Research Statement
Antoine Mellet

This is a (brief) description of my research in various areas of PDE. Nonlinear PDE of elliptic and parabolic type, in particular free boundary problems and reaction diffusion equations, have plaid a central role in my research. But I have also worked on compressible Navier-Stokes equations (existence, stability and a priori estimates). And my PhD was devoted to kinetic equations and their asymptotic regimes. I have recently gone back to that subject with some work on anomalous diffusion limit for transport equations (derivation of fractional diffusion equations).

In the first part of this research statement, I will describe my work on the homogenization of free boundary problems. The second part is devoted to some existence and stability results for compressible Navier-Stokes equations. The third part concerns kinetic equations and their asymptotic regimes, which include some of my early PhD work (briefly summarized) as well as recent developments. Finally, the last part describes some recent new directions of research.

1 Homogenization of some free boundary problems

1.1 Introduction

Homogenization is the process of replacing a PDE involving rapidly varying coefficients by a PDE involving only slowly varying ones in such a way that the corresponding solutions are close (in some topology). For instance, consider u_ε solution of

$$(a(x/\varepsilon)u_\varepsilon')' = f \qquad \text{for } x \in (0,1) \tag{1}$$

with homogeneous Neumann boundary conditions. The rapidly varying diffusion coefficient takes into account variations arising in the medium at a microscopic scale $\varepsilon \ll 1$. When $a(y)$ is periodic and bounded below by a positive constant, it is well known that as ε goes to zero, u_ε converges weakly in $H^1(0,1)$ to u solution of

$$(\overline{a}u')' = f \qquad \text{for } x \in (0,1) \tag{2}$$

where $\overline{a}$ is the harmonic average of $a(y)$: $\overline{a} = \left(\int_0^1 a(y)^{-1}\,dy\right)^{-1}$. Equation (2) is usually referred to as the homogenized equation. In this simple example, the coefficients of the effective equation are given by some kind of averaging of the original coefficients. In more complicated situation, the effect of small scale inhomogeneities may be hard to predict. For instance, a geodesic in an irregular medium will try to avoid unfavorable areas.

During my post-doc with Luis Caffarelli, I started a research program devoted to the homogenization of free boundary problems, when the rapid oscillations occur along the free boundary. Typically, we want to study the asymptotic behavior as ε goes to zero of $u_\varepsilon \geq 0$ solution of the free boundary problem

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \{u_\varepsilon > 0\} \cap \Omega \\ |\nabla u_\varepsilon|^2 = 2g(x/\varepsilon) & \text{on } \partial\{u_\varepsilon > 0\} \cap \Omega \end{cases} \tag{3}$$

(supplemented by some boundary conditions on $\partial\Omega$) for some periodic function $g(y)$. In this problem the oscillations are only seen by the solution along the free boundary $\Gamma(u_\varepsilon) = \partial\{u^\varepsilon > 0\}$. In a series of paper, we obtained some results on the homogenization of (3) and other free boundary problems, both in the periodic and random framework.

The difficulties in attacking this problem are numerous. The existence of a solution for a fixed $\varepsilon > 0$ is already a delicate issue. Classical solutions may not exists, and one needs to work with weak notions of solutions (for which the free boundary may not be regular enough to give a meaning to the free boundary condition in a classical sense). Furthermore, one typically does not expect to have uniqueness for general boundary data, and finally, the free boundary might oscillates rapidly as ε goes to zero and be very difficult to control.

Variational approach. One way to approach this problem is to think of (3) as the Euler equation for the minimization of

$$J_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + g(x/\varepsilon) 1_{\{u>0\}}\, dx.$$

We rely on a similar variational formulation in the study of equilibrium capillary surfaces and contact angle hysteresis which we will describe later on.

Global minimizers of this functional have good properties (see Alt-Caffarelli [1] and Caffarelli-Jerison-Kenig [9] for more recent developments), and it is easy to show that as ε goes to zero, they converge to a solution of

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap \Omega \\ |\nabla u|^2 = 2\langle g\rangle & \text{on } \partial\{u > 0\} \cap \Omega \end{cases} \tag{4}$$

where $\langle g\rangle$ is the average value of g. However, this is not the end of the story, because (3) has other solutions that are not global minimizers of J_ε (but only local minimizers). In particular, in one dimension it is easy to show that any function satisfying

$$\begin{cases} u_{xx} = 0 & \text{in } \{u > 0\} \cap \Omega \\ |u_x|^2 \in [2\inf g, 2\sup g] & \text{on } \partial\{u > 0\} \cap \Omega \end{cases}$$

can be obtained as limit of solutions of (3) as ε goes to zero.

This suggests that for the multi-dimensional problem, the homogenization of the free boundary condition

$$|\nabla u_\varepsilon|^2 = 2g(x/\varepsilon) \qquad \text{on } \partial\{u_\varepsilon > 0\}$$

leads to

$$|\nabla u|^2 \in [\alpha_{\min}(\nu), \alpha_{\max}(\nu)] \qquad \text{on } \partial\{u > 0\} \tag{5}$$

where ν denotes the normal vector to $\partial\{u > 0\}$ (it is easy to see that $\alpha_{\min}(\nu)$ and $\alpha_{\max}(\nu)$ should depend on ν, though it is not easy to determine how they do). This is of course very delicate to justify rigorously in dimension other than 1.

Singular perturbation approach. An alternative way of dealing with (3) is to think of it as the limit as δ goes to zero of the singular perturbation problem

$$u_{xx} = g(x/\varepsilon)\beta_\delta(u) \tag{6}$$

where $\beta_\delta = \frac{1}{\delta}\beta(\frac{u}{\delta})$ is an approximation of a Dirac at $u = 0$. This limit can easily be justified in dimension 1, but is much more delicate in higher dimension (see Berestycki-Caffarelli-Nirenberg [2]). It is easy to see that the limit $\varepsilon \to 0$ for fixed $\delta > 0$ leads to a simple averaging of the function $g(y)$. We are thus interested in the behavior of the solutions of (6) for $\delta \ll \varepsilon \ll 1$. This singular perturbation approach is very natural when working on combustion related free boundary problems as discussed below.

1.2 Main results

Among our results, we rigorously justify the derivation of (5) for particular solutions (plane-like solutions) and investigate in details two problems related to (3): A capillary drop problem (in which the Laplace equation is replaced by a mean-curvature equation) and a combustion problem (in which the Laplace equation is replaced by the heat equation). We present those results below. I have also studied other types of free boundary problems which exhibit very different behaviors. In particular, with I. Kim, we studied the homogenization of Hele-Shaw and Stefan problems. As we will see, the homogenization in that case leads to a nice averaging of the coefficients. Finally, I will present some related results concerning the obstacle problem in perforated domain (joint work with L. Caffarelli).

Capillary drops. In [14, 13], we investigate the effects of homogenization on the shape of equilibrium capillary drops. The shape of a liquid drop lying on a solid support is determined by Young-Laplace's law. When the free surface of the drop is a graph, it leads to the following mean-curvature free boundary problem:

$$-\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda \qquad \text{in } \{u > 0\}$$

$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nu = \beta \qquad \text{on } \partial\{u > 0\}. \tag{7}$$

The free boundary condition above (known as contact angle condition) plays a very important role in applications. However, in many experiments, the measured contact angle is quite far from the one predicted by this model and it usually depends on the history of the drop (whether the drop was formed by evaporation or by condensation). This is referred to as contact angle hysteresis, and it amounts to replacing (7) by

$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot \nu \in [\beta_{\min}, \beta_{\max}] \qquad \text{on } \partial\{u>0\}. \tag{8}$$

What we learned about the homogenization of (3) suggests that the discrepancy between (7) and (8) could be justified by replacing the constant coefficient β in (7) with a coefficient $\beta(x/\varepsilon)$ depending on the position via a microscopic variable (thus taking into account the roughness of the solid support or chemical contaminations).

This is what we established with Luis Caffarelli in [14, 13]. In those papers, we consider a slightly more general problem, allowing drops that cannot be described as graphs of a function u. The drop is thus described by a set E which minimizes the energy

$$\mathscr{J}_\varepsilon(E) = \sigma \int_{\{z>0\}} |D\varphi_E| - \sigma \int_{\{z=0\}} \beta(x/\varepsilon)\varphi_E \, dx$$

(the first term is the surface tension energy, proportional to the perimeter of E, the second term is the wetting energy). We prove that global minimizers of $\mathscr{J}_\varepsilon$ with a volume constraint are almost spherical (they are uniformly close to a minimizer of the energy functional $\mathscr{J}_0$ defined as $\mathscr{J}_\varepsilon$ with $\langle\beta\rangle$ instead of $\beta(x/\varepsilon)$), but that there exists some non-spherical local minimizers that exhibit contact angle hysteresis (satisfying the free boundary condition (8)).

The proof relies on the use of Schwartz symmetrization and some delicate non-degeneracy estimates for minimizers of the appropriate energy functional, which allows us to show, in particular, that the contact line (the free boundary) has finite $(n-1)-$Hausdorff measure (this partial regularity result is interesting in itself).

Combustion fronts. In another series of papers [10, 11, 12, 38, 37], we studied the following free boundary problem which arise in combustion theory and describes the propagation of combustion fronts:

$$\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon = 0 & \text{in } \{u_\varepsilon > 0\} \\ |\nabla u_\varepsilon|^2 = 2g(x/\varepsilon) & \text{in } \partial\{u_\varepsilon>0\} \end{cases} \tag{9}$$

(note that stationary solutions of (9) solve (3)). This equation arises as the limit of singular reaction-diffusion equations as in (6):

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = -g(x/\varepsilon)\beta_\delta(u_\varepsilon) \tag{10}$$

In our work, we were interested in particular solutions of (9) and (10) that describe fronts propagation, namely, solutions that are defined globally in time and satisfy, for a given $e \in S^{N-1}$

$$u_\varepsilon(x,t) \longrightarrow 0 \quad \text{as } x\cdot e \to -\infty$$

$$u_\varepsilon(x,t) \longrightarrow 1 \quad \text{as } x\cdot e \to +\infty.$$

When g is periodic, the existence of such solutions, known as Pulsating Traveling Fronts, follows from the work of Berestycki-Hamel [3]. Together with Luis Caffarelli and Ki-Ahm Lee, we studied the asymptotic behavior of those solutions as ε goes to zero. The first result, presented in [10], shows that these solutions are uniformly close to traveling waves propagating with a speed c_ε. The second paper, [11], investigates the value of the effective speed of propagation $\lim_{\varepsilon\to 0} c_\varepsilon$ (for the solutions of (9), or those of (10) when $\delta \ll \varepsilon \ll 1$). In particular, in one dimension, we show that

$$\lim_{\varepsilon\to 0} c_\varepsilon = \sqrt{2\inf g}.$$

This means that the worst property of the medium (and not some kind of average) determines the effective speed of propagation of fronts in periodic media and it yields the homogenized free boundary condition

$$|u_x|^2 = 2\inf g.$$

The fact that we capture the infimum of g rather than its supremum (or another value) is mainly due to the fact that the particular solutions under consideration are decreasing in time. In general, one can show that the homogenized free boundary condition for (9) would be (still in 1d)

$$|u_x|^2 \begin{cases} = 2\inf g & \text{if } u_t < 0 \\ \in [2\inf g, 2\sup g] & \text{if } u_t = 0 \\ = 2\sup g & \text{if } u_t > 0 \end{cases}$$

In higher dimension, a similar result is shown to hold, but there are no explicit formula for the effective speed of propagation. Instead we show that $\lim_{\varepsilon\to 0} c_\varepsilon$ is the smallest slope of the plane-like solutions of the elliptic problem (3) (this slope may depend on the direction).

Finally, with Y. Sire and J.-M. Roquejoffre [38], we investigate the propagation of fronts in one-dimensional case without any assumption on g other than bounded above and below (i.e. no periodicity). We prove that there exist some global in time solutions which generalize the notion of traveling waves in inhomogeneous media (these solutions, also called *generalized fronts*, were first introduced by Berestycki and Hamel in [4]).

In a recent paper with J. Nolen, J.-M. Roquejoffre and L. Ryzhik [37], we establish the stability and uniqueness of these fronts.

Hele-Shaw and Stefan problems. Hele-Shaw and Stefan free boundary problems are very different from the models presented above and exhibit different homogenization behaviors. Hele-Shaw problems read

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \{u_\varepsilon > 0\} \\ \partial_t u_\varepsilon = g(x/\varepsilon)|\nabla u_\varepsilon|^2 & \text{on } \partial\{u_\varepsilon > 0\} \end{cases} \tag{11}$$

while Stefan problems are of the form

$$\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon = 0 & \text{in } \{u_\varepsilon > 0\} \\ \partial_t u_\varepsilon = g(x/\varepsilon)|\nabla u_\varepsilon|^2 & \text{on } \partial\{u_\varepsilon > 0\}. \end{cases} \tag{12}$$

In both cases, the free boundary condition yields

$$V = g(x/\varepsilon)|\nabla u_\varepsilon|$$

where V is the velocity of the free boundary $\partial\{u_\varepsilon > 0\}$ in the normal direction.

The existence theory for these problems is somewhat easier because it can be shown that the time integral of classical solutions of (11) (respectively (12)) are solutions of an obstacle problem (respectively a parabolic obstacle problem). This leads to the notion of weak (or variational) solutions, which are the time derivatives of the solutions of the appropriate obstacle problem. Another natural notion of solutions for these equations is that of viscosity solutions, studied in particular by I. Kim [30].

The homogenization of (12) was first studied by J.-F. Rodrigues [46]. Using the variational formulation, one can show (both in the periodic and random case) that the homogenized free boundary condition reads

$$\partial_t u = \overline{g}|\nabla u|^2 \qquad \text{on } \partial\{u > 0\} \tag{13}$$

where $\overline{g}$ is the harmonic average of g. However, this variational approach does not provide good control on the convergence of the free boundary $\partial\{u_\varepsilon > 0\}$ to $\partial\{u > 0\}$ (except in 1d and for star shaped configurations). This is the question that we investigate with I. Kim in [32, 31]. To achieve this, we first prove that the notion of variational and viscosity solutions coincide. While the variational formulation leads to the formula (13), the use of comparison principles for viscosity solutions allow us to control the free boundary. We are then able to prove that the free boundaries of the ε-problem converge locally uniformly to the free boundary of the homogenized problem (with respect to the Hausdorff distance).

Obstacle problems in perforated domain. Finally, in a series of paper [8, 15] (with Luis Caffarelli), we investigate the homogenization of an obstacle problem in a perforated domain. The obstacle problem is also a free boundary problem, though it is very different from the ones discussed so far. It can be written as

$$\min(-\Delta u_\varepsilon, u_\varepsilon - \varphi_\varepsilon) = 0 \qquad \text{in } \Omega$$

where

$$\varphi_\varepsilon(x) = \begin{cases} \varphi(x) & \text{for all } x \in T_\varepsilon \\ -\infty & \text{for all } x \notin T_\varepsilon \end{cases}$$

for a given function φ.

This is a classical homogenization problem and the asymptotic behavior of u_ε strongly depends on the properties of the set $T_\varepsilon \subset \Omega$. Results were first obtained by L. Carbone and F. Colombini [17] in periodic settings and then in more general frameworks by G. Dal Maso et al. [23, 22, 21]. Using a slightly different approach, D. Cioranescu and F. Murat [19, 20] studied the particular case of a periodic repartition of holes:

$$T_\varepsilon = \bigcup_{k\in\mathbb{Z}^n} B_{a_\varepsilon}(\varepsilon k). \tag{14}$$

In that case, it can be proved that there exists a critical radius $a_\varepsilon \ll \varepsilon$ such that the limiting problem is no longer an obstacle problem, but a simple elliptic boundary value problem with a new term that takes into account the effect of the holes. More precisely, in dimension $n \geq 3$, when $a_\varepsilon = r_0 \varepsilon^{\frac{n}{n-2}}$ then there exists a constant $\mu(r_0) > 0$ such that $u = \lim_{\varepsilon\to 0} u_\varepsilon$ solves

$$-\Delta u - \mu(u - \varphi)_- = 0$$

where $u_- = \max(0, -u)$.

In [8], we generalize this result to the case where the holes are still located in small neighborhoods of lattice points $\varepsilon\mathbb{Z}^n$ but have random size and shape. The key assumption concerns the capacity of the holes, which must scale properly and have some averaging behavior (stationary ergodicity). We introduce a new approach based on the construction of a corrector using techniques reminiscent of those introduced by Caffarelli-Souganidis-Wang [16] for the homogenization of fully nonlinear equations in random media. A similar result, using Γ-convergence, was recently established by M. Focardi [26].

In [15], we generalize those results to fractional obstacle problems (which lead to a different critical radius) and boundary obstacle problems (i.e. when the obstacle is concentrated on the bounday of the domain).

2 Compressible Navier-Stokes equations

In a very different direction, I have studied, in collaboration with A. Vasseur, the compressible Navier-Stokes system of equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma - \operatorname{div}(\mu D(u)) - \nabla(\lambda \operatorname{div} u) = 0. \end{cases}$$

Classically, the stability of weak solutions (i.e. limit of weak solutions is a weak solution) is considered as the main step to prove the existence of weak solutions (it is usually not very difficult to show the existence of solutions for a regularized system).

When the viscosity coefficients are constant, this was proved by P.L. Lions [33] in the early 90's for isentropic gas. The result was later improved by Feireisl, Novotny, and Petzeltova [25] and more recently extended to heat conducting fluids by E. Feireisl [24]. In this framework, the natural energy estimates gives control on u in H^1 and ρ in L^γ. One of the main difficulty is then to pass the limit in the pressure term ρ^γ.

However, some physically relevant models feature coefficients that depends on the density or temperature. In that case, the techniques developed by P.L. Lions break down. And if those coefficients vanish when the density is zero (vacuum), we lose the H^1 regularity for the velocity which is usually crucial in the stability analysis.

Degenerate viscosity coefficients. In [40], we studied isentropic compressible Navier-Stokes equations when the viscosity coefficients $\mu(\rho)$ and $\lambda(\rho)$ are degenerate for $\rho = 0$. We prove the stability of a class of weak solutions without restriction on the size of the initial data and in presence of vacuum. The result holds for a wide range of pressure law (without the restriction $\gamma > 3/2$ that arise in Lions and Feireisl results) and in any dimension.

The main tool is a new entropy-like inequality satisfied by classical solutions when μ and λ satisfy

$$\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho)$$

(this relation, together with the condition $\mu + N\lambda \geq 0$ imposes that the coefficients are degenerate in dimension $N \geq 2$). This includes well-known models arising in shallow water theory corresponding to $\mu = \rho$ and $\lambda = 0$. This entropy was discovered by D. Bresch and B. Desjardins [5, 6] in the framework of Korteweg's equations and it provides additional regularity and compactness for the density. It thus becomes trivial to pass to the limit in the pressure term. The new difficulty, however, is to deal with the vacuum set (since we have no control on u on this set) and to pass to the limit in the terms involving the velocity u. This is done by establishing a new estimate of the energy $\rho|u|^2$ in a space better than the usual L^1.

Let us point out that constructing a sequence of approximated solutions verifying all the a priori estimates is actually highly non trivial in this case because of the complexity of the additional entropy inequality. Our result has nevertheless been used to obtain the existence of weak solutions in certain situations (see for instance [27] for the radially symmetric case).

In [41], we consider the one-dimensional problem in a very different framework: We consider smooth initial density bounded away from zero (i.e. no vacuum at initial time). We then prove global existence of strong solutions even when the viscosity coefficient is degenerate (the only restriction is that $\mu(\rho) \geq \mu_0\rho^{1/2}$ for small ρ). The result also relies on the entropy of [5] which is valid without restriction in one dimension (there is only one viscosity coefficient in one dimension) and gives control on some negative powers of the density. The framework of this result is similar to that considered by D. Hoff in [28] for weak solutions, but we obtain the existence and uniqueness of a strong solution and do not require the viscosity coefficient to be constant, not even strictly positive.

De-Giorgi's techniques for compressible Navier-Stokes equations. In a different direction, using some classical methods from the Calculus of Variations in the framework of fluid dynamics we derive new estimates for the solutions of Navier-Stokes equations. In [44] we obtain L^p a priori estimates for the velocity field in compressible Navier-Stokes equations. The result is actually quite general and applies to many other quantities (scalar or vector) that are advected by a compressible flow and subject to some diffusion phenomena (such as the density of pollutant in a compressible flow or the temperature of that flow). Similar technics can be used to control the temperature uniformly by below in the full compressible Navier-Stokes system of equations (see [43]).

3 Asymptotic analysis of kinetic models

Typically, a kinetic equation models the evolution of a cloud of particles at the microscopic level via the particles distribution function $f(x, v, t)$ (which gives the velocity distribution of the particles near a point x in space and at time t). Kinetic equations have many advantages; in particular it is possible to model very specific microscopic phenomena. However, these equations are very costly to simulate numerically since they involve 6 variables (3 space and 3 velocity variables). It is thus important to study asymptotic regimes that lead to hydrodynamics type equations.

Diffusion and homogenization limits. Much of my early works were devoted to the study of kinetic equations in various asymptotic regimes. In particular, I was interested in regimes in which the interactions of the particles with a surrounding medium were responsible for the relaxation of the velocity distribution of the particles toward thermodynamical equilibrium. For appropriate time and space scaling, one can derive hydrodynamics equations which describe the evolution of macroscopic quantities such as the density and flux of particles.

Typically, one considers the following rescaled kinetic equation:

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon) \tag{15}$$

as $\varepsilon \to 0$. The choice of scaling in (15) corresponds to small mean free path ($\varepsilon \bar{L}$, where $\bar{L}$ is the macroscopic length scale) and long time (of order $\varepsilon^{-2}\bar{\tau}$). The asymptotic behavior of f^ε strongly depends on the operator Q which models the interactions with the surrounding medium. In simple cases (linear Boltzmann equation), one can show that $f^\varepsilon(x, v, t)$ converges to $\rho(x, t)F(v)$ where F is a given function, and ρ is the density of the particles, solution of a diffusion equation

$$\partial_t \rho - \mathrm{div}(D\nabla\rho + U\rho) = 0.$$

I have studied this type of diffusion limits for kinetic equations in various frameworks, principally, but not only, in semiconductor devices. I have in particular investigated situations in which the convergence to the equilibrium is driven by phenomena taking place at the boundary of the domain or along interfaces. The main difficulty in that case is to control the trace of the solutions of the kinetic equation. With S. Mischler [35], we obtain some existence results for kinetic equations in this framework (with diffusive boundary conditions) and with various collaborators, we derive some asymptotic models in two different frameworks: the semiconductor super-lattices, where scattering phenomena arise along infinitely many hyper-surfaces, and the plasma propellors for satellites, where electrons are confined between two parallel planes by a strong magnetic field.

Other works have been devoted to diffusive regime with nonlinear collision operator (Pauli operator), and asymptotic regimes combining diffusion behavior and homogenization.

Anomalous diffusion limits. More recently, I have studied anomalous diffusion regimes for kinetic equations. These arise when the mean squared displacement of the particles is not a linear function of time, but rather a power of time. In that case, one must use a different time scale ($t \sim \varepsilon^{-\alpha}\bar{\tau}$ instead of $t \sim \varepsilon^{-2}\bar{\tau}$), and the limiting macroscopic equations are of fractional order.

For equation (15) with a linear collision operator Q, such regimes occur when the equilibrium distribution function is a heavy tail distribution (decreases as a power of $|v|$ for large v) or when the collision frequency is degenerate for finite v.

With S. Mischler and C. Mouhot [36], we developed a simple method to investigate such regimes, based on Fourier analysis. In [34], I developed a different approach which allows us to consider more general situations (space varying collision frequencies, nonlinear collision operators, boundary conditions...). The treatment of boundary conditions is an important problem currently under investigation. Other applications include the derivation of fractional Navier-Stokes equations and the study of weak turbulence.

Kinetic/Fluid coupling: Valsov-Fokker-Planck and Navier-Stokes equations. With A. Vasseur, [39, 42], we consider a system of coupled Kinetic and Fluid equations. This system models the evolution of a cloud of particles (described by Vlasov-Fokker-Planck equation) which is subjected to a drag force exerted by a surrounding fluid (described by incompressible Navier-Stokes equations). This model was formally introduced and studied by J. Carrillo and T. Goudon [18]. Together with A. Vasseur, we proved the existence of weak solutions for the coupled system of equations and rigorously justified the asymptotic analysis formally performed in [18] using a relative entropy method.

4 Other works

A thin films like equation arising in crack dynamics. Together with C. Imbert [29], we study the following equation:

$$\partial_t u + \partial_x(u^n \partial_x I(u)) = 0 \qquad \text{for } x \in \Omega \tag{16}$$

where Ω is a bounded interval of $\mathbb{R}$, n is a positive real number and I is a non-local elliptic operator of order 1 satisfying $I \circ I = -\partial_{xx}$ (more precisely, I is defined as the square root of the Laplace operator with Neumann boundary conditions). When $n = 3$, (16) arises in the modeling of hydraulic fractures [47]. Because of the importance of this model in various field (i.e. mining industry), there is an extensive literature devoted to formal asymptotics and numerical analysis for this equation, but, to my knowledge, no rigorous results.

This equation is very similar to the well-known thin film equation (which corresponds to $I = \partial_{xx}$). In particular, like the thin-film equation it lacks a comparison principle, and the existence of a non-negative solution (for non-negative initial data) is thus non-trivial. However, compared with the thin film equation, the analysis of (16) presents some additional difficulties: First, the operator I is non-local and the algebra is not as simple as with the Laplace operator. Second, because of the lower order of the operator I, the natural regularity given by the energy inequality ($u \in H^{\frac{1}{2}}$ rather than $u \in H^1$) does not give the boundedness and continuity of weak solutions even in dimension 1 (the power 1/2 of the laplacian is critical in dimension 1 in that respect).

In [29], we develop the functional framework suitable to the study of (16) and prove the existence of non-negative solutions for (16) for all $n \geq 1$. There are still many interesting open problems, such as the existence of solutions with compactly supported initial data in the physically relevant case $n = 3$ and the finite speed expansion of the support.

Existence and regularity of an extremal solution for a mean-curvature problem. In [40, 41] (in collaboration with J. Vovelle), we study the following problem:

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = g(\lambda, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{17}$$

where

$$g(\lambda, u) = \lambda f(u), \quad \text{or} \quad g(\lambda, u) = H + \lambda f(u).$$

The function f is a convex function such that $f(0) \geq 0$ ($f(0) > 0$ when $g(\lambda, u) = \lambda f(u)$), $f'(0) \geq 0$ and with at least linear growth at infinity, that is:

$$\lim_{t\to\infty} \frac{f(t)}{t} > 0. \tag{18}$$

Typical examples include $f(u) = (1+u)^p$ with $p \geq 1$ and $f(u) = e^u$, but our results apply to more general nonlinearities, including some singular functions of the form $f(u) = \frac{1}{(1-u)^q}$, $q > 1$ (in such a case, we will naturally look for solutions satisfying $0 \leq u(x) \leq 1$ in Ω).

The corresponding problem with the Laplace operator:

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{19}$$

has been extensively studied for various nonlinearity f satisfying similar conditions, with f superlinear. For that problem, it is well known that there exists a critical value $\lambda^* \in (0, \infty)$ for the parameter λ such that one (or more) solution exists for $\lambda < \lambda^*$, a unique weak solution u^* exists for $\lambda = \lambda^*$ while there is no solution for $\lambda > \lambda^*$ (see [7]). In [45], we establish similar result for (17). The delicate part is to prove the existence of the extremal solution u^*. In doing so, we prove in particular that the minimal solution u_λ (minimal means smallest) is bounded in L^∞ in all dimension and for a wide range of convex nonlinearity $f(u)$. This is very different from (19), for which such results strongly depend on the dimension and on f. On the other hand, unlike (19), this L^∞ bound does not imply that u_λ is a classical solution (because of the degenerate nature of the mean-curvature operator). We prove however that for power like nonlinearities and in any dimension, radially symmetric solutions are Lipschitz and thus classical solutions of (17).

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