply the u.s.c. envelope $\tilde{h}$ of the entropy function $h$ on $\mathcal{M}_D$.

The proof of this [BD2] uses the functional analytic characterization of the sex entropy function, and a study of the lift of $h_{\text{sex}}$ from $\mathcal{M}_D$ to a function on the Bauer simplex whose boundary is the closure of the ergodic measures in $\mathcal{M}_D$.

**Sex Entropy Variational Principle** [BD1]. The topological sex entropy is the max of its sex entropy function.

So for $D$, the topological sex entropy equals its topological entropy.
Another Misiurewicz example.

Another (much easier) Misiurewicz example (1971): a smooth system \((W \times S_1, R)\) with the entropy function on \(\mathcal{M}_R\) not lower semicontinuous:

- \(R\) is \(C^\infty\)
- Each \(W \times \{t\} := W_t\) is \(R\)-invariant
  \[ R_t : W_t \to W_t \]
- \(h(R_t) = 0\) if \(t \neq 0\)
- \(h(R_0) > 0\).
Because $W$ is $C^\infty$, it is asymptotically $h$-expansive. The sex entropy function on $\mathcal{M}_W$ is simply the entropy function, and the residual entropy is zero.

We will combine the two Misiurewicz examples in a fiber product to get an explicit example of a $C^r$ diffeo with positive topological sex entropy.
Smooth examples with positive residual entropy.

- Set $X = V \times W \times S^1$.

- Define $T : X \rightarrow X$,
  $T : (v, w, t) \mapsto (D_t(v), R_t(w), t)$.

- $h_{\text{top}}(R_t) = 0$ if $t \neq 0$, and $h_{\text{top}}(D_0) = 0$.

- Thus $h_{\text{top}}(T) = \max \{ h_{\text{top}}(D), h_{\text{top}}(R) \}$.

- To prove $T$ has positive topological residual entropy: by the Sex Entropy Variational Principle, it suffices to show the sup of $h^T_{\text{sex}}$ is larger than the max above.
• $T : (v, w, t) \mapsto (D_t(v), R_t(w), t)$.

• $T$ is a fiber product of $V$ and $W$ over $S^1$. Apply the functorial fiber product result (3) to $\mu \in \mathcal{M}_T$ with projections $\mu_D, \mu_R$:

$$h_{\text{sex}}(T, \mu) = h_{\text{sex}}(D, \mu_D) + h_{\text{sex}}(R, \mu_R) = \tilde{h}(\mu_D) + h(\mu_R)$$

where we used $h_{\text{sex}}^R(\mu_R) = h(\mu_R)$, which holds because $R$ is asymptotically $h$-expansive, which holds because $R$ is $C^\infty$.

• Now choose a $\mu_D$ and $\mu_R$ on $V_0$ and $W_0$ to maximize the $\tilde{h}(\mu_D)$ and $h(\mu_R)$ above, respectively at $h_{\text{top}}(D)$ and $h_{\text{top}}(R)$, and let $\mu$ be their product measure on $V \times W \times \{0\}$. We get

$$h_{\text{sex}}^T(\mu) = h_{\text{top}}(D) + h_{\text{top}}(R) > \max\{h_{\text{top}}(D), h_{\text{top}}(R)\}.$$  

This finishes the proof.