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DIFFUSION DYNAMICS OF AN ELECTRON GAS CONFINED BETWEEN TWO PLATES*

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Abstract. We consider electrons constrained to move in the gap between two plane parallel plates, confined by a magnetic field perpendicular to the plates and accelerated by an electric field parallel to them. The electrons are subject to elastic collisions against the solid plates on the one hand and against atoms or ions in the gap between the plates on the other hand. Under the assumption that the dynamics is dominated by the collisions, we derive a diffusion type model for the energy distribution function of the electrons. The present paper is an extension of a previous one [17] where only collisions against the solid boundaries were considered.

1. Introduction. In a previous paper [17], we considered electrons constrained to move in the gap between two plane parallel plates, confined by a magnetic field perpendicular to the plates and accelerated by an electric field parallel to them. The electrons were subject to elastic collisions against the solid plates. Under the assumptions that the distance between the plates is small compared with the typical lateral extension of the phenomenon and that the dynamics is dominated by the collisions against the wall, we derived a diffusion type model for the dynamics of the electrons in the lateral directions. This obtained diffusion model is of 'Spherical Harmonics Expansion' (SHE) type as the main macroscopic quantity which is evolved by the model is the energy distribution function. The present paper is an extension of [17] to the case where collisions against atoms or ions in the gap between the plates are considered in addition to collisions against the solid boundaries. The situation so-depicted is typical of certain plasma devices like plasma propellors for satellites (see [10], [20] for more details about these devices, [13], [14] for a formal derivation of these models, [15] for numerical applications and [29], [30] for related physical approaches).

Like in [17], following the formal approach [13], [14], we consider that the ratio (denoted by α) of the distance between the two plates to the typical longitudinal length scale (i.e. along the planes) is small and simultaneously that the magnetic field is large (of order α^{-1}) so that the Larmor radius (or gyration radius in the magnetic field) stays of order unity. Besides, we suppose that the electrons are subject to elastically diffusive collisions when they hit the solid plates i.e. they are reemitted with their incident energy but random velocity directions. In [17], collisions with the boundary was the only source of collisions and particle motion in the gap between the plates was supposed collisionless. We showed that the large time behaviour (on time scales of order α^{-2}) of the particle distribution function is, to leading order, given by $F(\underline{\xi}, \varepsilon, t)$, where $\underline{\xi}$ denotes the longitudinal position variable (i.e. the vector $\underline{\xi}$ is parallel to the plates), $\varepsilon = |v|^2/2$ is the electron kinetic energy (with v the velocity) and t the time. The function F was proved to satisfy a diffusion equation in position and energy space known in the literature as the SHE model (see references below).

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In the present paper, we consider the more realistic situation where electrons moving in the gap between the plates can undergo elastic collisions against atoms or ions. These collisions add another source of diffusion which combines in a somehow complex way to collisions with the plates and yield a compound diffusivity in the SHE model. We rigorously prove the convergence of the kinetic model towards the SHE equation when the parameter α tends to 0, and additionally, we provide an explicit expression of the diffusivity in the case of isotropic collisions, which is extremely useful for numerical applications of the model [16], [25]. The proof differs from that of [17] in several instances which will be outlined in the course of the paper.

The SHE model has first been derived by a formal truncation of the Spherical Harmonics Expansion of the Boltzmann equation (hence the terminology). We refer to [5], [12] for applications to semiconductor modeling and [18], [34] for applications to plasmas and discharge physics. In the SHE model, energy diffusion is caused by the combined effects of the electric field and of the collisions. Position and energy diffusions are not independent (in other words, the diffusion is degenerate) because total energy is preserved during both free flight and collisions.

A related problem had previously been considered by [1] and [2]. The authors considered a collisionless neutral gas flowing in a thin domain (e.g. the gap between two plane parallel plates) and subject to accomodation at the boundary (i.e. reemission according to a given distribution function after collisions against the solid plates). In this case, since collisions against the boundary were inelastic, the large time dynamics was that of an ordinary diffusion equation in position space only. We refer to [17] for a detailed description of the analogies and differences between this earlier approach and ours. Let us just point out that, in the present plane parallel geometry, the diffusivity of [2] was infinite. This is due to the too large proportion of particles reemitted with velocities tangent to the plates and which travel a very large distance between two encounters with the plates. It was later shown in [8] and in [21], [19] that a logarithmic time rescaling can restore a finite diffusivity. In our case, the diffusivity stays finite without logarithmic time rescaling. This is because Larmor gyration in the magnetic field on the one hand [17] and collisions with atoms on the other hand limit the distance travelled by a particle between two encounters with the plates.

Mathematically, the present problem belongs to the class of diffusion approximation problems for kinetic equations. We refer to [17] for a summary and bibliography of the various mathematical techniques that are used in this area. Let us just mention the important references that are landmarks of this field: [7], [4], [22] as well as [11] for a general exposition of kinetic theory.

The paper is organized as follows. First, an introduction to the kinetic model is given in Section 2 and the main result is stated. Then, properties of the collision operators are reviewed in Section 3. An existence theorem for the kinetic model is proved in Section 4. The convergence of the kinetic model to the SHE model is then investigated in Section 5 and some properties of the diffusivity tensor are outlined. Eventually, an explicit formula for the diffusivity in the case of isotropic collision operators is given in Section 6 and a short conclusion is drawn in Section 7.

2. The kinetic model. We introduce a coordinate system (x, y, z) such that the two solid plates are located at $\{x = 0\}$ and $\{x = 1\}$. The electron gas occupies the domain $X := (x, y, z) \in [0, 1] \times \mathbb{R}^2$. We denote by $\underline{\xi} = (y, z) \in \mathbb{R}^2$ the component of the position vector parallel to the plates. Let $v \in \mathbb{R}^3$ denote the particle velocity. We decompose: $v = (v_x; \underline{v})$, where v_x is the velocity component parallel to the x-axis and

$\underline{v} = (v_y, v_z) \in \mathbb{R}^2$ is the component parallel to the plates. The electrons are subject to a magnetic field transverse to the plates $(B(\underline{\xi}), 0, 0)$ and to an electric field parallel to the plates, $(0, E_y(\underline{\xi}), E_z(\underline{\xi})) = (0; \underline{E}(\underline{\xi}))$, where B and $\underline{E}$ are supposed to depend only upon $\underline{\xi}$ due to the divergence-free and curl-free constraints respectively. We shall denote by Θ the domain of phase-space: $\Theta = \{(X, v),\ X \in [0,1] \times \mathbb{R}^2,\ v \in \mathbb{R}^3\}$.

Let $f^\alpha(X, v, t) = f^\alpha(x, \underline{\xi}, v_x, \underline{v}, t)$ denote the electron distribution function, i.e. the density of particles in an elementary phase-space volume element about the position X and velocity v at time $t > 0$. It obeys the following scaled Boltzmann equation

$$\alpha^2 \frac{\partial f^\alpha}{\partial t} + \alpha(\underline{v} \cdot \nabla_{\underline{\xi}} f^\alpha - \underline{E} \cdot \nabla_{\underline{v}} f^\alpha) + v_x \frac{\partial f^\alpha}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f^\alpha = \mathcal{L} f^\alpha, \tag{2.1}$$

where $(\underline{v} \times B) = (0, v_z B, -v_y B)$. The scaling parameter α measures the ratio between the microscopic length scale (typically, the distance between the plates) and the macroscopic length scale (typically, the lateral extension of the device in the direction $\underline{\xi}$). Note that the ratio between the microscopic time scale (typically the time elapsed between two encounter with the solid plates) and the macroscopic time scale (i.e. the observation time) is of order α^2. This ratio between the position and time scales is typical of diffusion processes. We shall see that this is the relevant scaling for the present problem. We also observe that the magnetic and electric field scales are such that the number of particle gyrations under the magnetic field between two encounters with the plates is of order 1, while the acceleration of the particles due to the electric field in the same interval is of order α.

The operator $\mathcal{L}$ models elastic collisions with atoms or ions present in the gap between the two plates. We first introduce the spherical coordinates in velocity space: $v = |v|\omega$, where ω belongs to the unit sphere $\mathbb{S}^2$. Then $\mathcal{L}$ is given by:

$$\mathcal{L}(f)(v) = \int_{\omega' \in \mathbb{S}^2} \Phi(\underline{\xi}, \omega, \omega')\,(f(x, |v|\omega', t) - f(x, |v|\omega, t))\, d\omega'\,. \tag{2.2}$$

The fact that the integration is carried over a sphere expresses the conservation of energy during an elementary collision which sends the electron from velocity $v = |v|\omega$ to $v' = |v|\omega'$. The factor $\Phi(\underline{\xi}, \omega, \omega')$ is related to the differential scattering cross-section associated with the collision mechanism and is supposed given and smooth. Moreover, Φ may depend on $\underline{\xi}$ as the characterisitics of the atoms or ion gases may depend on the spatial coordinate. However, we shall suppose that these characteristics are homogeneous within a given section $\{\underline{\xi} = \text{Constant}\ \}$ of the device, hence the dependence on $\underline{\xi}$ only.

The scaling of the collision operator is such that the number of elastic collisions with atoms or ions which a given electron suffers between two encounters with the plates is of order unity. We point out that restricting to elastic collisions amounts to neglecting the velocity of the atoms or ions as well as energy transfers between them and the electrons. This limitation could be overcome by considering inelastic collisions. Indeed, our analysis would apply if an inelastic collision operator $\alpha^2 \mathcal{I}(f^\alpha)$ was added as an order α^2 term at the right-hand side of (2.1). This scaling is valid as long as energy tranfers between electrons and atoms or ions are small, which is usually the case due to the very small electron mass. However, for the sake of simplicity, we shall disregard inelastic collisions in the present work.

Equation (2.1) is supplemented with conservative boundary conditions expressing the incoming flux of particles in terms of the outgoing flux. We first need to define the traces of f^α on the boundaries of the domain Θ. Therefore, we consider the boundary

$\Gamma = \{0,1\} \times \mathbb{R}^2 \times \mathbb{R}^3$ and the following *incoming* and *outgoing* sets respectively denoted by Γ^- and Γ^+:

$$\Gamma^{\pm} = \{(X,v) \in \Theta\,,\ \text{s.t. } (x=0 \text{ and } \mp v_x > 0) \text{ or } (x=1 \text{ and } \pm v_x > 0)\}\ .$$

Denoting by f_- and f_+ the traces of the distribution function f on the sets Γ^- and Γ^+, the boundary conditions read as follows:

$$f_- = \mathcal{B} f_+ := \beta \mathcal{J} f_+ + (1-\beta)\mathcal{K}(f_+), \quad (X,v) \in \Gamma^-, \tag{2.3}$$

where the accommodation coefficient $\beta = \beta(x, \underline{\xi}, |v|)$ is such that $0 \le \beta < 1$. $\mathcal{J}$ is the mirror reflection operator given according by $\mathcal{J} f_+(v) = f_+(v_*)$ where $v_* = (-v_x, v_y, v_z)$ is the specular reflection of the vector v with respect to the boundary. The operator $\mathcal{K}$ operates from functions defined on Γ^+ towards functions defined on Γ^- according to:

$$\mathcal{K}(f_+)(X,v) = \int_{\{\omega' \in \mathbb{S}^2, (X,|v|\omega') \in \Gamma^+\}} K(X,|v|;\omega' \to \omega)\, f_+(X,|v|\omega')\, |\omega'_x|\, d\omega'. \tag{2.4}$$

The dependence of the kernel K with respect to $X = (x;\underline{\xi})$, $x \in \{0,1\}$, $\underline{\xi} \in \mathbb{R}^2$ and $|v|$ will be omitted, otherwise specified. The quantity $K(\omega' \to \omega)\, |\omega_x| d\omega$ is the probability than an electron impinging on a plane at a position (y,z) with velocity modulus $|v|$ and velocity direction ω' will be reflected with new velocity angle ω belonging to the solid angle $d\omega$ (and the same velocity modulus). Hence, the boundary condition (2.3) states that an electron colliding with the boundary can undergo a mirror reflection with probability β and a diffuse reflection with probability $(1-\beta)$. The diffuse reflection is obviously elastic as the reflected velocity $v = |v|\omega$ has the same modulus as the incoming one $v' = |v'|\omega$.

The same remark as for the collisions with neutrals atoms or ions can be made here. In realistic physical situations, collisions with the boundary are not elastic. However, there is a range of energies for which the relative energy change is small and can be neglected or accounted for by a (boundary) inelastic collision operator of magnitude α^2. This occurence has been considered in [13]. Again, for the sake of simplicity, these inelastic collisions will not be considered here.

In this paper, we investigate the limit $\alpha \to 0$ of (2.1). To avoid the treatment of initial layers, we first assume that the initial data are well prepared:

Hypothesis 2.1. *We suppose that there exists a function F_I such that $f_I(x,\underline{\xi},v) = F_I(\underline{\xi}, |v|^2/2)$ and that f_I satisfies: $f_I \in L^2(\Theta)$, $(\underline{v}\cdot\nabla_{\underline{\xi}} - \underline{E}\cdot\nabla_{\underline{v}}) f_I \in L^2(\Theta)$.*

Then, we prove the following result:

Theorem 2.1. *Under a certain number of hypotheses listed in the following sections, problem (2.1), (2.2), (2.3) admits a solution f^α, for every $\alpha > 0$. Moreover, when α tends to zero, f^α converges to f^0 in the weak star topology of $L^\infty([0,T], L^2(\Theta))$ for any $T > 0$, where $f^0(X,v,t) = F(\underline{\xi}, |v|^2/2, t)$, and $F(\underline{\xi}, \varepsilon, t)$ is a distributional solution of the problem:*

$$4\pi\sqrt{2\varepsilon}\frac{\partial F}{\partial t} + \left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right)\cdot \underline{J} = 0, \tag{2.5}$$

$$\underline{J}(\underline{\xi}, \varepsilon, t) = -\mathbb{D}\left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right) F(\underline{\xi}, \varepsilon, t), \tag{2.6}$$

$$F|_{t=0} = F_I. \tag{2.7}$$

The diffusion tensor $\mathbb{D} = \mathbb{D}(\underline{\xi}, \varepsilon)$ *is given by*

$$\mathbb{D}(\underline{\xi}, \varepsilon) = (2\varepsilon)^{3/2} \int_0^1 \int_{\mathbb{S}^2} \underline{D}(x, \omega; \underline{\xi}, \varepsilon)\underline{\omega}\, dx\, d\omega, \tag{2.8}$$

where $\underline{\omega} = (\omega_y, \omega_z)$, $\underline{D} = (D_y, D_z)$, *and the* 2×2 *matrix* $\underline{D}\,\underline{\omega}$ *is the tensor product of* $\underline{D}$ *and* $\underline{\omega}$. *The functions* $D_i(x, \omega; \underline{\xi}, \varepsilon)$ *with* $i = y, z$ *are solutions of the problem*

$$-v_x \frac{\partial D_i}{\partial x} + (\underline{v} \times B) \cdot \nabla_{\underline{v}} D_i - \mathcal{L} D_i = \omega_i\,, \text{ in } \Theta \tag{2.9}$$

$$(D_i)_+ = \mathcal{B}^*(D_i)_-\,, \text{ on } \Gamma \tag{2.10}$$

and are unique up to addition of an arbitrary function of $\underline{\xi}$ *and* ε.

Moreover, we shall see that the matrix $\mathbb{D}$ is definite positive (Proposition 5.3) and satisfies the Onsager relation (Proposition 5.4). Note that $\mathbb{D}$ is not symmetric in general (see the explicit computation Proposition 6.1). System (2.5)-(2.6) is the "so called" Spherical Harmonic Equation (SHE) model (see references in the introduction).

The proof of Theorem 2.1 will be divided in the following steps. In the next section, we recall some properties of the collision operators (see [17] for the boundary operator, and [5], [27] for operator $\mathcal{L}$). In Section 4, we prove the existence and uniqueness of the solution f^α of the evolution problem (2.1) with boundary conditions (2.3). Then, we establish estimates on f^α which show the existence of a weak limit f^0 which additionally does not depend on x and ω. Then, we show that the current converges weakly and we establish equation (2.6). Finally, we derive the continuity equation (2.5), first for f^α, then letting $\alpha \to 0$, for its limit f^0, which concludes the proof.

3. The collision operators.

3.1. Collisions with the boundary. The study of the boundary operator $\mathcal{B}$ has been extensively developped in [17]. In the present section, we summarize the main assumptions and results. Let $\mathcal{S}_\pm(x)$, $x = 0, 1$ be the following half-spheres:

$$\mathcal{S}_\pm(0) = \{\omega \in \mathbb{S}^2, \pm\omega_x < 0\}\,,\ \mathcal{S}_\pm(1) = \{\omega \in \mathbb{S}^2, \pm\omega_x > 0\}\,. \tag{3.1}$$

Introduce the domain $\mathcal{S} = [0, 1] \times \mathbb{S}^2$, with *incoming* and *outgoing* sets respectively denoted by $\mathcal{S}^-$ and $\mathcal{S}^+$ and given by $\mathcal{S}^\pm = (\{0\} \times \mathcal{S}_\mp(0)) \cup (\{1\} \times \mathcal{S}_\pm(1))$.

We shall consider L^2-based functional spaces with the associated inner products on $L^2(\Theta)$, $L^2(\Gamma^+)$ and $L^2(\Gamma^-)$:

$$(f, g)_\Theta = \int_\Theta fg\, d\theta\,, \quad (f, g)_{\Gamma^\pm} = \int_{\Gamma^\pm} fg\, |v_x|\, d\Gamma,$$

where $d\theta = dxd\underline{\xi}dv$ is the volume element in phase space, and $d\Gamma = \sum_{x=0,1} d\underline{\xi}dv$ is the surface element. The inner products on $L^2(\mathcal{S})$, $L^2(\mathcal{S}^\pm)$ are defined analogously by:

$$(f, g)_\mathcal{S} = \int_0^1 \int_{\mathbb{S}^2} fg(x, \omega)\, dx\, d\omega, \quad (f, g)_{\mathcal{S}^\pm} = \sum_{x=0,1} \int_{\mathcal{S}_\pm(x)} |\omega_x|\, fg(x, \omega)\, d\omega\,. \tag{3.2}$$

Let us define (as in [17]) the operator Q_+ as the orthogonal projection (for the inner product $(\cdot,\cdot)_{\mathcal{S}^+}$) of $L^2(\mathcal{S}^+)$ on the space $\mathcal{C}^+$ of constant functions on each connected component, i.e.:

$$Q_+ f(x,\omega) = \frac{1}{\pi}\int_{\mathcal{S}_+(x)} |\omega_x|\, f(x,\omega)\, d\omega\,,\ \omega\in\mathcal{S}_+(x)\,,\ x\in\{0,1\}, \tag{3.3}$$

and the operator P_+ as the orthogonal complement of Q_+: $P_+ = I - Q_+$. In the same way, we define the operators Q_- and P_- on the space $\mathcal{C}^-$ of constant functions on each connected component of $\mathcal{S}^-$.

We now list the required assumptions on the operator $\mathcal{K}$. For that purpose, we consider that the operator $\mathcal{K}$ operates on the variable ω only, i.e. we consider that $x\in\{0,1\}$, $\underline{\xi}\in\mathbb{R}^2$ and $|v|>0$ as parameters.

Hypothesis 3.1. *We assume that the kernel K satisfies the following properties:*
(o) $\mathcal{K}$ bounded from $L^2(\mathcal{S}_+(x))$ to $L^2(\mathcal{S}_-(x))$, for any $x\in\{0,1\}$,
(i) positivity: $K(\omega'\to\omega)>0$,
(ii) flux conservation: for any $x\in\{0,1\}$ we have

$$\int_{\mathcal{S}_-(x)} K(\omega'\to\omega)\,|\omega_x|\,d\omega = 1, \tag{3.4}$$

(iii) reciprocity relation:

$$K(\omega'\to\omega) = K(-\omega\to-\omega'),\quad \forall\,\omega\in\mathcal{S}_-(x),\,\omega'\in\mathcal{S}_+(x),\,\forall x\in\{0,1\}\,. \tag{3.5}$$

As proved in [17], from Hypothesis 3.1 follows the *normalization identity*: For any $x\in\{0,1\}$, we have

$$\int_{\mathcal{S}_+(x)} K(\omega'\to\omega)\,|\omega'_x|\,d\omega' = 1\,, \tag{3.6}$$

and the *Darrozes-Guiraud inequality*: For any $x\in\{0,1\}$, we have

$$\int_{\mathcal{S}_-(x)} |f_-(x,\omega)|^2\,|\omega_x|\,d\omega \le \int_{\mathcal{S}_+(x)} |f_+(x,\omega)|^2\,|\omega_x|\,d\omega\,. \tag{3.7}$$

We consider the adjoint operator $\mathcal{K}^*$ of $\mathcal{K}$. Obviously, $\mathcal{K}^*$ operates from $L^2(\mathcal{S}_-(x))$ to $L^2(\mathcal{S}_+(x))$ and is given by

$$\mathcal{K}^*(f)(x,\omega) = \int_{\mathcal{S}_-(x)} K(\omega\to\omega')\,|\omega'_x|\,f(x,\omega')\,d\omega',\ \omega\in\mathcal{S}_+(x)\,. \tag{3.8}$$

Hence, the adjoint operator $\mathcal{B}^*$ of $\mathcal{B}$ operates on the same spaces as $\mathcal{K}^*$ and is given by:

$$\mathcal{B}^* = \beta\mathcal{J}^* + (1-\beta)\mathcal{K}^*\,, \tag{3.9}$$

where $\mathcal{J}^*$ is the adjoint of $\mathcal{J}$ and is the mirror reflection operator acting from functions defined on $\mathcal{S}_-(x)$ to functions defined on $\mathcal{S}_+(x)$. Additionnally to Hypothesis 3.1, we assume that:

HYPOTHESIS 3.2. *(i) The operator $\mathcal{K}$ is a compact operator from $L^2(\mathcal{S}^+)$ to $L^2(\mathcal{S}^-)$. Analogously, the operator $\mathcal{K}^*$ is compact from $L^2(\mathcal{S}^-)$ to $L^2(\mathcal{S}^+)$,*
(ii) there exists a constant $k < 1$ such that:

$$\|\mathcal{K}P_+\|_{\mathcal{L}(L^2(\mathcal{S}^+),L^2(\mathcal{S}^-))} \leq k < 1\,, |v| \in \mathbb{R}^+\,,\, \underline{\xi} \in \mathbb{R}^2\,, \tag{3.10}$$

(iii) there exists $\beta_0 < 1$ such that $0 \leq \beta \leq \beta_0 < 1$, $\underline{\xi} \in \mathbb{R}^2$, $|v| > 0$, $x = 0, 1$.

It follows that:

$$\|\mathcal{B}P_+\|_{\mathcal{L}(L^2(\mathcal{S}^+),L^2(\mathcal{S}^-))} \leq \sqrt{\beta_0 + (1-\beta_0)k^2} = k_0 < 1\,. \tag{3.11}$$

As consequences we get the two following lemmas (see [17]),

LEMMA 3.1. *Under Hypotheses 3.1, 3.2, we have:*
(i) The Null-Space of $I - \mathcal{J}\mathcal{B}^*$ and of $I - \mathcal{J}^*\mathcal{B}$ are respectively given by:

$$N(I - \mathcal{J}\mathcal{B}^*) = \mathcal{C}^-\,, \quad N(I - \mathcal{J}^*\mathcal{B}) = \mathcal{C}^+\,,$$

(ii) $\mathcal{B}$ as an operator from $L^2(\mathcal{S}^+)$ to $L^2(\mathcal{S}^-)$ is of norm 1,
(iii) if $K(\omega' \to \omega) \geq C > 0$, where C is a constant, then assumption (3.10) holds.

LEMMA 3.2. *Let the projection operators $Q_\pm$, $P_\pm$, $\mathcal{B}$ be defined as above. Then the following equalities hold:*

$$\mathcal{B}Q_+ = Q_-\mathcal{B} = \mathcal{J}Q_+ = Q_-\mathcal{J}\,, \quad \mathcal{B}P_+ = P_-\mathcal{B}. \tag{3.12}$$

and similarly (mutatis mutandis) for $\mathcal{B}^*$.

3.2. Collisions with atoms or ions. The properties of the collision operator $\mathcal{L}$ are fairly classical. We refer to [5], [12], [27] for details of the proofs. We assume that the scattering cross section related quantity Φ satisfies the following:

HYPOTHESIS 3.3. *(i) There exist two constants, c_1 and c_2, such that*

$$0 < c_1 \leq \Phi \leq c_2\,,$$

(ii) Φ is invariant by exchange of v and v', i.e. for every $\omega, \omega' \in \mathbb{S}^2$,

$$\Phi(\omega', \omega) = \Phi(\omega, \omega')\,.$$

Let define:

$$\lambda(\omega) = \int_{\mathbb{S}^2} \Phi(\omega, \omega')\, d\omega'\,,$$

(remark that λ may depends on $\underline{\xi}$ and $|v|$), then, the operator $\mathcal{L}f$ may be split in a 'gain' part $\mathcal{L}^+f$ and a 'loss' part $-\lambda f$:

$$\mathcal{L}f = \mathcal{L}^+f - \lambda f \quad , \qquad \mathcal{L}^+f = \int_{\mathbb{S}^2} \Phi(\omega, \omega') f(\omega')\, d\omega'. \tag{3.13}$$

Again, we note that $\mathcal{L}$ can be viewed as an operator acting on functions of ω only. Therefore, the other variables can be treated as simple parameters. We define the space $L^2(\mathbb{S}^2)$ equipped with the usual inner product $(f,g)_{\mathbb{S}^2} = \int_{\mathbb{S}^2} f(\omega)g(\omega)d\omega$. We deduce ([5], [12], [27]):

LEMMA 3.3. *Under Hypothesis 3.3, we have:*
(i) For every $f, g \in L^2(\mathbb{S}^2)$:

$$(\mathcal{L}(f), g)_{\mathbb{S}^2} = -\frac{1}{2}\int_{\mathbb{S}^2} \Phi(f'-f)(g'-g)\, d\omega\, d\omega' \,, \tag{3.14}$$

(ii) $\mathcal{L}$ is uniformly bounded, i.e. $\exists M$ such that $\|\mathcal{L}\|_{L^2(\mathbb{S}^2)} \leq M$, independently of X and $|v|$,
(iii) $\mathcal{L}$ is self-adjoint, i.e. $(\mathcal{L}f, g)_{\mathbb{S}^2} = (f, \mathcal{L}g)_{\mathbb{S}^2} \quad \forall f, g \in L^2(\mathbb{S}^2)$,
(iv) $\mathcal{L}$ is a dissipative operator, i.e. $(\mathcal{L}f, f)_{\mathbb{S}^2} \leq 0$ for every $f \in L^2(\mathbb{S}^2)$,
(v) the Null-Space $N(\mathcal{L})$ of $\mathcal{L}$ is the one dimensional space of constant functions.

Moreover, considering the projection operator onto constant functions over the sphere $\mathbb{S}^2$, defined for every $f \in L^2(\mathbb{S}^2)$ by :

$$[f] = \frac{1}{4\pi}\int_{\mathbb{S}^2} f(\omega)\, d\omega \,, \tag{3.15}$$

one can derive the following coercivity property and characterization of the range $R(\mathcal{L})$ of $\mathcal{L}$ (see [5] and [12]):

LEMMA 3.4. *(i) For every $f \in L^2(\mathbb{S}^2)$,*

$$-(\mathcal{L}f, f)_{\mathbb{S}^2} \geq c_1 4\pi |f - [f]|^2_{L^2(\mathbb{S}^2)} \,, \tag{3.16}$$

(ii) $R(\mathcal{L}) = N(\mathcal{L})^\perp = \{f \in L^2(\mathbb{S}^2),\ s.t.\ [f] = 0\}$.

4. The evolution problem. The following hypotheses make the transport operator in equation (2.1) easily solvable. Some of the regularity assumptions could be relaxed at the expense of technicalities, but we shall avoid them in the present paper.

HYPOTHESIS 4.1. *(i) The electric field satisfies $\underline{E} = \underline{E}(\underline{\xi}) \in (W^{1,\infty}(\mathbb{R}^2))^2$ and is independent of t,*
(ii) similarly $B = B(\underline{\xi}) \in C^1(\mathbb{R}^2)$ and is independent of t,
(iii) there exists a constant $B_0 > 0$ such that $|B(\underline{\xi})| \geq B_0 > 0$, for every $\underline{\xi} \in \mathbb{R}^2$.

We now define the following operator on $L^2(\Theta)$:

$$\mathcal{A}^\alpha f = \underline{v}\cdot\nabla_{\underline{\xi}} f - \underline{E}\cdot\nabla_{\underline{v}} f + \frac{1}{\alpha}\left(v_x\frac{\partial f}{\partial x} - (\underline{v}\times B)\cdot\nabla_{\underline{v}} f\right), \tag{4.1}$$

with domain $D(\mathcal{A}^\alpha)$ defined by:

$$D(\mathcal{A}^\alpha) = \{f \in L^2(\Theta),\, \mathcal{A}^\alpha f \in L^2(\Theta),\, f_+ \in L^2(\Gamma^+),\, f_- = \mathcal{B}f_+\}\,.$$

We denote by $\mathcal{A}$ the bare differential operator (4.1) when no indication of the domain is needed. Following [3], [33], we define the spaces:

$$H(\mathcal{A}^\alpha) = \{f \in L^2(\Theta), \mathcal{A}^\alpha f \in L^2(\Theta)\}\,, \tag{4.2}$$

$$H_0(\mathcal{A}^\alpha) = \{f \in H(\mathcal{A}^\alpha),\, f_- \in L^2(\Gamma_-)\} = \{f \in H(\mathcal{A}^\alpha),\, f_+ \in L^2(\Gamma_+)\}. \tag{4.3}$$

Then, from [3], [33], we deduce:

LEMMA 4.1 (Green's Formula). *Under Hypothesis 4.1, for f and g in $H_0(\mathcal{A}^\alpha)$ with compact support with respect to v, we have:*

$$(\mathcal{A}^\alpha f, g)_\Theta + (f, \mathcal{A}^\alpha g)_\Theta = \frac{1}{\alpha}\left((f_+, g_+)_{\Gamma_+} - (f_-, g_-)_{\Gamma_-}\right). \tag{4.4}$$

It is difficult to show that $D(\mathcal{A}^\alpha)$ is closed for the graph norm $|f|^2_{\mathcal{A}^\alpha} = |f|^2_{L^2(\Theta)} + |\mathcal{A}^\alpha f|^2_{L^2(\Theta)}$ because we are lacking a control of $|f_+|^2_{L^2(\Gamma_+)}$ (and more precisely of $|Q_+ f_+|^2_{L^2(\Gamma_+)}$ since $\mathcal{B}$ acts like the identity on the spaces $\mathcal{C}^\pm$). In [17], following [6], we perturb the boundary operator in order to control $|Q_+ f_+|^2_{L^2(\Gamma_+)}$ and, by passing to the limit in the perturbation parameter, we can show the existence (but not the uniqueness) of a solution of the kinetic problem (2.1) (with $\mathcal{L} = 0$), (2.3). Since the collision operator $\mathcal{L}$ is a bounded perturbation of the transport operator, the same proof can be applied to the present case without any modification. Therefore, we simply summarize the main steps of the proof (in order to fix the notations) and state the main estimates below. We shall refer to [17] for details.

For $\eta > 0$, we introduce the operator:

$$\mathcal{B}_\eta = \mathcal{B} P_+ + \frac{1}{1+\eta} J Q_+ ,$$

and the operator $\mathcal{A}^\alpha_\eta = \mathcal{A}$ with domain

$$D(\mathcal{A}^\alpha_\eta) = \{u \in H(\mathcal{A}^\alpha)\ ,\ u_+ \in L^2(\Gamma_+)\, ,\, u_- = \mathcal{B}_\eta u_+\}\, , \tag{4.5}$$

We now have

$$||\mathcal{B}_\eta||_{\mathcal{L}(L^2(\mathcal{S}^+), L^2(\mathcal{S}^-))} < 1\, ,\ \forall \eta > 0\, , \tag{4.6}$$

It is readily seen that the operator $\mathcal{A}^\alpha_\eta$ is closed. In [17], we prove that it generates a strongly continuous semigroup of contraction:

PROPOSITION 4.1. *$\mathcal{A}^\alpha_\eta$, with domain $D(\mathcal{A}^\alpha_\eta)$, is a maximal accretive operator.*

Since $\mathcal{L}$ is a bounded perturbation and $-\mathcal{L}$ is accretive, it follows that $\mathcal{A}^\alpha_\eta - \alpha^{-1}\mathcal{L}$ also generates a strongly continuous semigroup of contractions. We point out that $\alpha > 0$ is kept fixed in this part. However, for future use, we shall keep track of the dependences of the various estimates upon α. Then, we apply Hille-Yosida's theorem (see [9]) and get:

LEMMA 4.2. *For all $\eta > 0$, for all $F_\eta \in D(\mathcal{A}^\alpha_\eta)$, there exits a unique function $f^\alpha_\eta \in \mathcal{C}([0,T]; D(\mathcal{A}^\alpha_\eta)) \cap \mathcal{C}^1([0,T]; L^2(\Theta))$, solution of*

$$\alpha \partial_t f^\alpha_\eta + \mathcal{A}_\eta f^\alpha_\eta = \frac{1}{\alpha}\mathcal{L} f^\alpha_\eta\, , \quad f^\alpha_\eta|_{t=0} = F_\eta \tag{4.7}$$

Moreover, we have the following estimates:

$$|f^\alpha_\eta|_{L^2(\Theta)} \le |F_\eta|_{L^2(\Theta)}\, , \tag{4.8}$$

$$|\alpha \partial_t f^\alpha_\eta|_{L^2(\Theta)} = |(\mathcal{A}_\eta - \alpha^{-1}\mathcal{L}) f^\alpha_\eta|_{L^2(\Theta)} \le |(\mathcal{A}_\eta - \alpha^{-1}\mathcal{L}) F_\eta|_{L^2(\Theta)}\, . \tag{4.9}$$

We then notice [6]:

LEMMA 4.3. *Let F_I be as in Hypothesis 2.1. There exists a sequence $(F_\eta)_{\eta>0}$ such that $F_\eta \in D(\mathcal{A}^\alpha_\eta)$ and*

$$F_\eta \longrightarrow F_I\,, \quad \mathcal{A}F_\eta \longrightarrow \mathcal{A}F_I \quad \text{in } L^2(\Theta)\text{-weak},$$

as $\eta \to 0$.

We now give estimates of the boundary values $|f_\pm|_{L^2(\Gamma^\pm)}$ in terms of $|f|_{\mathcal{A}^\alpha}$ for functions of $D(\mathcal{A}^\alpha_\eta)$. We first state that the projection P_+ of the trace at the boundary of a function of $D(\mathcal{A}^\alpha_\eta)$ is controlled by the graph norm. This is done by evaluating $(\mathcal{A}^\alpha_\eta f, f)_\Theta$ using Green's formula (4.4):

LEMMA 4.4. *If $f \in D(\mathcal{A}^\alpha_\eta)$, then there exists a constant $C > 0$ such that:*

$$|P_- f_-|^2_{L^2(\Gamma^-)} \le |P_+ f_+|^2_{L^2(\Gamma^+)} \le \frac{2\alpha}{1-k_0^2}(\mathcal{A}^\alpha f, f)_\Theta \le C\alpha|f|^2_{\mathcal{A}^\alpha}. \tag{4.10}$$

We now notice that, if $f \in D(\mathcal{A}^\alpha_\eta)$, then $(1+\eta)Q_- f_- = JQ_+ f_+$ (thanks to Lemma 3.2 and to the definition of $\mathcal{B}_\eta$). Thus, there exists a single function $q(f) = q(x, \xi, |v|)$, $x = 0, 1, \xi \in \mathbb{R}^2, |v| > 0$, such that

$$q = (1+\eta)Q_- f_-\ , \ \text{on}\,\Gamma^-, \quad q = Q_+ f_+\ , \ \text{on}\,\Gamma^+ \tag{4.11}$$

The following estimate is obtained in a similar way as (4.10), but using a suitable multiplier prior to the application of Green's formula (4.4):

LEMMA 4.5. *Let $f \in D(\mathcal{A}^\alpha_\eta)$, then:*

$$|q(f)|^2_{L^2_R(\Gamma)} \le C\left(\alpha|f|^2_{\mathcal{A}^\alpha} + R|f|^2_{L^2(\Theta)}\right), \tag{4.12}$$

where the family of semi-norms $|\varphi|^2_{L^2_R(\Gamma)}$, for $R > 0$, is defined by:

$$|\varphi|^2_{L^2_R(\Gamma)} = \int_{\Gamma, |v|\le R} |v_x|\,|\varphi|^2\, d\Gamma, \tag{4.13}$$

and the associated function space is denoted $L^2_R(\Gamma)$.

With all this material, it is now easy to pass to the limit in (4.7) as $\eta \to 0$ and obtain the following existence result and estimates:

PROPOSITION 4.2. *Under Hypotheses 2.1, 3.1, 3.2, 3.3, 4.1, there exists a solution f^α to problem (2.1), (2.3), such that $f^\alpha \in L^\infty(0,T;L^2(\Theta))$, $\mathcal{A}f^\alpha \in L^\infty(0,T,L^2(\Theta))$, $P_+ f^\alpha_+ \in L^\infty(0,T,L^2(\Gamma^+))$, $Q_+ f^\alpha_+ \in L^\infty(0,T,L^2_R(\Gamma^+))$, for all $R > 0$, and the boundary condition is satisfied in the sense that:*

$$P_- f^\alpha_- = \mathcal{B}P_+ f^\alpha_+\,, \quad Q_- f^\alpha_- = \mathcal{J}Q_+ f^\alpha_+\,.$$

Moreover, we have:

$$|f^\alpha|_{C^0([O,T],L^2(\Theta))} \leq |F_I|_{L^2(\Theta)} \ , \tag{4.14}$$

$$\left|\mathcal{A}^\alpha f^\alpha - \alpha^{-1}\mathcal{L}f^\alpha\right|_{C^0([O,T],L^2(\Theta))} \leq |\mathcal{A}^\alpha F_I|_{L^2(\Theta)} \ , \tag{4.15}$$

$$\int_0^T \left|P_+ f_+^\alpha(t)\right|^2_{L^2(\Gamma^+)} dt \leq C\alpha^2 \left|F_I\right|^2_{L^2(\Theta)} \ , \tag{4.16}$$

$$\int_0^T \left|P_- f_-^\alpha(t)\right|^2_{L^2(\Gamma^-)} dt \leq C\alpha^2 \left|F_I\right|^2_{L^2(\Theta)} \ , \tag{4.17}$$

$$\int_0^T |q(f^\alpha(t))|^2_{L^2_R(\Gamma)} dt \leq C_R |F_I|_{\mathcal{A}^\alpha} \ , \quad \forall R > 0 \, , \tag{4.18}$$

$$\int_0^T |f^\alpha - [f^\alpha]|^2_{L^2(\Theta)} ds \leq -C \int_0^T (\mathcal{L}f^\alpha, f^\alpha)_\Theta ds \leq C\alpha^2 |F_I|^2_{L^2(\Theta)} \, . \tag{4.19}$$

where we denote by $q(f^\alpha) = Q_- f_-^\alpha = \mathcal{J} Q_+ f_+^\alpha$.

Proof. Only estimate (4.19) does not follow directly from Lemmas 4.4 and 4.5. But it is an easy consequence of the coercivity estimate (3.16). □

5. Convergence towards the asymptotic model.

5.1. Weak limit of f^α. As a consequence of Proposition 4.2, as α tends to zero, there exists a subsequence, still denoted by f^α, which converges in $L^\infty(0,T;L^2(\Theta))$ weak star to a function f^0. Furthermore, using the diagonal extraction process, the subsequence $q(f^\alpha)$ converges to a function $q(x, \underline{\xi}, |v|, t)$ with $x = 0, 1$ in $L^2(0,T,L^2(\Gamma_R))$ weak star for any R, where $\Gamma_R = \{(X,v) \in \Gamma, \text{ s.t. } |v| < R\}$. Also, from (4.16), (4.17), the traces $P_+ f_+^\alpha$ (resp. $P_- f_-^\alpha$) converge in $L^2(0,T;L^2(\Gamma^+))$ (resp. $L^2(0,T;L^2(\Gamma^-))$), strongly towards zero. Finally, from estimates (4.19) we also obtain that the limit function f^0 is independent on ω in Θ, i.e. $f^0 = f^0(x, \underline{\xi}, |v|, t)$.

We now introduce the weak formulation of problem (2.1), (2.3):

LEMMA 5.1. *Let f^α be a solution of problem (2.1), (2.3) given by Proposition 4.2. Then, for any test function $\phi \in C_0^1([0,T] \times \Theta)$, compactly supported in Θ such that $\phi(\cdot,\cdot,T) = 0$, we have:*

$$\begin{aligned}\int_0^T \int_\Theta f^\alpha \left(\alpha \frac{\partial}{\partial t}\phi + (\underline{v} \cdot \nabla_{\underline{\xi}}\phi - \underline{E} \cdot \nabla_{\underline{v}}\phi)\right) dt\, d\theta + \alpha \int_\Theta f_I \phi|_{t=0}\, d\theta \\ + \frac{1}{\alpha}\int_0^T \int_\Theta f^\alpha \left(v_x \frac{\partial}{\partial x}\phi - (\underline{v} \times B) \cdot \nabla_{\underline{v}}\phi + \mathcal{L}\phi\right) dt\, d\theta \\ = \frac{1}{\alpha}\left(\int_0^T \int_{\Gamma^+} |v_x|\, f_+^\alpha(\phi_+ - \mathcal{B}^*\phi_-)\, dt\, d\Gamma\right) .\end{aligned} \tag{5.1}$$

Proof. Multiplying equation (2.1), using the Green Formula (4.4) and the boundary conditions (2.3) yields equation (5.1), which is nothing but the weak formulation of (2.1)-(2.3). □

We now prove that the limit function f^0 does not depend on x nor on ω. By using the collision operator $\mathcal{L}$, the proof is considerably simpler than in [17].

LEMMA 5.2. *The limit function f^0 is a function of $\underline{\xi}, |v|, t$ only, i.e. $f^0 = f^0(\underline{\xi}, |v|, t)$.*

Proof. We already know that f^0 does not depend on ω. In particular we have $\mathcal{L}(f^0) = 0$. Then, multiplying (5.1) by α, using a test function ϕ with compact support in Θ, and letting $\alpha \to 0$, we get:

$$\int_0^T \int_\Theta f^0 \left(v_x \frac{\partial}{\partial x}\phi - (\underline{v} \times B) \cdot \nabla_{\underline{v}}\phi + \mathcal{L}\phi \right) dt\, d\theta = 0\,. \tag{5.2}$$

This is equivalent to saying that f^0 is a distributional solution of the equation $\mathcal{A}^0 f^0 = \mathcal{L} f^0$, where $\mathcal{A}^0$ is given by:

$$\mathcal{A}^0 f = v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f. \tag{5.3}$$

But $\mathcal{L} f^0 = 0$ and f^0 does not depend on ω, so that $(\underline{v} \times B) \cdot \nabla_{\underline{v}} f^0 = 0$. Thus, f^0 satisfies

$$v_x \frac{\partial f^0}{\partial x} = 0.$$

Hence, f^0 is independent on x too, and is given by $f^0(X, v, t) = f^0(\underline{\xi}, |v|, t)$. ☐

From now on we shall denote $F(\underline{\xi}, \varepsilon, t) = f^0(\underline{\xi}, |v|, t)$, where $\varepsilon = |v|^2/2$ is the kinetic energy.

5.2. Auxiliary equation. This is the part which differs the most substantially from [17] as we shall develop below. Let us define the following operator, for $f \in L^2(\Theta)$:

$$\mathcal{T} f = -v_x \frac{\partial f}{\partial x} + (\underline{v} \times B) \cdot \nabla_{\underline{v}} f = -|v|\omega_x \frac{\partial f}{\partial x} - B \frac{\partial f}{\partial \omega}(e_x \times \omega) \tag{5.4}$$

where $\frac{\partial f}{\partial \omega}(e_x \times \omega)$ is the differential of f with respect to $\omega \in \mathbb{S}^2$ acting on the tangent vector to $\mathbb{S}^2$, $e_x \times \omega$. Note that we formally have $\mathcal{T} = \mathcal{A}^{0*}$. Nevertheless, we shall avoid this notation, since the determination of the domain $\mathcal{D}(\mathcal{A}^{0*})$ is not clear.

In this section we are concerned with solving the following equation:

$$\mathcal{T} f - \mathcal{L} f = g\,, \quad f_+ = \mathcal{B}^* f_-\,, \tag{5.5}$$

where g is a given function, which is intended to be equal to ω_y and ω_z. In [17] where $\mathcal{L} = 0$, equation (5.5) reduced to a first order differential equation. Therefore, it was possible to integrate it along the characteristics and to reduce it to a Fredholm fixed point problem for the boundary values of f. This procedure was borrowed from [2]. In the present case, this method is no longer operative, because of the presence of the collision operator $\mathcal{L}$. The most direct extension of the method of [17] would make use of stochastic trajectories. However, we will rather solve the problem in a deterministic framework.

It is easy to check that $\mathcal{T}$ and $\mathcal{L}$ only operate on the variables $(x, \omega) \in [0, 1] \times \mathbb{S}^2$, leaving $\underline{\xi} \in \mathbb{R}^2$ and $|v| \geq 0$ as parameters. Therefore, we only consider the dependence of f on $(x, \omega) \in \mathcal{S} = [0, 1] \times \mathbb{S}^2$, and assume $|v|$ to be a real positive number. We introduce the following domain:

$$\mathcal{D}(\mathcal{T}) = \{ f \in L^2(\mathcal{S})\,;\, \mathcal{T} f \in L^2(\mathcal{S}),\, f_- \in L^2(\mathcal{S}^-),\, f_+ = \mathcal{B}^* f_- \}\,. \tag{5.6}$$

An easy computation shows that if (5.5) holds, then we have $\int_0^1 \int_{\mathbb{S}^2} g(x,\omega) d\omega\, dx = 0$. The following proposition asserts that this condition is sufficient to ensure the existence of f:

PROPOSITION 5.1. *For all g in $L^2(\mathcal{S})$ such that*

$$\int_0^1 \int_{\mathbb{S}^2} g(x,\omega) d\omega\, dx = 0\,, \tag{5.7}$$

there exists a unique function f in $\mathcal{D}(\mathcal{T})$, solving (5.5) and such that

$$\int_0^1 \int_{\mathbb{S}^2} f(x,\omega) d\omega\, dx = 0\,. \tag{5.8}$$

Furthermore, all solutions in this space are equal to f, up to addition of an arbitrary function of $\underline{\xi}$ and $|v|$.

Proof. The proof relies on a procedure first developed in [28]. We introduce the operator

$$T = -|v|\omega_x \frac{\partial}{\partial x} - B\frac{\partial}{\partial \omega}(e_x \times \omega) - \mathcal{L} = \mathcal{T} - \mathcal{L}\,,$$

with domain $\mathcal{D}(T) = \mathcal{D}(\mathcal{T})$. As a first step, we recall that $\mathcal{D}(T)$ is closed for the graph norm $|f|_{L^2} + |T(f)|_{L^2}$. Indeed, we have the following trace estimate, which follows directly from Lemmas 4.4 and 4.5:

$$|v| \left(|f_-|^2_{L^2(\mathcal{S}^-)} + |f_+|^2_{L^2(\mathcal{S}^+)} \right) \le C|v|\, |f|^2_{L^2(\mathcal{S})} + C\, |\mathcal{T} f|^2_{L^2(\mathcal{S})}\,. \tag{5.9}$$

From this estimate, using similar methods as for the proof of Proposition 4.2, we deduce (see [28] for further details):

LEMMA 5.3. *The operator $\mathcal{T} = -|v|\omega_x \frac{\partial}{\partial x} - B\frac{\partial}{\partial \omega}(e_x \times \omega)$ is maximal accretive on the domain $\mathcal{D}(\mathcal{T})$.*

We can now get to the very heart of the proof of Proposition 5.1, with the following lemma:

LEMMA 5.4. *The range $R(T)$ of the operator T satisfies:*

$$\overline{R(T)} = \left\{ g \in L^2(\mathcal{S}) \,:\, \int_0^1 \int_{\mathbb{S}^2} g(x,\omega) d\omega dx = 0 \right\}\,.$$

Proof. By the fact that $\overline{R(T)} = N(T^*)^\perp$, it is equivalent to show that the kernel $N(T^*)$ is reduced to the constant functions on $L^2(\mathcal{S})$. First, notice that since $\mathcal{L}$ is a self adjoint bounded operator, one has

$$T^* = \mathcal{T}^* - \mathcal{L} = |v|\omega_x \frac{\partial}{\partial x} + B\frac{\partial}{\partial \omega}(e_x \times \omega) - \mathcal{L}\,,$$

with $\mathcal{D}(T^*) = \mathcal{D}(\mathcal{T}^*)$, since $\mathcal{L}$ is bounded. The operator $\mathcal{T}$ being maximal accretive (Lemma 5.3), its formal adjoint $\mathcal{T}^*$ is also maximal accretive on $\mathcal{D}(T^*)$ (see [9]). Note that we do not need to characterize $\mathcal{D}(T^*)$. Lemma 3.4 then yields for all f in $\mathcal{D}(T^*)$:

$$(T^* f, f)_{L^2(\mathcal{S})} \ge (\mathcal{T}^* f, f)_{L^2(\mathcal{S})} - (\mathcal{L} f, f)_{L^2(\mathcal{S})} \ge 4\pi c_1 |f - [f]|^2_{L^2(\mathcal{S})}$$

Let now f belongs to the Null-Space $N(T^*)$. Then it also belongs to $N(\mathcal{L})$, and thus only depends on x (since on $\mathbb{S}^2$ the kernel of $\mathcal{L}$ is reduced to the constants, by Lemma 3.3). Now, by the same argument as in the proof of Lemma 5.2, f satisfies,

$$\left(|v|\omega_x \frac{\partial}{\partial x} + B\frac{\partial}{\partial \omega}(e_x \times \omega)\right) f = +|v|\omega_x \frac{\partial}{\partial x} f = 0\,,$$

which implies that f is also independent of x and is therefore a constant on $\mathbb{S}^2 \times [0,1]$. Conversely, it is readily seen that the constant functions lie in $N(T^*)$, which concludes the proof of the lemma. □

Note that in the same way, we have $N(T) = \mathbb{R}$, and the uniqueness condition (5.8) amounts to choose $f \in N(T)^\perp$. We now prove

LEMMA 5.5. *$R(T)$ is closed in $L^2(\mathcal{S})$.*

Proof. Of course, by proving this lemma, we shall also complete the proof of Proposition 5.1. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $R(T)$ such that:

$$g_n \overset{n\to\infty}{\longrightarrow} g \quad \text{ in } L^2(\mathcal{S}) \text{ strong}\,,$$

and let $(f_n)_{n\in\mathbb{N}}$ be the associated sequence in $\mathcal{D}(T) \cap N(T)^\perp$ such that $Tf_n = g_n$. If we prove that $|f_n|_{L^2(\mathcal{S})}$ is bounded, there exists a function $f \in L^2(\mathcal{S})$ such that $f_n \rightharpoonup f$ in L^2 weakly and $Tf_n \longrightarrow g$ in L^2 strongly. Thanks to estimate (5.9), we then check that $f \in \mathcal{D}(T)$ and thus $g \in R(T)$, which is the result to be proved.

To show that $|f_n|_{L^2(\mathcal{S})}$ is bounded, we proceed by contradiction. Suppose that $|f_n|_{L^2(\mathcal{S})}$ is not bounded. First notice that, up to a subsequence, we have $|f_n|_{L^2(\mathcal{S})} \to \infty$. Setting $F_n = f_n/|f_n|_{L^2(\mathcal{S})}$, we have:

$$\begin{cases} TF_n \longrightarrow 0 \quad \text{ in } L^2(\mathcal{S}) \text{ strongly,} \\ |F_n|_{L^2(\mathcal{S})} = 1\,, \\ \int_{\mathbb{S}^2}\int_0^1 F_n(x,\omega)\,dx\,d\omega = 0\,, \end{cases} \tag{5.10}$$

the last condition following from the choice $f_n \in N(T)^\perp$. This implies $F_n \rightharpoonup F$ weakly in $L^2(\mathcal{S})$. Furthermore, since we have:

$$(TF_n, F_n)_{L^2(\mathcal{S})} \geq 4\pi c_1 |F_n - [F_n]|^2_{L^2(\mathcal{S})}\,,$$

we deduce from (5.10) that $F_n - [F_n] \to 0$ in $L^2(\mathcal{S})$ strongly.

Writing $F_n = [F_n] + (F_n - [F_n])$, we readily obtain:

$$|F_n|^2_{L^2(\mathcal{S})} = 1 = |F_n - [F_n]|^2_{L^2(\mathcal{S})} + |[F_n]|^2_{L^2(\mathcal{S})}\,,$$

and

$$|[F_n]|_{L^2(\mathcal{S})} \longrightarrow 1\,. \tag{5.11}$$

We will have a contradiction if we prove

$$[F_n] \longrightarrow 0 \qquad \text{in} \quad L^2(\mathcal{S}) \text{ strongly}\,. \tag{5.12}$$

To this purpose, we write, following P.L. Lions and G. Toscani [26] (see also T. Goudon and F. Poupaud [24]):

$$\begin{aligned} -\mathcal{T}[F_n] = \omega_x|v|\partial_x[F_n] &= \mathcal{T}(F_n - [F_n]) - \mathcal{T}F_n \\ &= \mathcal{T}(F_n - [F_n]) - T(F_n) - \mathcal{L}(F_n)\,. \end{aligned}$$

Multiplying the equality by ω_x, and integrating with respect to $\omega \in \mathbb{S}^2$ (we recall that $[F_n]$ is constant on $\mathbb{S}^2$), we get:

$$(5.13)\quad |v| \left(\int_{\mathbb{S}^2} |\omega_x|^2 d\omega \right) \partial_x [F_n] = \int_{\mathbb{S}^2} \omega_x \mathcal{T}(F_n - [F_n]) d\omega - \int_{\mathbb{S}^2} \omega_x (T(F_n) + \mathcal{L}(F_n)) d\omega \,.$$

We now notice that, for all f with sufficient regularity, we have:

$$\int_{\mathbb{S}^2} \omega_x \frac{\partial f}{\partial \omega}(e_x \times \omega) d\omega = 0 \,, \qquad \int_{\mathbb{S}^2} \omega_x \mathcal{T} f d\omega = -\frac{\partial}{\partial x} \int_{\mathbb{S}^2} |v| |\omega_x|^2 f d\omega \,.$$

Therefore, since $F_n - [F_n]$ converges towards 0 in $L^2(\mathcal{S})$, the first term in the right-hand side of (5.13) converges to 0 in $H^{-1}([0,1])$, and the second one obviously strongly converges to 0 in $L^2([0,1])$. Moreover we obviously have:

$$(5.14)\qquad |v| \int_{\mathbb{S}^2} |\omega_x|^2 d\omega > 0 \,.$$

Hence $|\partial_x [F_n]|_{H^{-1}([0,1])}$ tends to 0. Defining $\varphi(x) = \int_0^x [F_n](z) dz$, the last condition (5.10) is equivalent to say that $\varphi \in H_0^1([0,1])$. We deduce:

$$(5.15)\qquad -\langle \partial_x [F_n], \varphi \rangle_{H^{-1}, H_0^1} = |[F_n]|^2_{L^2([0,1])} \le |\partial_x [F_n]|_{H^{-1}} |\varphi|_{H_0^1} \,,$$

and since $|\varphi|_{H_0^1} \le C |\partial_x \varphi|_{L^2([0,1])} = C |[F_n]|_{L^2([0,1])}$, (5.15) yields:

$$(5.16)\qquad |[F_n]|_{L^2([0,1])} \longrightarrow 0 \,,$$

which contradicts (5.11) and ends the proof of Lemma 5.5 as well as that of Proposition 5.1. □

We notice that the above proof is not constructive. We now give an alternate proof of Proposition 5.1 in the case of isotropic collision operators $\mathcal{L}$ and rotationally invariant boundary collision operators $\mathcal{K}$. This proof is more constructive and is an adaptation of the proof of [17]. However, it cannot be used in the general case. We assume (see [27] for more details):

HYPOTHESIS 5.1. *(i) K is rotationally invariant under rotations about the x-axis, i.e. there exists a function $\tilde{K}(\omega_x, \omega_x', \theta)$ such that:*

$$(5.17)\qquad K(\omega' \to \omega) = \tilde{K}(\omega_x, \omega_x', \frac{\underline{\omega}}{|\underline{\omega}|} \cdot \frac{\underline{\omega}'}{|\underline{\omega}'|}) \,.$$

(ii) Φ is constant.

From Hypothesis 5.1 follows that:

$$(5.18)\qquad \mathcal{L} f = \lambda([f] - f) \,, \quad [f] = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(\omega)\, d\omega \,, \quad \lambda = 4\pi\Phi \,.$$

We note that $\mathbb{S}^2$ can be parametrized by $\omega = (\sigma, \underline{\omega})$, where $\sigma = \omega_x / |\omega_x| \in \{-1, +1\}$. The fact that σ is equal to ± 1 recalls that we need two maps to parametrize the sphere in this way. Next, we note $R^+_{(x,\sigma)}(\underline{\omega})$ the rotation of $\underline{\omega}$ about the x-axis of angle bx, where:

$$b = \frac{B}{|v| \omega_x} \,, \quad \omega_x = \sigma \sqrt{1 - \omega_y^2 - \omega_z^2}.$$

In other words, $\omega^\dagger = R^+_{(x,\sigma)}(\underline{\omega}) = (\omega^\dagger_y, \omega^\dagger_z)$ is given by:

$$\begin{cases} \omega^\dagger_y = \omega_y \cos bx - \omega_z \sin bx \\ \omega^\dagger_z = \omega_y \sin bx + \omega_z \cos bx \end{cases} \tag{5.19}$$

We note that $\omega^\dagger$ also depends on $|v|$ and $\underline{\xi}$, but we shall not stress this dependence otherwise needed. Similarly, $R^-_{(x,\sigma)}(\underline{\omega})$ is the rotation of angle $-bx$. We have: $\omega^\dagger = R^+_{(x,\sigma)}(\underline{\omega})$ if and only if $\underline{\omega} = R^-_{(x,\sigma)}(\omega^\dagger)$, and

$$R^-_{(x',\sigma)} R^+_{(x,\sigma)} = R^+_{(x-x',\sigma)} = R^-_{(x'-x,\sigma)}.$$

We also define the operator $\mathcal{I}$ such that $\mathcal{I} f(x,\sigma,\underline{\omega}) = f(x,\sigma,-\underline{\omega})$. With Hypothesis 5.1, we clearly have:

Lemma 5.6. *(i) The boundary operators $\mathcal{B}$ and $\mathcal{B}^*$ commute with $\mathcal{I}$:*

$$\mathcal{B}\mathcal{I} = \mathcal{I}\mathcal{B}\,, \quad \mathcal{B}^*\mathcal{I} = \mathcal{I}\mathcal{B}^*.$$

(ii) The transport operator $\mathcal{T}$ commutes with $\mathcal{I}$.

$$\mathcal{T}\mathcal{I} = \mathcal{I}\mathcal{T}$$

Now, we have the

Lemma 5.7. *Let g be a function such that $\mathcal{I}g = -g$, and suppose that the problem*

$$\mathcal{T} f + \lambda f = g\,, \quad f_+ = \mathcal{B}^* f_-\,, \quad \int_0^1 \int_{\mathbb{S}^2} f(x,\omega)\, dx\, d\omega = 0\,, \tag{5.20}$$

has a unique solution $f \in L^2(\mathcal{S})$. Then, f is the solution of problem (5.5) as given by Proposition 5.1.

Proof. By uniqueness, we have $\mathcal{I} f = -f$. Therefore, f is odd with respect to $\underline{\omega}$ and consequently $[f] = 0$. Then, it follows that f is also a solution of problem

$$\mathcal{T} f + \lambda f = \lambda[f] + g\,, \quad f_+ = \mathcal{B}^* f_-\,, \quad \int_0^1 \int_{\mathbb{S}^2} f(x,\omega)\, dx\, d\omega = 0\,,$$

i.e. is the solution of problem (5.5) as given by proposition 5.1. ◻

We now show

Proposition 5.2. *The problem (5.20) has a unique solution $f \in L^2(\mathcal{S})$.*

Proof. Applying the change of variables (5.19), equation (5.20) reads:

$$-|v|\omega_x \frac{\partial f^\dagger}{\partial x} + \lambda f^\dagger = g^\dagger \tag{5.21}$$

where $f^\dagger(x,\sigma,\underline{\omega}^\dagger) = f(x,\sigma,R^+_{(x,\sigma)}(\underline{\omega}^\dagger))$ and $g^\dagger(x,\sigma,\underline{\omega}^\dagger) = g(x,\sigma,R^+_{(x,\sigma)}(\underline{\omega}^\dagger))$. Integrating with respect to x we get:

$$f^\dagger(x,\sigma,\omega^\dagger) = \begin{cases} e^{-\gamma(1-x)} f^\dagger(1,\sigma,\omega^\dagger) + G^\dagger(x,\sigma,\omega^\dagger)\,, & \sigma = +1\,, \\ e^{\gamma x} f^\dagger(0,\sigma,\omega^\dagger) + G^\dagger(x,\sigma,\omega^\dagger)\,, & \sigma = -1\,, \end{cases} \tag{5.22}$$

where $\gamma = \lambda/|v|\omega_x$ and $G^\dagger$ is given by

$$(5.23)\qquad G^\dagger(x,\sigma,\omega^\dagger) = \begin{cases} \dfrac{1}{|v|\,|\omega_x|}\displaystyle\int_x^1 e^{-\gamma(x'-x)} g^\dagger(x',\sigma,\omega^\dagger)\,dx' \;, & \sigma = +1\,,\\ \dfrac{1}{|v|\,|\omega_x|}\displaystyle\int_0^x e^{\gamma(x-x')} g^\dagger(x',\sigma,\omega^\dagger)\,dx' \;, & \sigma = -1\,. \end{cases}$$

Back to the original variables we have:

$$(5.24)\qquad f(x,\sigma,\underline{\omega}) = \begin{cases} e^{-\gamma(1-x)} f_+(1,\sigma,R^+_{(1-x,\sigma)}(\underline{\omega})) + G(x,\sigma,\underline{\omega}) \;, & \sigma = +1\,,\\ e^{\gamma x} f_+(0,\sigma,R^-_{(x,\sigma)}(\underline{\omega})) + G(x,\sigma,\underline{\omega}) \;, & \sigma = -1\,, \end{cases}$$

where G is defined by

$$(5.25)\qquad G(x,\sigma,\underline{\omega}) = \begin{cases} \dfrac{1}{|v|\,|\omega_x|}\displaystyle\int_x^1 e^{-\gamma(x'-x)} g(x',\sigma,R^+_{(x'-x,\sigma)}(\underline{\omega}))\,dx' \;, & \sigma = +1\,,\\ \dfrac{1}{|v|\,|\omega_x|}\displaystyle\int_0^x e^{\gamma(x-x')} g(x',\sigma,R^-_{(x-x',\sigma)}(\underline{\omega}))\,dx' \;, & \sigma = -1\,. \end{cases}$$

Note that G also depends on $|v|$ and $\underline{\xi}$, and that $f_+(0)$ and $f_+(1)$ have to be determined by means of the boundary conditions. Evaluating (5.24) for $x = 0, 1$, we get:

$$(5.26)\qquad f_-(1,\sigma,\underline{\omega}) = e^{-|\gamma|} f_+(0,\sigma,R^-_{(1,\sigma)}(\underline{\omega})) + G_-(1,\sigma,\underline{\omega}) \;, \quad \sigma = -1\,,$$

$$(5.27)\qquad f_-(0,\sigma,\underline{\omega}) = e^{-|\gamma|} f_+(1,\sigma,R^+_{(1,\sigma)}(\underline{\omega})) + G_-(0,\sigma,\underline{\omega}) \;, \quad \sigma = +1\,,$$

where,

$$G_-(1,\sigma,\underline{\omega}) = \frac{1}{|v|\,|\omega_x|}\int_0^1 e^{-|\gamma|(1-x')} g(x',\sigma,R^-_{(1-x',\sigma)}(\underline{\omega}))\,dx' \;, \quad \sigma = -1\,,$$
$$G_-(0,\sigma,\underline{\omega}) = \frac{1}{|v|\,|\omega_x|}\int_0^1 e^{-|\gamma| x'} g(x',\sigma,R^+_{(x',\sigma)}(\underline{\omega}))\,dx' \;, \quad \sigma = +1\,.$$

Equs. (5.26) and (5.27) can be written compactly:

$$(5.28)\qquad f_- = M_+ f_+ + G_-$$

where M_+ is a bounded operator of norm strictly smaller than 1 from $L^2(\mathcal{S}^+)$ onto $L^2(\mathcal{S}^-)$. Now, thanks to the boundary conditions, we have, omitting the dependence on σ and $\underline{\omega}$:

$$f_+(1) = \mathcal{B}_1^* f_-(1) = \mathcal{B}_1^* M_+ f_+(0) + \mathcal{B}_1^* G_-(1)\,,$$
$$f_+(0) = \mathcal{B}_0^* f_-(0) = \mathcal{B}_0^* M_+ f_+(1) + \mathcal{B}_0^* G_-(0)\,,$$

where $\mathcal{B}_1^*$ is the boundary operator defined on the plane $x = 1$ and $\mathcal{B}_0^*$ is the one defined on $x = 0$. Iterating, we get

$$(\mathrm{I} - \mathcal{B}_1^* M_+ \mathcal{B}_0^* M_+) f_+(1) = \mathcal{B}_1^* M_+ \mathcal{B}_0^* G_-(0) + \mathcal{B}_1^* G_-(1)\,,$$
$$(\mathrm{I} - \mathcal{B}_0^* M_+ \mathcal{B}_1^* M_+) f_+(0) = \mathcal{B}_0^* M_+ \mathcal{B}_1^* G_-(1) + \mathcal{B}_0^* G_-(0)\,,$$

where I is the identity operator. Now, since M_+ is of norm smaller than 1, so are $\mathcal{B}_1^* M_+ \mathcal{B}_0^* M_+$ and $\mathcal{B}_0^* M_+ \mathcal{B}_1^* M_+$ (since $\mathcal{B}$ is of norm 1). Therefore, the operators $(\mathrm{I} - \mathcal{B}_1^* M_+ \mathcal{B}_0^* M_+)$ and $(\mathrm{I} - \mathcal{B}_0^* M_+ \mathcal{B}_1^* M_+)$ are invertible, and provide a unique expression of $f_+(1)$ and $f_+(0)$ in terms of g. Then, with (5.24), we find a uniquely defined expression of the solution of problem (5.20) which is easily proved to belong to $L^2([0,1]\times\mathbb{S}^2)$. This ends the proof of Proposition 5.2. $\square$

For $g\in L^2(\mathcal{S})$ satisfying $\int_0^1\int_{\mathbb{S}^2} g(x,\omega)d\omega dx = 0$, we denote by $T_v^{-1}(g)$ the unique function $f\in\mathcal{D}(T)$ given by Proposition 5.1. Then the previous proof can be adapted in order to provide the following estimate (see [28] for further details):

LEMMA 5.8. *For all $\delta>0$, there exists a constant C_δ such that:*

$$\forall |v|\ge\delta\,,\quad \forall g\in Im(T_v)\,,\quad |T_v^{-1}(g)|_{L^2(\mathcal{S})}\le C_\delta |g|_{L^2(\mathcal{S})}\,. \tag{5.29}$$

We point out that the restriction that $|v|$ should be far enough from zero is needed by the coercivity condition (5.14). In the remainder of the paper, we only need g to be equal to ω_y and ω_z. We obviously have (since these functions are odd with respect to $\omega\in\mathbb{S}^2$):

LEMMA 5.9. *The functions $g=\omega_y$ and $g=\omega_z$ satisfy the assumptions of Propositions 5.1 and 5.2.*

Thus there exist functions $D_y(x,\omega;\underline{\xi},\varepsilon)$, $D_z(x,\omega;\underline{\xi},\varepsilon)$, solutions of problem (5.5) with right-hand sides $g=\omega_y$ and $g=\omega_z$ respectively, unique up to additive functions of $\underline{\xi}$ and ε. In addition, we need the following regularity for D_y, D_z:

HYPOTHESIS 5.2. *(i) D_y, D_z are bounded functions with bounded derivatives with respect to $\underline{v}$, $\underline{\xi}$ on $\Theta\backslash\{v=0\}$.*
(ii) The functions $\omega_i D_j(x,\omega;\underline{\xi},\varepsilon)$ belongs to $L^1(\mathcal{S})$ and

$$\int_0^1\int_{\mathbb{S}^2}\omega_i D_j\,dx\,d\omega$$

is a C^1 function of $(\underline{\xi},\varepsilon)\in\mathbb{R}^2_{\underline{\xi}}\times\mathbb{R}^+_\varepsilon$.

REMARK 5.1. Hypothesis 5.2 can be viewed as a regularity assumption on the data: the magnetic field B, the boundary scattering kernel K and the accommodation coefficient β. We do not look for explicit condition on these data because the developments would be technical and of rather limited interest. $\square$

5.3. Obtention of the SHE model. The rest of the proof of Theorem 2.1 follows the same route as [17]. We just summarize the main steps. Given D_y (respectively D_z) the solutions of problem (5.5) with $g=\omega_y$ (respectively $g=\omega_z$), we define the diffusivity tensor as follows:

$$\mathbb{D}_{ij}=(2\varepsilon)^{3/2}\int_0^1\int_{\mathbb{S}^2}\omega_j\,D_i(x,\omega;\underline{\xi},\varepsilon)\,dx\,d\omega\ ,\quad i,j\in\{y,z\}.$$

We remark that $\mathbb{D}$ is a C^1 function of $(\underline{\xi},\varepsilon)\in\mathbb{R}^2\times\mathbb{R}^+$. We also note that the definition of $\mathbb{D}_{ij}$ does not depend on the arbitrary additive function of $\underline{\xi}$ and ε which enters in the definition of D_j. Let us introduce the current $\underline{J}^\alpha(\underline{\xi},\varepsilon,t)=(J^\alpha_y,J^\alpha_z)$ as follows:

$$\underline{J}^\alpha(\underline{\xi},\varepsilon,t)=\frac{|v|}{\alpha}\int_0^1\int_{\mathbb{S}^2}\underline{v}f^\alpha(x,\underline{\xi},|v|,\omega,t)\,dx\,d\omega \tag{5.30}$$

Now, denoting by Θ' the position-energy space $\Theta' = \mathbb{R}^2 \times \mathbb{R}^+$ and by $d\theta'$ its volume element $d\theta' = d\underline{\xi}\, d\varepsilon$ (note that $dv = |v|^2 d|v|\, d\omega = \sqrt{2\varepsilon}\, d\varepsilon\, d\omega$), we have:

LEMMA 5.10. *As α goes to 0, the current $\underline{J}^\alpha(\underline{\xi}, \varepsilon, t)$ converges in the distributional sense towards $\underline{J}^0(\underline{\xi}, \varepsilon, t)$. More precisely, for every $\psi \in C^1(\Theta' \times [0, T], \mathbb{R}^2)$ with compact support in $\mathbb{R}^2 \times \mathbb{R}^+_* \times [0, T[$, we have:*

$$\begin{aligned}\lim_{\alpha\to 0} \int_0^T \int_{\Theta'} \underline{J}^\alpha \cdot \underline{\psi}\, d\theta'\, dt &= \int_0^T \int_{\Theta'} \underline{J}^0 \cdot \underline{\psi}\, dt\, d\theta' \\ &= \int_0^T \int_{\Theta'} F \left(\nabla_{\underline{\xi}} - \underline{E} \frac{\partial}{\partial \varepsilon} \right) \cdot (\mathbb{D}^T \underline{\psi})\, dt\, d\theta',\end{aligned} \tag{5.31}$$

and thus

$$\underline{J}^0 = -\mathbb{D} \cdot \left(\nabla_{\underline{\xi}} - \underline{E} \frac{\partial}{\partial \varepsilon} \right) F^0. \tag{5.32}$$

We note that the right-hand side of equation (5.31) is the weak form of that of equation (2.6).

Proof. We recall that the proof consists in using the weak formulation (5.1) with $\phi = \sqrt{2\varepsilon}\underline{\psi}(\underline{\xi}, \varepsilon, t) \cdot \underline{D}(x, \omega; \underline{\xi}, \varepsilon)$ as a test function (Hypothesis 5.2 provides all the necessary assumptions to allow this computation). The terms of the order α^{-1} in the weak formulation (5.1) exactly give the integral at the left-hand side of (5.31). By passing to the limit as $\alpha \to 0$, and after some computations, the order 1 terms of (5.1) lead to the expression at the right-hand side of (5.31). The computations are detailed in [17]. $\square$

To end the proof of Theorem 2.1, there remains to prove that equation (2.5) holds in a weak sense. The result is just stated below. The proof relies on the use of the weak formulation (5.1) with $\phi = \underline{\psi}(\underline{\xi}, \varepsilon, t)$ as a test function and is detailed in [17].

LEMMA 5.11. *For any test function ψ belonging to $C^2(\Theta' \times [0, T])$, with compact support in $\mathbb{R}^2 \times \mathbb{R}^+ \times [0, T[$, we have:*

$$\int_0^T \int_{\Theta'} \left(4\pi\sqrt{2\varepsilon} F^0 \frac{\partial \psi}{\partial t} + \underline{J}^0 \cdot \left(\nabla_{\underline{\xi}} - \underline{E} \frac{\partial}{\partial \varepsilon} \right) \psi \right) dt\, d\theta' + \int_{\Theta'} 4\pi\sqrt{2\varepsilon} F_I \psi|_{t=0}\, d\theta' = 0\,. \tag{5.33}$$

5.4. Properties of the diffusivity. In this section, we underline that the diffusion tensor $\mathbb{D}$ is positive definite. Moreover, under a certain invariant property of the collision operator $\mathcal{L}$ (always satisfied in practice), it satisfies the Onsager reciprocity relation saying that the transpose of $\mathbb{D}(B)$ for a given magnetic field B equals the diffusion tensor associated with $-B$.

PROPOSITION 5.3. *The diffusion tensor $\mathbb{D}$ is positive definite: there exists $C > 0$ such that:*

$$(\mathbb{D}Y, Y) = \sum_{i,j=1}^{2} \mathbb{D}_{ij} Y_i Y_j \geq C|Y|^2 = C \sum_{i=1}^{2} Y_i^2. \tag{5.34}$$

Proof. The proof is a minor modification of the proof of [17] and is left to the reader. $\square$

Proposition 5.4. *Suppose that the collision operator $\mathcal{L}$ satisfies the following invariance property :*

$$\mathcal{L}(f(-\omega)) = (\mathcal{L}f)(-\omega)\,,$$

then the diffusion tensor $\mathbb{D}$ satisfies the Onsager relation:

$$\mathbb{D}(B)^T = \mathbb{D}(-B).$$

Proof. Again, the proof follows that of [17]. □

6. Explicit expression of the diffusivity for isotropic collisions operators. In this section, we are concerned with a subcase of Hypothesis 5.1, the case where both the collision operator with atoms or ions $\mathcal{L}$ and the boundary collision operator $\mathcal{K}$ are isotropic. $\mathcal{L}$ being isotropic means that it is given by (5.18) i.e.

$$\mathcal{L}f = \lambda([f]-f)\,, \quad [f] = \frac{1}{4\pi}\int_{\mathbb{S}^2} f(\omega)\,d\omega\,. \tag{6.1}$$

The boundary collision operator $\mathcal{K}$ is isotropic if its kernel $K(\omega' \to \omega)$ does not depend on the outgoing angle ω. Given the constraints of the flux conservation and reciprocity (see Hypothesis 3.1), we find that $K = \pi^{-1}$ and the boundary operator $\mathcal{B}$ is written, using the coordinate system $(\sigma, \underline{\omega})$ on the sphere $\mathbb{S}^2$ (see Section 5.2):

$$\begin{aligned} f(x=0,\sigma=+1,\underline{\omega}) = \beta_0 f(x=0,\sigma=-1,\underline{\omega}) + \\ +(1-\beta_0)\pi^{-1}\int_{|\underline{\omega}|\le 1} f(x=0,\sigma=-1,\underline{\omega})\,d\underline{\omega}\,, \end{aligned} \tag{6.2}$$

$$\begin{aligned} f(x=1,\sigma=-1,\underline{\omega}) = \beta_1 f(x=1,\sigma=1,\underline{\omega}) + \\ +(1-\beta_1)\pi^{-1}\int_{|\underline{\omega}|\le 1} f(x=1,\sigma=1,\underline{\omega})\,d\underline{\omega}\,, \end{aligned} \tag{6.3}$$

where β_0 stands for $\beta(x=0)$ and similarly for β_1. We note that in the parametrization $(\sigma,\underline{\omega})$, the surface element is given by $d\omega = |\omega_x|^{-1}\,d\underline{\omega}$ so that the integrals at the right-hand sides of (6.2) and (6.3) are actually integrals $\int_{\mathcal{S}_+(x)} f_+(x,\omega)|\omega_x|d\omega$.

We concentrate on the explicit computation of the solutions D_j, $j \in \{y,z\}$ of the auxiliary equation (5.5) with right-hand side $g = \omega_j$. By using the same invariance arguments under the action of the operator $\mathcal{I}$ as in Section 5.2, it is readily seen that the unique solution of (5.5) satisfying the additional condition (5.8) is actually a solution $D_j(x,\sigma,\omega_y,\omega_z)$ of the problem

$$-|v|\omega_x \frac{\partial D_j}{\partial x} + B\left(\omega_z\frac{\partial D_j}{\partial \omega_y} - \omega_y\frac{\partial D_j}{\partial \omega_z}\right) + \lambda D_j = \omega_j\,, \tag{6.4}$$

$$D_j(x=0,\sigma=-1,\underline{\omega}) = \beta_0 D_j(x=0,\sigma=1,\underline{\omega})\,, \tag{6.5}$$

$$D_j(x=1,\sigma=1,\underline{\omega}) = \beta_1 D_j(x=1,\sigma=-1,\underline{\omega})\,. \tag{6.6}$$

We recall that we denote by $\sigma = \omega_x/|\omega_x| \in \{-1,1\}$ and consider the parametrization $(\sigma,\omega_y,\omega_z)$ of the sphere $\mathbb{S}^2$, for which the surface element is $d\omega = |\omega_x|^{-1}d\omega_y d\omega_z$. We shall also be concerned with finding an explicit formula for the diffusivity tensor $\mathbb{D}$ (2.8).

We use the notations:

$$b = \frac{B}{|v|\,|\omega_x|}\,, \quad \gamma = \frac{\lambda}{|v|\,|\omega_x|}\,,$$

where we note a slight change compared with the definitions used in Section 5.2. We shall rather consider $\tilde{D}_j(x') = 2D_j((x'+1)/2)$, which is defined for $x' \in [-1,1]$ and solves the problem (dropping the prime):

$$-|v|\omega_x \frac{\partial \tilde{D}_j}{\partial x} + \frac{B}{2}\left(\omega_z \frac{\partial \tilde{D}_j}{\partial \omega_y} - \omega_y \frac{\partial \tilde{D}_j}{\partial \omega_z}\right) + \frac{\lambda}{2}\tilde{D}_j = \omega_j\,, \tag{6.7}$$

$$\tilde{D}_j(x=-1, \sigma=-1, \underline{\omega}) = \beta_{-1}\tilde{D}_j(x=-1, \sigma=1, \underline{\omega})\,, \tag{6.8}$$

$$\tilde{D}_j(x=1, \sigma=1, \underline{\omega}) = \beta_1\tilde{D}_j(x=1, \sigma=-1, \underline{\omega})\,, \tag{6.9}$$

(where we use β_{-1} instead of β_0), and the diffusivity is given by

$$\mathbb{D} = (2\varepsilon)^{3/2}\frac{1}{4}\int_{-1}^{1}\int_{\mathbb{S}^2} \underline{\tilde{D}}(x,\omega)\underline{\omega}\,dx\,d\omega\,. \tag{6.10}$$

We shall drop the tildes in the remainder of this section. We introduce the change of variables

$$\omega_y^\dagger = \omega_y\cos(\sigma b x/2) + \omega_z\sin(\sigma b x/2)\,, \quad \omega_z^\dagger = -\omega_y\sin(\sigma b x/2) + \omega_z\cos(\sigma b x/2)\,, \tag{6.11}$$

and $D_j^\dagger(x,\sigma,\omega_y^\dagger,\omega_z^\dagger) = D_j(x,\sigma,\omega_y,\omega_z)$, . We introduce the following complex numbers (with i such that $i^2=-1$):

$$\Lambda = \lambda - iB\,, \quad \Gamma = \frac{\Lambda}{|v|\,|\omega_x|} = \gamma - ib\,, \quad \Omega^\dagger = \omega_y^\dagger + i\omega_z^\dagger\,, \quad D^\dagger = D_y^\dagger + iD_z^\dagger\,.$$

Then, $D^\dagger(x,\sigma,\Omega^\dagger)$ is a solution of

$$\frac{\partial D^\dagger}{\partial x} - \frac{\sigma\gamma}{2}D^\dagger = -\frac{1}{\sigma\,|v|\,|\omega_x|}\Omega^\dagger e^{i\sigma b x/2}\,, \tag{6.12}$$

together with the boundary conditions (6.8) and (6.9). For $D^\dagger$, we prove:

LEMMA 6.1. *We have*

$$D^\dagger(x,\sigma,\Omega^\dagger) = \frac{\Omega^\dagger}{\Lambda/2}\,e^{\sigma\gamma x/2}\left[2A\,\sinh(\Gamma/2)\,(e^{-\gamma}\beta_\sigma + 1) + e^{-\sigma\Gamma x/2} - e^{\Gamma/2}\right]\,, \tag{6.13}$$

with $A = (1-\beta_1\beta_{-1}e^{-2\gamma})^{-1}$.

Proof. By integrating (6.12), we find

$$D^\dagger(x,\sigma,\Omega^\dagger) = e^{\sigma\gamma(x+1)/2}D^\dagger(-1,\sigma,\Omega^\dagger) + \psi_{-1}(x,\sigma,\Omega^\dagger)\,, \tag{6.14}$$

$$= e^{\sigma\gamma(x-1)/2}D^\dagger(1,\sigma,\Omega^\dagger) + \psi_1(x,\sigma,\Omega^\dagger)\,, \tag{6.15}$$

with

$$\psi_{\mp 1}(x,\sigma,\Omega^\dagger) = \frac{\Omega^\dagger}{\Lambda/2}\,e^{\sigma\gamma x/2}\,(e^{-\sigma\Gamma x/2} - e^{\pm\sigma\Gamma/2})\,. \tag{6.16}$$

Then, iterating formulae (6.14), (6.15) with the boundary conditions (6.8), (6.9), we obtain:

$$D^\dagger(x=-\sigma,\sigma,\Omega^\dagger) = A\left[e^{-\gamma}\beta_\sigma\psi_{-\sigma}(x=\sigma,-\sigma,\Omega^\dagger)+\psi_\sigma(x=-\sigma,\sigma,\Omega^\dagger)\right]\,. \tag{6.17}$$

Inserting the boundary values (6.17) into (6.15) and using (6.16) for $\psi_{\pm 1}$, we find (6.13). □

Then, we prove

PROPOSITION 6.1. *The diffusion matrix $\mathbb{D}$ is given by:*

$$\mathbb{D} = (2\varepsilon)^{3/2}\begin{pmatrix} d_r & -d_i \\ d_i & d_r \end{pmatrix}, \tag{6.18}$$

where the complex number $d = d_r + i d_i$ has the expression:

$$d = \frac{1}{\Lambda}\int_0^1 \left[1 - \frac{\sinh(\Gamma/2)e^{\Gamma/2}}{\Gamma/2} + A\frac{\sinh^2(\Gamma/2)}{\Gamma/2}\left(e^{-\gamma}(\beta_{-1}+\beta_1)+2\right)\right](1-w^2)\,2\pi\,dw\,, \tag{6.19}$$

with

$$\Gamma = \Gamma(w) = \frac{\Lambda}{\sqrt{2\varepsilon}\,w}\,,\quad \gamma=\gamma(w) = \frac{\lambda}{\sqrt{2\varepsilon}\,w}\,,\quad A = A(w) = \frac{1}{1-\beta_1\beta_{-1}e^{-2\gamma(w)}}\,,$$

and $\Lambda = \lambda - iB$.

We remark that the formula is considerably more complex than in the case of no collisions with the atoms or ions ($\lambda = 0$, see [13]). The collisions with the boundary and inside the gap combine in a fairly complex way. The coefficients d_r and d_i can be compared with the so-called Pedersen and Hall conductivities of collisional magnetized plasmas (see e.g. [31]). However, there is no such simple formula for the diffusivities in the present case like that of the Pedersen and Hall conductivities.

We note that the value of d can be computed by numerical integrations: it can be expressed as a combination of elementary integrals which are functions of the three parameters $\lambda/\sqrt{2\varepsilon}$, $B/\sqrt{2\varepsilon}$ and $\beta_1\beta_{-1}$. This expression is used in [16] for numerical computations of the electron fluid in a real plasma propellor for satellites.

Proof. By an argument involving rotational invariance, it is easy to show that the diffusion matrix $\mathbb{D}$ has the form (6.18) and that

$$\begin{aligned} d_r &= \frac{1}{4}\int_{-1}^{1}\sum_{\sigma=\pm1}\int_{|\Omega|<1} D_y\omega_y\,|\omega_x|^{-1}\,d\Omega\,dx \\ &= \frac{1}{4}\int_{-1}^{1}\sum_{\sigma=\pm1}\int_{|\Omega|<1} D_z\omega_z\,|\omega_x|^{-1}\,d\Omega\,dx \\ &= \frac{1}{8}\int_{-1}^{1}\sum_{\sigma=\pm1}\int_{|\Omega|<1} (D_y\omega_y + D_z\omega_z)\,|\omega_x|^{-1}\,d\Omega\,dx\,, \end{aligned}$$

and similarly

$$d_i = \frac{1}{8}\int_{-1}^{1}\sum_{\sigma=\pm1}\int_{|\Omega|<1} (D_z\omega_y - D_y\omega_z)\,|\omega_x|^{-1}\,d\Omega\,dx\,.$$

Therefore

$$d = d_r + i d_i = \frac{1}{8} \int_{-1}^{1} \sum_{\sigma=\pm 1} \int_{|\Omega|<1} D(\Omega)\, \Omega^* \, |\omega_x|^{-1} \, d\Omega \, dx \,,$$

where Ω^* denotes the complex conjugate of Ω, and, by the change of variables (6.11):

$$d = d_r + i d_i = \frac{1}{8} \int_{-1}^{1} \sum_{\sigma=\pm 1} \int_{|\Omega^\dagger|<1} D^\dagger(\Omega^\dagger) \left(\Omega^\dagger e^{i\sigma b x/2} \right)^* \, |\omega_x|^{-1} \, d\Omega^\dagger \, dx \,.$$

Now, inserting the expression (6.13) of $D^\dagger$, and performing the integration with respect to x, we obtain:

$$d = \frac{1}{\Lambda} \int_{|\Omega^\dagger|<1} \left[1 - \frac{\sinh(\Gamma/2) e^{\Gamma/2}}{\Gamma/2} + A \frac{\sinh^2(\Gamma/2)}{\Gamma/2} \left(e^{-\gamma} (\beta_{-1} + \beta_1) + 2 \right) \right] \times$$
$$\times |\Omega^\dagger|^2 \, |\omega_x|^{-1} \, d\Omega^\dagger \,.$$

Then, the expression (6.19) is obtained by using the change to polar coordinates in $\Omega^\dagger$: $|\Omega^\dagger|^2 = 1 - w^2$, $d\Omega^\dagger = 2\pi w \, dw$, $|\omega_x| = w$. $\square$

7. Conclusion. In this paper, we have studied the macroscopic behaviour of an electron fluid confined between two plane parallel solid plates and subject to crossed electric and magnetic fields (the magnetic field being normal to the plates). The electrons undergo collisions both with the solid plates and with the atoms or ions of a gas filling the gap between the plates. We showed that, under physically realistic assumptions on the nature of the collisions, the macroscopic behaviour of the gas is described by a 'Spherical Harmonics Expansion' (SHE) type model. A rigorous derivation of the model from the microscopic kinetic description has been given. An explicit computation of the diffusivity in the case of isotropic collisions, which is a very useful practical case, is also derived.

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