Uniqueness and Semigroup for the Vlasov Equation with Elastic-Diffusive Reflexion Boundary Conditions

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Abstract—In this paper, we investigate some uniqueness results for the Vlasov equation with elastic-diffusive boundary conditions. As an application, we build the associated semigroup in an $L^1$ setting. © 2004 Elsevier Ltd. All rights reserved.

1. INTRODUCTION

Let $\Omega$ be an open subset of $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, and define $\mathcal{O} = \Omega \times \mathbb{R}^N$, and $\Sigma = \partial \Omega \times \mathbb{R}^N$. We introduce the outgoing and incoming trace subset $\Sigma_\pm = \{(x,v) \in \Sigma; \pm n(x) \cdot v > 0\}$, where $n(x)$ denotes the unit outward normal vector on the boundary $\partial \Omega$, and we denote by $\gamma_\pm f$ the restriction of the trace of $f$ on $\Sigma_\pm$. The equation we are concerned with in this paper is the following Vlasov equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \quad (x,v) \in \mathcal{O}, \quad t \in [0,T],$$

$$f(x,v,0) = \varphi(x,v), \quad (x,v) \in \mathcal{O},$$

(1)

together with boundary conditions of the form

$$\gamma_- f(t) = K \gamma_+ f(t), \quad \forall (x,v) \in \Sigma_-, \quad \forall t \in (0,T).$$

(2)

This equation describes the evolution of the distribution function of a cloud of particles confined in the domain $\Omega$. $v \in \mathbb{R}^N$ is the velocity of the particles, $E(x)$ is the electric field which satisfies $(H0)$ $E(x)$ is given, time-independent, and belongs to $W^{1,1}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$.

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The expression and properties of the operator \( K \) depend on the model we choose for the reflexion, absorption, and emission of particles. This note is concerned with the case of diffusive reflexion by the boundary, which gives rise to the following expression:

\[
K(\varphi)(x,v) = \int_{v \in \mathbb{R}^N, n(x) \cdot v' > 0} k(x,v',v)\varphi(x,v')n(x) \cdot v' \, dv', \quad \forall (x,v) \in \Sigma_+.
\]

From physical considerations, the kernel (or cross-section) \( k(x,v',v) \) has to satisfy:

(H1) positivity: \( k(x,v',v) \geq 0 \);

(H2) mass conservation: \( \int_{n(x) \cdot v < 0} k(x,v',v) |n(x) \cdot v| \, dv = 1 \), for \( n(x) \cdot v > 0 \).

One usually also adds the following hypothesis, which ensures the existence of a thermodynamical equilibrium:

(H3) there exists a Maxwellian distribution \( M(x,v) \) satisfying \( KM = M \).

The Maxwellian distribution reads \( M(x,v) = (1/2\pi \Theta^2) \exp(-|v|^2/2\Theta), \) where \( \Theta(x) \) is the temperature of the boundary. This last hypothesis is actually a consequence of (H2) when the following detailed balance principle (or reciprocity relation) holds:

\[
k(x,-v,-v')M(x,v) = k(x,v',v)M(x,v') \tag{3}
\]

However, in this paper, we investigate the case of elastic reflexion. Therefore, from now on, we shall assume that the cross-section reads:

(H3') \( k(x,v',v) = k_0(x,v',v)\delta(|v|^2-|v'|^2) \).

In this context, (H3) gives rise to the following normalization condition:

\[
\int_{n(x) \cdot v < 0} k(x,v',v) |n(x) \cdot v'| \, dv' = 1, \quad \text{for } n(x) \cdot v < 0.
\]

In particular, \( K\Phi = \Phi \) holds for every function depending on the velocity through the energy only, i.e., \( \Phi = \Phi(|v|^2) \).

Under these hypotheses, we investigate in this paper the properties of equations (1),(2): first, we establish in Proposition 1 the existence and uniqueness of solutions for initial data in \( L^1(\mathcal{O}) \cap L^2(\mathcal{O}) \), that satisfies \( \gamma f \in L^1_{\text{loc}}([0,T] \times \Sigma, |n(x) \cdot v| \, dv \, d\sigma_x \, dt) \). Then, in Proposition 2, we deduce the existence of a semigroup \( S(t) \) on \( L^1(\mathcal{O}) \), such that \( S(t)f \in L^\infty(0,T;L^1(\mathcal{O})) \) is a weak solution of (1), and satisfies (2) in a sense that has to be precise (see Remark 2.1). However, we do not know whether the so-constructed solution, the trace of which belongs to \( L^1_{\text{loc}}([0,T] \times \Sigma, |n(x) \cdot v|^2 \, dv \, d\sigma_x \, dt) \), is unique in the class of weak solutions in \( L^1(\mathcal{O}) \).

Many ideas used below have been first developed by Mischler in [4,5] in the case of specular and Maxwellian reflexion on the boundary. We refer to these papers for reference about the Vlasov equation and boundary conditions. We also stress the fact that the proof of Proposition 1 can be adapted in order to provide the uniqueness of the solution in [2].

The main tool that will be used throughout this note is the so-called Darrozès-Guiraud inequality [3], which reads, under the general framework of (H1)-(H3): for all convex nonnegative functions \( \beta \in C^0(\mathbb{R}) \),

\[
\int_{n(x) \cdot v < 0} \beta \left( \frac{K(\varphi)}{M} \right) M(v) |n(x) \cdot v| \, dv \leq \int_{n(x) \cdot v > 0} \beta \left( \frac{\varphi}{M} \right) M(v) |n(x) \cdot v| \, dv, \tag{4}
\]

with equality for \( \beta(y) = y \) (this is the expression of the flux conservation).

As a consequence, any function \( f \) satisfying boundary conditions (2) satisfies (at least formally)

\[
\int_{\Sigma} \beta \left( \frac{\gamma f}{M} \right) M(v) (n(x) \cdot v) \, dv \, d\sigma_x \geq 0.
\]
In the next section, we state our main results, the proofs of which are detailed in Sections 3 and 4.

2. MAIN RESULTS

From now on, we assume that (H0)–(H3) and (H3') hold. We also assume that \( n(x) \) can be extended to \( \mathbb{R}^N \) in a regular way (such that \( n(x) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N) \)). For technical purposes, it is convenient to decompose the velocity set \( \mathbb{R}^N \) as \( \mathbb{R}^+ \times S^{N-1} \) by writing \( v = |v|\omega \), where \( \omega \) is the angular velocity. With these notations, the operator \( K \) reads

\[
K(\varphi)(x,v) = \int_{n(x)\omega > 0} k_0(x,|v|,\omega) \varphi(x,|v|\omega) |v| n(x) \cdot \omega' \, d\omega', \quad \forall (x,v) \in \Sigma_-,
\]

and the normalization condition yields

\[
\int_{n(x)\omega > 0} k_0(x,u,\omega') n(x) \cdot \omega' \, d\omega' = 1.
\]

Let us also rewrite the Darrozès-Guiraud inequality (4) in this case: for all convex nonnegative functions \( \beta \in C^0(\mathbb{R}) \),

\[
\int_{n(x)\omega < 0} \beta(K\varphi(|v|\omega)) |n(x) \cdot \omega| \, d\omega \leq \int_{n(x)\omega > 0} \beta(\varphi(|v|\omega)) |n(x) \cdot \omega| \, d\omega. \tag{6}
\]

In particular, with \( \beta(y) = y^p \), we deduce

\[
\|K(\varphi)\|_{L^p(\Sigma_-)} \leq \|\varphi\|_{L^p(\Sigma_+)} \quad \forall p < +\infty.
\]

REMARK 2.1. From Theorem 1 in [4], for any function \( f \in L^\infty(0,T;L^p_{\text{loc}}(\mathcal{O})) \) solution of (1), we can define its trace \( \gamma f \), which belongs to \( L^1_{\text{loc}}([0,T] \times \Sigma, (n(x) \cdot v)^2 \, dv \, d\sigma_x \, dt) \). However, we need more integrability in order to give sense to (2). In view of (6), \( K(x,|v|) \) is a bounded operator on \( L^1(S^{N-1}) \), for any \( x, |v| \in \partial\mathcal{O} \times \mathbb{R}^+ \). Therefore, \( K\gamma f \) is well defined (and (2) has a meaning) as soon as \( \gamma f \in L^1_{\text{loc}}([0,T] \times \partial\mathcal{O} \times \mathbb{R}^+;L^1(S^{N-1});|n(x) \cdot v| \, dv \, d\sigma_x \, dt) \).

The first result we are aiming at is the following proposition.

**PROPOSITION 1.** For all initial data \( \varphi \in L^1(\mathcal{O}) \cap L^2(\mathcal{O}) \), there exists a unique solution \( f(x,v,t) \) of (1),(2) in \( L^\infty(0,T;L^1(\mathcal{O}) \cap L^2(\mathcal{O})) \) satisfying \( \gamma f \in L^1(0,T;L^1_{\text{loc}}(\Sigma,|n(x) \cdot v| \, dv \, d\sigma_x)) \).

Moreover, we have

\[
\|f(t)\|_{L^1(\mathcal{O})} \leq \|\varphi\|_{L^1(\mathcal{O})} \tag{7}
\]

(with equality if \( \varphi \geq 0 \), and for all \( U \) compact subset of \( \mathcal{O} \),

\[
\int_0^T \int_{U \cap \Sigma} |\gamma f| |n(x) \cdot v| \, dv \, d\sigma_x \, dt \leq C_U \left( 1 + \|E\|_{L^\infty(U_s)} + \|
abla_x n\|_{L^\infty(U_s)} \right) \|\varphi\|_{L^2(\mathcal{O})}.
\]

This first result defines a semigroup \( S(t) \) on \( L^1(\mathcal{O}) \cap L^2(\mathcal{O}) \), which satisfies \( \|S(t)\|_{L(L^1(\mathcal{O}))} \leq 1 \). \( L^1(\mathcal{O}) \cap L^2(\mathcal{O}) \) being a dense subset of \( L^1(\mathcal{O}) \), there exists a unique extension \( S(t): L^1(\mathcal{O}) \rightarrow L^1(\mathcal{O}) \). However, in Proposition 1, we control the \( L^1_{\text{loc}} \)-norm of the trace by the \( L^2 \)-norm of the initial data, and we shall therefore need further estimates in order to take the limit in (2). To that purpose, we introduce the following hypothesis.

(H4) For all compact set \( U \subseteq \partial\mathcal{O} \times \mathbb{R}^+ \), there exists a constant \( \beta_U > 0 \), such that

\[
\int_{n(x)\omega < 0} k_0(x,u,\omega') |n(x) \cdot \omega'|^2 \, d\omega \geq \beta_U, \quad \forall (x,u) \in U, \quad \omega' \cdot n(x) > 0.
\]

This so-called 'spreading condition', which can also be written \( K^*|n(x) \cdot \omega| \geq \beta_U \), is often used to get control on the trace. It is satisfied in particular when \( k_0 \) is bounded by below by a Maxwellian distribution.
PROPOSITION 2. Assume (H0)-(H4). For any initial data $\varphi \in L^1(\mathcal{O})$, there exists a function $f(t) = S(t)\varphi \in L^\infty(0, T; L^1(\mathcal{O}))$, solution of (1) which satisfies
\[
\|S(t)\varphi\|_{L^1(\mathcal{O})} \leq \|\varphi\|_{L^1(\mathcal{O})}.
\]
Moreover, its trace is such that $\gamma_+ S(t)\varphi \in L^1([0, T] \times V \times S^{N-1}; |n(x) \cdot v| dv \, d\sigma_x \, dt)$ for all compact subset $V$ of $\partial \Omega \times \mathbb{R}^+$. and satisfies (2).

REMARK 2.2. Both Propositions 1 and 2 may be generalized to a boundary condition which is a convex combination of an elastic-diffusive reflexion and a local reflexion (specular reflexion for instance).

3. PROOF OF PROPOSITION 1

UNIQUENESS. Let $f$ be a solution satisfying $f|_{t=0} \in L^1(\mathcal{O})$, and $f \in L^1(0, T; L^1_{\text{loc}}(|n(x) \cdot v| dv \, d\sigma_x))$. First of all, it has been proved in [4, Theorem 1] that such a solution actually belongs to $C^0(0, T; L^1_{\text{loc}}(\mathcal{O}))$. From [4], we also know that $f(t)$ is a renormalized solution of the Vlasov equation (1): for all $\beta \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$, we have
\[
\Lambda_{E}\beta(f) = 0 \quad \text{and} \quad \gamma\beta(f) = \beta(\gamma f).
\]

We define a sequence of smooth convex and nonnegative functions $\beta_\varepsilon$ as follows: $\beta_\varepsilon(y) = |y| - \varepsilon$ for $|y| \geq 2\varepsilon$ and $\beta_\varepsilon(y) = y^2/(4\varepsilon)$ for $|y| \leq 2\varepsilon$. We also introduce $\chi_R(x, |v|) = \chi(x/R, |v|/R)$, with $\chi$ a smooth function satisfying $0 \leq \chi \leq 1$, $\chi = 1$ on $B_1 \times [0, 1]$, and $\text{supp} \chi_R \subset B_2 \times [0, 2]$ (where $B_r$ denotes the ball of radius $r$, center at the origin in $\mathbb{R}^N$). Then the Green formula leads to
\[
\left[ \int_\mathcal{O} \beta_\varepsilon(f)\chi_R \, dv \, dx \right]^t_0 = \int_0^t \int_\mathcal{O} \beta_\varepsilon(f)\Lambda_{E}\chi_R \, dv \, dx \, ds
\]
\[
+ \int_0^t \int_{\Sigma_-} \beta_\varepsilon(\gamma f)\chi_R(|v|)|n(x) \cdot v| \, dv \, d\sigma_x \, ds - \int_0^t \int_{\Sigma_+} \beta_\varepsilon(\gamma f)\chi_R(|v|)|n(x) \cdot v| \, dv \, d\sigma_x \, ds.
\]

Since $\chi_R$ does not depend on the angular velocity $\omega$, Fubini’s theorem and (6) yield
\[
\left[ \int_\mathcal{O} \beta_\varepsilon(f)\chi_R \, dv \, dx \right]^t_0 \leq \int_0^t \int_\mathcal{O} \beta_\varepsilon(f)\Lambda_{E}\chi_R \, dv \, dx \, ds,
\]
with equality if $f \geq 0$. We deduce (7) by taking successively the limits $\varepsilon \to 0$ and $R \to \infty$ (see [4] for details). The uniqueness follows by standard argument.

EXISTENCE. Let now $\varphi$ be in $L^1(\mathcal{O}) \cap L^2(\mathcal{O})$. In order to prove the existence of a solution, we first assume that $\varphi$ is positive (for the general function, we decompose into positive and negative parts), and we define a sequence $(f_n)_{n\in\mathbb{N}}$ of solutions of the Vlasov equation (1) in $L^2(\mathcal{O})$, with initial data $\varphi$, and the following boundary conditions:
\[
\gamma_0 = 0, \quad \gamma_{-} f_n = K\gamma_{+} f_{n-1}, \quad \forall n \geq 1.
\]
Such a sequence is well defined since $\gamma_{+} f_{n-1}$, and therefore, $K\gamma_{+} f_{n-1}$ lies in $L^2(\Sigma_-)$ for all $n \geq 1$ (see [7]).

Thanks to the monotonicity of the operator $K$ and the maximum principle for the transport equation, it is easy to check that the sequence $(f_n)_{n\in\mathbb{N}}$ is nondecreasing. We deduce that for all convex functions $\beta$ such that $\beta$ is nondecreasing on $\mathbb{R}^+$, we have
\[
\int_{\Sigma_-} \beta(\gamma_{-} f_n)|n(x) \cdot v| \, dv \, d\sigma_x = \int_{\Sigma_-} \beta(K\gamma_{+} f_{n-1})|n(x) \cdot v| \, dv \, d\sigma_x
\]
\[
\leq \int_{\Sigma_+} \beta(\gamma_{+} f_{n-1})|n(x) \cdot v| \, dv \, d\sigma_x
\]
\[
= \int_{\Sigma_+} \beta(\gamma_{+} f_n)|n(x) \cdot v| \, dv \, d\sigma_x.
\]
The first inequality is a consequence of the Darrozès-Guiraud inequality (6), and the second one a consequence of the monotonicity of the sequence. Now, multiplying (1) by $f_n$ and integrating, the Green formula yields
\[
\left[ \int_{\Omega} |f_n|^2 \, dv \, dx \right]_0^t = - \int_0^t \int_{\Sigma} (\gamma f_n)^2 (n(x) \cdot v) \, dv \, d\sigma_x \, ds,
\]
and (8) with $\beta(y) = y^2$ implies
\[
\|f_n(t)\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)}.
\]
The similar $L^1$ estimate is obtained by proceeding as in the proof of the uniqueness, using inequality (8).

It remains to show that we can control the trace, at least locally. Let $U$ be a compact subset of $\Omega$, and let $\psi(x,|v|)$ be a compactly supported function on $\Omega$ such that $\psi = 1$ on $U$. In the spirit of [4], we multiply (1) by $(n(x) \cdot v) \psi(x,|v|)$; using (9), it yields
\[
\int_0^T \int_{\Sigma \cap U} (\gamma f_n)^2 |n(x) \cdot v|^2 \psi(x,|v|) \, dv \, d\sigma_x \, dt \leq C_U \left( 1 + \|E\|_{L^\infty(u_*)} + \|
abla \psi\|_{L^\infty(u_*)} \right) \|\varphi\|_{L^2(\Omega)}^2,
\]
and therefore, $\gamma f_n \in L^1(0,T;L^1_{loc}(\Omega), n(x) \cdot v \, dv \, d\sigma_x \, dt)$.

4. PROOF OF PROPOSITION 2

The main issue in the proof of Proposition 2 is concerned with boundary condition (2): let $\varphi$ be in $L^1(\Omega)$, and $\varphi_n \in L^1(\Omega) \cap L^2(\Omega)$ be a sequence of function such that $\varphi_n \to \varphi$ in $L^1(\Omega)$. Assume moreover, as in the previous section, that $\varphi$ and $\varphi_n$ are nonnegative. Thanks to Proposition 1, the sequence $f_n = S(t) \varphi_n$ strongly converges in $L^1(\Omega)$ toward $f(t) = S(t) \varphi$. Moreover, $\gamma f_n \in L^1(0,T;L^1_{loc}(\Omega), n(x) \cdot v \, dv \, d\sigma_x \, dt)$ and $\gamma f_n$ satisfies (2). It remains to prove that we can take the limit in (2).

First of all, multiplying (1) by $(n(x) \cdot v) \psi(x,|v|)$ for some compactly supported function $\psi$, we prove (as in the proof of Proposition 1) that, for all $U$ compact subsets of $\Omega$, we have
\[
\int_0^T \int_{\Sigma \cap U} |\gamma f_n| |n(x) \cdot v|^2 \, dv \, d\sigma_x \, dt \leq C_U \left( 1 + \|E\|_{L^\infty(u_*)} + \|
abla \psi\|_{L^\infty(u_*)} \right) \|\varphi_n\|_{L^1(\Omega)}.
\]
and therefore, $\gamma f \in L^1_{loc}(0,T) \times \Sigma, |n(x) \cdot v|^2 \, dv \, d\sigma_x \, dt)$. As discussed previously, this is not enough to take the limit in (2); however, thanks to (H4), we are going to derive another a priori estimate. Let now $V$ be a compact subset of $\partial \Omega \times \mathbb{R}^+_+$, from (2) and the previous estimate, we get
\[
\int_0^T \int_{(V \times S^{N-1}) \cap \Sigma_+} |K \gamma f_n| |n(x) \cdot v|^2 \, dv \, d\sigma_x \, dt \leq C_V \|\varphi_n\|_{L^1(\Omega)},
\]
for some constant $C_V$. Hence, $\gamma f_n$ being nonnegative and using Fubini’s theorem, we have
\[
\int_0^T \int_{V \cap \Sigma_+} \gamma f_n(v') |n(x) \cdot v'| |v'|^2 \int_{n(x)\cdot\omega<0} k_0(\omega',\omega)(n(x) \cdot \omega)^2 \, dw \, dv' \, d\sigma_x \, dt \leq C_V \|\varphi_n\|_{L^1(\Omega)}.
\]
Noticing that $|v'|^2$ is bounded by below on $V$, it follows from (H4) that
\[
\int_0^T \int_{(V \times S^{N-1}) \cap \Sigma_+} \gamma f_n(v') |n(x) \cdot v'| \, dv' \, d\sigma_x \, dt \leq \frac{C_V}{\beta_V} \|\varphi_n\|_{L^1(\Omega)}.
\]
We deduce that $\gamma f_n$ strongly converges to $\gamma f$ in $L^1([0,T] \times V \times S^{N-1}, |n(x) \cdot v| \, dv \, d\sigma_x \, dt)$ for all compact subsets $V$ of $\partial \Omega \times \mathbb{R}^+_+$ (and so does $\gamma f_n$ thanks to (2)), and, in view of Remark 2.1, this is enough to give sense to $K \gamma f$, and pass to the limit in (2).
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