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The Toric Geometry of Triangulated Polygons in Euclidean Space

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Abstract. Speyer and Sturmfels associated Gröbner toric degenerations $\mathrm{Gr}_2(\mathbb{C}^n)^{\mathcal{T}}$ of $\mathrm{Gr}_2(\mathbb{C}^n)$ with each trivalent tree $\mathcal{T}$ having n leaves. These degenerations induce toric degenerations $M_{\mathbf{r}}^{\mathcal{T}}$ of $M_{\mathbf{r}}$, the space of n ordered, weighted (by $\mathbf{r}$) points on the projective line. Our goal in this paper is to give a geometric (Euclidean polygon) description of the toric fibers and describe the action of the compact part of the torus as "bendings of polygons". We prove the conjecture of Foth and Hu that the toric fibers are homeomorphic to the spaces defined by Kamiyama and Yoshida.

1 Introduction

In [SpSt] the authors associated Gröbner toric degenerations $\mathrm{Gr}_2(\mathbb{C}^n)^{\mathcal{T}}$ of $\mathrm{Gr}_2(\mathbb{C}^n)$ with each trivalent tree $\mathcal{T}$ having n leaves (see definition of Gröbner toric degeneration below). These degenerations induce toric degenerations $M_{\mathbf{r}}^{\mathcal{T}}$ of $M_{\mathbf{r}}$, the space of n ordered, weighted (by $\mathbf{r}$) points on the projective line. We denote the corresponding toric fibers by $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ and $(M_{\mathbf{r}})_0^{\mathcal{T}}$, respectively. Our goal in this paper is to give a geometric (Euclidean polygon) description of the toric fibers and describe the action of the compact part of the torus as "bendings of polygons".

A toric degeneration of a variety X to a toric variety X' is a flat family over the line with general fiber isomorphic to X and special fiber isomorphic to the toric variety X'. A special class of toric degenerations, called Gröbner degenerations, arise from taking an ideal I (*e.g.,* the ideal cutting out X out of either affine space resp. projective space if X is affine resp. projective) and a weight vector w of integers defined on the set of variables such that the initial ideal $\mathrm{in}_w(I)$ (see [St]) is generated by binomials. In such a setting, there exists a flat family over the line with general fiber isomorphic to $X = \mathcal{Z}(I)$ and special fiber isomorphic to $X' = \mathcal{Z}(\mathrm{in}_w(I))$.

1.1 The Grassmannian and Imploded Spin-framed Polygons

We start by identifying the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ with the moduli space of "imploded spin-framed" n-gons in $\mathbb{R}^3$. We define the space of *imploded framed vectors*, which is topologically the cone $C\mathrm{SO}(3,\mathbb{R})$ of $\mathrm{SO}(3,\mathbb{R})$, as the space

$$\left\{ (F, e) \in \mathrm{SO}(3,\mathbb{R}) \times \mathbb{R}^3 \;\middle|\; e = tF(\epsilon_1) \text{ for some } t \in \mathbb{R}_{\geq 0} \right\}$$

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modulo the equivalence relation $(F_1, 0) \sim (F_2, 0)$ for all $F_1, F_2 \in \mathrm{SO}(3, \mathbb{R})$, where $\epsilon_1 = (1, 0, 0)$ is the first standard basis vector of $\mathbb{R}^3$. We will see later that this equivalence relation is "implosion" in the sense of [GJS]. We call the equivalence class of (F, e) an "imploded framed vector". Note that the isotropy $T_{\mathrm{SO}(3,\mathbb{R})}$ of ϵ_1 in $\mathrm{SO}(3, \mathbb{R})$ has a natural right action on the space of imploded framed vectors.

Now fix a covering homomorphism $\pi\colon \mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$ such that the Cartan subgroup of diagonal matrices $T_{\mathrm{SU}(2)}$ in $\mathrm{SU}(2)$ maps onto $T_{\mathrm{SO}(3,\mathbb{R})}$. We define the space of imploded *spin*-framed vectors, which is topologically the cone $C\mathrm{SU}(2) \cong \mathbb{C}^2$, as the space

$$\left\{ (F, e) \in \mathrm{SU}(2) \times \mathbb{R}^3 \;\middle|\; e = t\pi(F)(\epsilon_1) \text{ for some } t \in \mathbb{R}_{\geq 0} \right\}$$

modulo the equivalence relation $(F_1, 0) \sim (F_2, 0)$ for all $F_1, F_2 \in \mathrm{SU}(2)$. We call the equivalence class of (F, e) an "imploded spin-framed vector". An *imploded spin-framed n-gon* is an n-tuple $((F_1, e_1), \ldots, (F_n, e_n))$ of imploded spin-framed vectors such that $e_1 + e_2 + \cdots + e_n = 0$. Let $\widetilde{P}_n(\mathrm{SU}(2))$ denote the space of imploded spin-framed n-gons. There is an action of $\mathrm{SU}(2)$ on $\widetilde{P}_n(\mathrm{SU}(2))$ given by

$$F \cdot \big((F_1, e_1), \ldots, (F_n, e_n) \big) = \big((FF_1, \pi(F)(e_1) \big), \ldots, \big(FF_n, \pi(F)(e_n) \big) \,.$$

We let $P_n(\mathrm{SU}(2))$ denote the quotient space. Note that since we may scale the edges of an n-gon, the space $P_n(\mathrm{SU}(2))$ is a cone with vertex the zero n-gon (all the edges are the zero vector in $\mathbb{R}^3$). Finally, note that there is a natural right action of $T_{\mathrm{SU}(2)^n}$ on $\widetilde{P}_n(\mathrm{SU}(2))$ that rotates frames but fixes the vectors

$$(t_1, \ldots, t_n) \cdot \big((F_1, e_1), \ldots, (F_n, e_n) \big) = \big((F_1 t_1, e_1), \ldots, (F_n t_n, e_n) \big) \,.$$

Remark 1.1 We will see later that $P_n(\mathrm{SU}(2))$ occurs naturally in equivariant symplectic geometry ([GJS]). It is the symplectic quotient by the left diagonal action of $\mathrm{SU}(2)$ on the right imploded cotangent bundle of $\mathrm{SU}(2)^n$. The space $\widetilde{P}_n(\mathrm{SU}(2))$ is the zero level set of the momentum map for the left diagonal action on the right imploded product of cotangent bundles. Hence we find that the space $P_n(\mathrm{SU}(2))$ has a (residual) right action of an n–torus $T_{\mathrm{SU}(2)^n}$, the maximal torus in $\mathrm{SU}(2)^n$ which rotates the imploded spin-frames.

Let $Q_n(\mathrm{SU}(2))$ be the quotient of the subspace of imploded spin-framed n-gons of perimeter 1 by the action of the diagonal embedded circle $T_{\mathrm{SU}(2)} \xrightarrow{\Delta} T_{\mathrm{SU}(2)^n}$. In what follows $\mathrm{AffGr}_2(\mathbb{C}^n)$ denotes the affine cone over the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ for the Plücker embedding. Thus $\mathrm{AffGr}_2(\mathbb{C}^n)$ is the subcone of $\bigwedge^2(\mathbb{C}^n)$ consisting of the decomposable bivectors (the zero locus of the Plücker equations). The reason we consider $Q_n(\mathrm{SU}(2))$ here is the following theorem (proved in Section 3) that gives a polygonal interpretation of $\mathrm{Gr}_2(\mathbb{C}^n)$. It is the starting point of our work.

Theorem 1.2 (i) *$P_n(\mathrm{SU}(2))$ and $\mathrm{AffGr}_2(\mathbb{C}^n)$ are homeomorphic.*
(ii) *This homeomorphism induces a homeomorphism between $Q_n(\mathrm{SU}(2))$ and $\mathrm{Gr}_2(\mathbb{C}^n)$.*

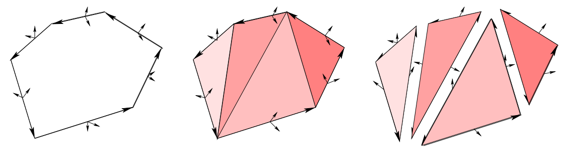

Figure 1.1: A framed spatial polygon with chosen triangulation.

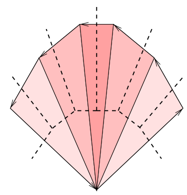

Figure 1.2: The standard triangulation of the model 7-gon with dual caterpillar tree.

1.2 Triangulations, Trivalent Trees, and the Construction of Kamiyama–Yoshida

Let P denote a fixed convex planar n-gon. Throughout the paper we will use the symbol $\mathcal{T}$ to denote either a triangulation of P or its dual trivalent tree. A trivalent tree is an acyclic graph where each vertex has valence three or valence one. Accordingly we fix a triangulation $\mathcal{T}$ of P. The points of $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ are "imploded spin-framed n–gons" with a fixed perimeter in $\mathbb{R}^3$ modulo an equivalence relation called $\mathcal{T}$-congruence and denoted $\sim_{\mathcal{T}}$ that depends on the triangulation $\mathcal{T}$.

Now that we have a polygonal interpretation of $\mathrm{Gr}_2(\mathbb{C}^n)$ we will impose the equivalence relation of $\mathcal{T}$-congruence (to be described below) on our space of framed polygons and obtain a polygonal interpretation of $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ corresponding to the triangulation (tree) $\mathcal{T}$. We now describe the equivalence relation of $\mathcal{T}$-congruence. Here we will discuss only the case of the *the standard triangulation* $\mathcal{T}_0$, that is the triangulation of P given by drawing the diagonals from the first vertex to the remaining nonadjacent vertices. The dual tree to the standard triangulation will be called the caterpillar or fan (see Figure 1.2). However, we will state our theorems in the generality in which they are proved in the paper namely for all trivalent trees $\mathcal{T}$ with n leaves.

The reader is urged to refer to Figure 1.3 to understand the following description. The equivalence relation for the standard triangulation (in the case of n-gon linkages) described below was first introduced in [KY]. We have extended their definition to all triangulations and to n-gons equipped with imploded spin-frames. Label the diagonals of the triangulation counterclockwise by 1 through $n-3$. For each $S \subset \{1, 2, \ldots, n-3\}$ let $\widetilde{P}_n(\mathrm{SU}(2))^{[S]}$ denote the subspace of $\widetilde{P}_n(\mathrm{SU}(2))$, where the diagonals corresponding to the elements in S are zero and all other diagonals are nonzero. Let $\mathbf{F} = ((F_1, e_1), (F_2, e_2), \ldots, (F_n, e_n))$ be a point in $\widetilde{P}_n(\mathrm{SU}(2))^{[S]}$. Suppose that $|S| = i$. Since i diagonals are zero, there will be $i+1$ sums of the form $e_j + e_{j+1} + \cdots + e_{j+k_j}$ that are zero, and the n-gon underlying $\mathbf{F}$ will be the wedge of $i+1$ closed subpolygons corresponding to the $i+1$ closed subpolygons in the collapsed reference polygon. Thus we can divide $\mathbf{F}$ into $i+1$ imploded spin-framed closed subpolygons (in terms of formulas we can break up the above n-tuple into $i+1$ sub k_j-tuples of edges for $1 \leq j \leq i+1$). We may act on each imploded spin-framed subpolygon (k_j-tuple) by a copy of $SU(2)$, which acts by rotating the k_j-th subpolygon and fixes the other subpolygons. For any two distinct subpolygons, the associated $SU(2)$ actions commute, and so there is an action of $SU(2)^{i+1}$ on $\widetilde{P}_n(SU(2))^{[S]}$. We pass to the quotient $\widetilde{P}_n(SU(2))/{\sim_{\mathcal{T}_0}}$ by dividing out $\widetilde{P}_n(SU(2))^{[S]}$ by the action of $SU(2)^{i+1}$. By definition $\mathcal{T}$-congruence is the equivalence relation induced by the above quotient operation; see Figure 1.3.

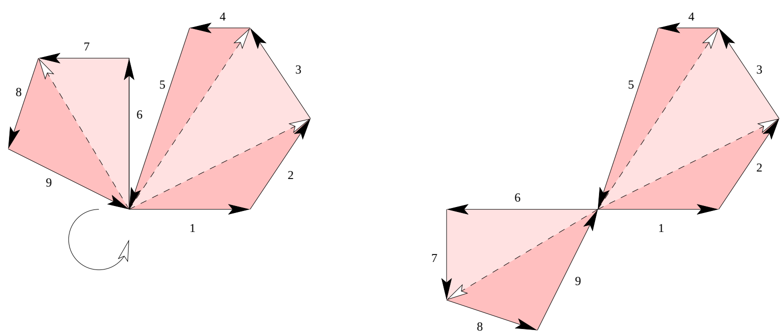

Figure 1.3: The numbered solid arrows indicate edges, and the dashed arrows are diagonals of the standard triangulation $\mathcal{T}_0$. The two polygons pictured above lie within $\widetilde{P}_9(SU(2))^{[\{4\}]}$, since the fourth special diagonal vanishes (the sum of the first five edges). Such polygons appear as a wedge of a 5-gon and a 4-gon, and there is a natural action of $SU(2)^2$ on them, where the first (resp. second) factor rotates the 5-gon (resp. 4-gon). The two polygons above lie in the same orbit under $SU(2)^2$; hence they are $\mathcal{T}_0$-congruent.

We let $V_n^{\mathcal{T}_0}$ denote the quotient space $\widetilde{P}_n(SU(2))/{\sim_{\mathcal{T}_0}}$. Thus $V_n^{\mathcal{T}_0}$ is decomposed into the pieces $(V_n^{\mathcal{T}_0})^{[S]} = \widetilde{P}_n(SU(2))^{[S]}/{\sim_{\mathcal{T}_0}}$. We will call the resulting decomposition in the special case of $\mathcal{T}_0$ the Kamiyama–Yoshida decomposition (or KY-decomposition). In this paper we use a weakened definition of the term *decomposition*; we will refer to decompositions of spaces where the pieces are products of spaces with isolated singularities.

Remark 1.3 The equivalence relation of $\mathcal{T}$-congruence can be defined analogously for any triangulation of P (equivalently any trivalent tree with n-leaves) and induces an equivalence relation on $Q_n(SU(2))$ (and many other spaces associated to spaces of n-gons in $\mathbb{R}^3$, for example, the space of n-gons itself or the space of n-gon linkages). We will use the symbol $\sim_{\mathcal{T}}$ to denote all such equivalence relations.

We will use $W_n^{\mathcal{T}}$ to denote the quotient of $Q_n(SU(2))$ by the equivalence relation $\sim_{\mathcal{T}}$. We can now state our first main result (this is proved in Section 8).

Theorem 1.4 (i) *The toric fiber of the toric degeneration of* $\mathrm{AffGr}_2(\mathbb{C}^n)$ *corresponding to the trivalent tree* $\mathcal{T}$ *is homeomorphic to* $V_n^{\mathcal{T}}$.

(ii) *The toric fiber of the toric degeneration of* $\mathrm{Gr}_2(\mathbb{C}^n)$ *corresponding to the trivalent tree* $\mathcal{T}$ *is homeomorphic to* $W_n^{\mathcal{T}}$.

Remark 1.5 The quotient map from $Q_n(\mathrm{SU}(2))$ to $W_n^{\mathcal{T}}$ given by passing to $\mathcal{T}$-congruence classes maps the generic fiber of the toric degeneration onto the special (toric) fiber.

Now fix an n-tuple of positive integers $\mathbf{r} = (r_1, \dots, r_n)$. The space $M_{\mathbf{r}}$ is the moduli space of spatial n-gons with side lengths $\mathbf{r}$ (see Section 2). The space $M_{\mathbf{r}}$ can be realized as a GIT quotient (or symplectic reduction) of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$. The toric degeneration of the Grassmannian and the notion of $\mathcal{T}$-congruence naturally descend to $M_{\mathbf{r}}$. Let $(M_{\mathbf{r}})_0^{\mathcal{T}_0}$ denote the toric fiber of the degeneration associated with the standard triangulation $\mathcal{T}_0$ (see Section 4 for details).

We now describe the space $V_{\mathbf{r}}^{\mathcal{T}_0}$ that will be proved later to be homeomorphic to the toric fiber $(M_{\mathbf{r}})_0^{\mathcal{T}_0}$. Again we will restrict ourselves to the standard triangulation in our description. Starting with the space

$$\widetilde{M}_{\mathbf{r}} = \{(e_1, \dots, e_n) \in (\mathbb{R}^3)^n \mid \sum_i e_i = 0,\ \|e_i\| = r_i\}$$

of closed n-gon linkages with side-lengths $\mathbf{r}$, we define $V_{\mathbf{r}}^{\mathcal{T}_0} = \widetilde{M}_{\mathbf{r}}/\sim_{\mathcal{T}_0}$. This is the construction of [KY]; see pictures below.

Theorem 1.6 *The toric fiber* $(M_{\mathbf{r}})_0^{\mathcal{T}}$ *of the toric degeneration of* $M_{\mathbf{r}}$ *corresponding to the trivalent tree* $\mathcal{T}$ *is homeomorphic to* $V_{\mathbf{r}}^{\mathcal{T}}$.

Remark 1.7 The quotient map from $M_{\mathbf{r}}$ to $V_{\mathbf{r}}^{\mathcal{T}}$ maps the generic fiber of the toric degeneration onto the special (toric) fiber.

The result in the previous theorem was conjectured by Philip Foth and Yi Hu in [FH].

1.3 Bending Flows, Edge Rotations, and the Toric Structure of $W_n^{\mathcal{T}_0}$

The motivation for the above construction becomes more clear once we introduce the bending flows of [KM] and [Kly]. The lengths of the $n-3$ diagonals created are continuous functions on $M_{\mathbf{r}}$ and are smooth where they are not zero. They give rise to Hamiltonian flows that were called *bending flows* in [KM]. The bending flow associated with a given diagonal has the following description. The part of the n-gon to one side of the diagonal does not move, while the other part rotates around the diagonal at constant speed. The lengths of the diagonals are action variables that generate the bending flows; the conjugate angle variables are the dihedral angles between the fixed and moving parts. However, the bending flow along the i-th diagonal is not defined at those n-gons where the i-th diagonal is zero. If the bending flows are everywhere defined (for example if one of the side-lengths is much larger than the rest), then we may apply the theorem of Delzant ([Del]) to conclude that $M_{\mathbf{r}}$ is toric. However, for many $\mathbf{r}$ (including the case of regular n -gons) the bending flows are not everywhere defined. The point of [KY] was to make the bending flows well-defined by dividing out the subspaces of $\widetilde{M}_{\mathbf{r}}$ where a collection of i diagonals vanish by $SO(3,\mathbb{R})^{i+1}$. We illustrate their construction with two examples.

First, let $\mathbf{r} = (1,1,1,1,1,1)$, so $M_{\mathbf{r}}$ is the space of regular hexagons with side-lengths all equal to 1. Let $M_{\mathbf{r}}^{(2)}$ be the subspace of $M_{\mathbf{r}}$ where the middle (second) diagonal vanishes. Thus $M_{\mathbf{r}}^{(2)}$ is the space of "bowties" (see Figure 1.4) modulo the diagonal action of $SO(3,\mathbb{R})$ on the two equilateral triangles. We can no longer bend on the second diagonal because we have no axis of rotation. Passing to $\mathcal{T}$-congruence classes collapses the space $M_{\mathbf{r}}^{(2)}$ to a point by dividing by the action of $SO(3,\mathbb{R}) \times SO(3,\mathbb{R})$. Bending along the second diagonal fixes this point by definition.

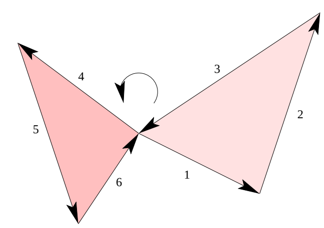

Figure 1.4: The bowties are all $\mathcal{T}_0$-congruent and so define a single point in $V_{\mathbf{r}}^{\mathcal{T}_0}$. However, in $M_{\mathbf{r}}$ the subspace of bowties is homeomorphic to $SO(3,\mathbb{R})$.

For our second example, we consider the space of regular octagons with all side-lengths equal to 1 and the subspace $M_{\mathbf{r}}^{(3)}$ where the middle diagonal vanishes. Thus $M_{\mathbf{r}}^{(3)}$ is the space of wedges of rhombi modulo the diagonal action of $SO(3,\mathbb{R})$ on the two rhombi; see Figure 1.5. Passing to $\mathcal{T}$-congruence classes amounts to dividing by the action of $SO(3,\mathbb{R}) \times SO(3,\mathbb{R})$ on the two rhombi.

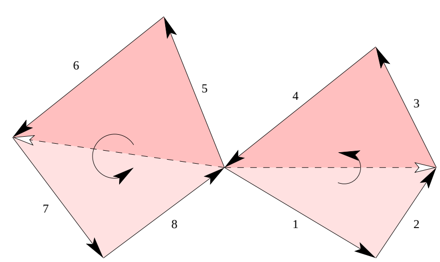

Figure 1.5: Here $n = 8$ and $S = \{3\}$. The two dashed arrows are the first and fifth diagonals of the standard triangulation. Recall that in the standard triangulation, the i-th diagonal is the sum of the first $i+1$ edges. The middle three components of the torus T^5 act trivially, and the quotient 2-torus acts by bending along the first and fifth diagonals.

We obtain the action of the bending flows on $V_n^{\mathcal{T}_0}$ as follows. In this case we will be given a lift of the one-parameter group bending along a diagonal to SU(2). The one parameter group acts through its quotient in SO$(3, \mathbb{R})$ by bending along the diagonal. If an edge moves under this bending, then the imploded spin-frame is moved by the one-parameter group in SU(2) in the same way. Hence, one part of the imploded spin-framed polygon is fixed and the other moves by a "rigid motion", *i.e.,* all the spin-framed edges of the second part are moved by the same one-parameter group in SU(2). The bendings give rise to an action of an $n-3$ torus T_{bend} on $V_n^{\mathcal{T}}$. There are also "edge-rotations" that apply a one-parameter group to the imploded spin-frame but do not move the edge. This action on frames is the action of the torus $T_{\mathrm{SU}(2)^n}$ coming from the theory of the imploded cotangent bundle of $\mathrm{SU}(2)^n$. This action is not faithful; the diagonal subtorus acts trivially. The bendings together with the edge rotations give rise to an action of a compact $2n-4$ torus $T = T_{bend} \times T_{\mathrm{SU}(2)^n}$.

1.4 A Sketch of the Proofs

The main step in proving Theorems 1.4 and 1.6 is to produce a space $P_n^{\mathcal{T}_0}(\mathrm{SU}(2))$ that "interpolates" between $V_n^{\mathcal{T}_0}$ and $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}_0}$. Our construction of $P_n^{\mathcal{T}_0}(\mathrm{SU}(2))$ was motivated by the construction of the toric varieties associated with SU(2)-character varieties of fundamental groups of surfaces given by Hurtubise and Jeffrey in [HJ]. The connection is that the space $M_{\mathbf{r}}$ can be interpreted as the (relative) character variety of the fundamental group of the n-punctured two-sphere with values in the translation subgroup of the Euclidean group $\mathbb{E}_3$, a small loop around the i-th puncture maps to translations by the i-th edge of the polygon (considered as a vector in $\mathbb{R}^3$).

Take the triangulated model (convex planar) n-gon P and break it apart into $n-2$ triangles $T_1, \ldots, T_{n-2}$. Equivalently we break apart the dual tree $\mathcal{T}$ into a forest $\mathcal{T}^D$ consisting of $n-2$ tripods.

Attach to each of the $3(n-2)$ edges of the $n-2$ triangles (or each edge of the forest $\mathcal{T}^D$) a copy of $T^*(\mathrm{SU}(2))$. Now right-implode each copy of $T^*(\mathrm{SU}(2))$ so that a copy of $\mathcal{E}T^*(\mathrm{SU}(2))) \cong \mathbb{C}^2$, the imploded cotangent bundle of SU(2), is attached to each edge. Since $\mathcal{E}T^*(\mathrm{SU}(2))$ admits an action of the circle (from the right) and SU(2) from the left, the resulting space admits an action of a torus $\mathbb{T}$ of dimension $3(n-2)$ and a commuting action of $\mathrm{SU}(2)^{3(n-2)}$. The torus has a product decomposition $\mathbb{T} = \mathbb{T}_e \times \mathbb{T}_d$, where $\mathbb{T}_e$ is the product of factors corresponding to the n edges of the polygon P and $\mathbb{T}_d$ is the product of factors corresponding to the $n-3$ diagonals of P. Note that each diagonal of P occurs in two triangles and so corresponds to two edges of $\mathcal{T}^D$. Hence each diagonal gives rise to a two-torus $S^1 \times S^1$ in $\mathbb{T}$ that we will refer to as the two-torus corresponding to that diagonal. Define a subtorus $\mathbb{T}_d^-$ of $\mathbb{T}_d$ of dimension $n-3$ by taking an antidiagonal embedding of S^1 in each two-torus corresponding to a diagonal.

Now take the symplectic quotient (at level zero) of (the product of) the three copies of $\mathbb{C}^2$ associated with the three sides of each triangle by SU(2) acting diagonally. For each triangle (or each tripod) we obtain a resulting copy of $\bigwedge^2(\mathbb{C}^3)$. The resulting product $(\bigwedge^2(\mathbb{C}^3))^{n-2}$ has an induced action of the torus $\mathbb{T}$. Glue the copies

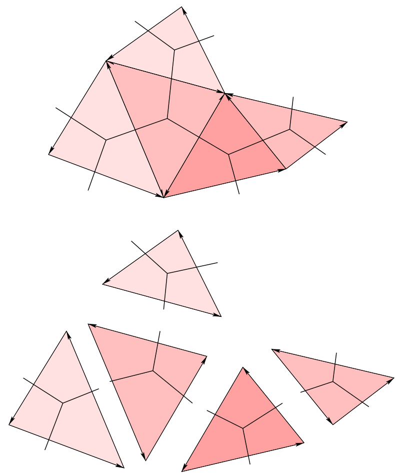

Figure 1.6: The decomposed polygon and the decomposed dual tree $\mathcal{T}^D$.

$\bigwedge^2(\mathbb{C}^3)$ associated with the triangles together along the edges of the triangles associated with diagonals by taking the symplectic quotient at level zero by the torus $\mathbb{T}_d^- \subset \mathbb{T}_d$ described above. Each of the two previous symplectic quotients has a corresponding GIT quotient. Taking both GIT quotients we obtain a space that is an affine torus quotient of affine space. Hence the combined symplectic quotient is the space underlying the affine toric variety

$$P_n^{\mathcal{T}_0}(\mathrm{SU}(2)) = \left(\bigwedge^2(\mathbb{C}^3)\right)^{n-2} //_0 \mathbb{T}_d^-.$$

Let $\mathbf{t}_e(\lambda)$ be the element in complexified torus $\underline{\mathbb{T}} \cong (\mathbb{C}^*)^{3n-6}$ of $\mathbb{T}$ such that all the edge components coincide with $\lambda \in \mathbb{C}^*$ and all the components corresponding to diagonals are 1. Then Theorem 1.4 follows by putting together items (ii) and (iv) in the next theorem, which is proved in Section 8.

Theorem 1.8 (i) *The toric varieties* $\mathrm{AffGr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ *and* $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ *are isomorphic as affine toric varieties.*

(ii) *The grading action of* $\lambda \in \mathbb{C}^*$ *on* $\mathrm{AffGr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ *corresponds to the action of* $\mathbf{t}_e(\sqrt{\lambda}^{-1})$ *on* $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ *(this is well defined). Consequently* $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ *is projectively isomorphic to the quotient of* $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ *by this* $\mathbb{C}^*$ *action. We will denote this quotient by* $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$.

(iii) *There is a homeomorphism (that creates imploded spin-frames along the diagonals of the triangulation)*

$$\Psi_n^{\mathcal{T}} : V_n^{\mathcal{T}} \to P_n^{\mathcal{T}}(\mathrm{SU}(2)).$$

(iv) *The homeomorphism* $\Psi_n^{\mathcal{T}}$ *induces a homeomorphism from* $W_n^{\mathcal{T}}$ *to the projective toric variety* $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$.

The quotient $P_n^{\mathcal{T}_0}(\mathrm{SU}(2)) = \bigwedge^2(\mathbb{C}^3)^{n-2}//\mathbb{T}_d^-$ admits a residual action by the quotient torus $\mathbb{T}/\mathbb{T}_d^-$. This quotient torus contains a factor that can be identified with $\mathbb{T}_e$. The toric fiber $(M_{\mathbf{r}})_0$ is obtained from $P_n^{\mathcal{T}_0}(\mathrm{SU}(2))$ by taking the symplectic quotient by $\mathbb{T}_e$ at level $\mathbf{r}$. Theorem 1.6 follows from the two statements of the following theorem.

Theorem 1.9 (i) *The toric variety $P_n^{\mathcal{T}}(\mathrm{SU}(2))//_{\mathbf{r}}\mathbb{T}_e$ is isomorphic to the toric variety $(M_{\mathbf{r}})_0^{\mathcal{T}}$.*

(ii) *For each $\mathbf{r}$ the homeomorphism $\Psi_n^{\mathcal{T}}$ induces a homeomorphism*

$$\Psi_{\mathbf{r}}^{\mathcal{T}} \colon V_{\mathbf{r}}^{\mathcal{T}} \to P_n^{\mathcal{T}}(\mathrm{SU}(2))//_{\mathbf{r}}\mathbb{T}_e.$$

Note that the toric varieties $(\bigwedge^2(\mathbb{C}^3))^{n-2}//\mathbb{T}_d^-$ (resp. $\bigwedge^2(\mathbb{C}^3)^{n-2}//_{\mathbf{r},0}(\mathbb{T}_e \times \mathbb{T}_d^-)$) mediate between the Kamiyama–Yoshida spaces $V_n^{\mathcal{T}}$ (resp. $V_{\mathbf{r}}^{\mathcal{T}}$) and the toric varieties $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ (resp. $(M_{\mathbf{r}})_0^{\mathcal{T}}$).

1.5 The Hamiltonian Nature of the Edge Rotations and the Bending Flows

It remains to place the edge rotations and bending flows in their proper context (in terms of symplectic and algebraic geometry). We do this with the following theorems. We note that since the spaces $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ and $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ are quotients of the affine space $(\bigwedge^2(\mathbb{C}^3))^{n-2}$ by tori, they inherit symplectic stratifications from the orbit type stratification of $(\bigwedge^2(\mathbb{C}^3))^{n-2}$ according to [SjL]. We identify the edge flows and bending flows with the action of the maximal compact subgroups (compact torus) of the complex tori that act holomorphically with open orbits on the toric varieties $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ and $(M_{\mathbf{r}})_0^{\mathcal{T}}$. This is accomplished by the following theorem, which is proved by Theorems 8.1 and 10.1.

Theorem 1.10 (i) *The action of the edge rotations corresponds under $\Psi_n^{\mathcal{T}}$ to the residual action of $\mathbb{T}/\mathbb{T}_d^-$.*

(ii) *The action of the bending flows corresponds under $\Psi_n^{\mathcal{T}}$ (resp. $\Psi_{\mathbf{r}}^{\mathcal{T}}$) to the residual action of $\mathbb{T}/\mathbb{T}_e \times \mathbb{T}_d^-$.*

We may now give the edge rotations and the bending flows on $V_n^{\mathcal{T}}$ and $V_{\mathbf{r}}^{\mathcal{T}}$ a natural Hamiltonian interpretation. This is proved by the last theorem along with Proposition 8.13 and Theorem 8.16.

Theorem 1.11 (i) *The edge rotations on $V_n^{\mathcal{T}}$ are the stratified symplectic Hamiltonian flows (in the sense of* [SjL]*) associated with the lengths of edges in $\mathcal{T}$-congruence classes of spin-framed polygons.*

(ii) *The bending flows on $V_n^{\mathcal{T}}$ (resp. $V_{\mathbf{r}}^{\mathcal{T}}$) are the stratified Hamiltonian flows associated to the lengths of the diagonals of $\mathcal{T}$-congruence classes of spin-framed polygons (resp. polygonal linkages).*

2 The Moduli Spaces of n-gons and n-gon Linkages in $\mathbb{R}^3$

Throughout this paper the term n-gon will mean a closed n-gon in $\mathbb{R}^3$ modulo translations. More precisely, an n-gon $\mathbf{e}$ will be an n-tuple $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of vectors in $\mathbb{R}^3$ satisfying the closing condition

$$e_1 + e_2 + \cdots + e_n = 0.$$

We will say the e_i is the i-th edge of $\mathbf{e}$. We will say two n-gons $\mathbf{e}$ and $\mathbf{e}'$ are congruent if there exists a rotation $g \in \mathrm{SO}(3, \mathbb{R})$ such that

$$e_i' = ge_i, 1 \leq i \leq n.$$

We will let Pol_n denote the space of closed n-gons in $\mathbb{R}^3$ and $\overline{\mathrm{Pol}}_n$ denote the quotient space of n-gons modulo congruence.

Now let $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ be an n-tuple of nonnegative real numbers. We will say an n-gon $\mathbf{e}$ is an n-gon *linkage* with side-lengths $\mathbf{r}$ if the i-th edge of $\mathbf{e}$ has length $r_i, 1 \leq i \leq n$. We will say an n-gon or n-gon linkage is *degenerate* if it is contained in a line.

We define the configuration space $\widetilde{M_\mathbf{r}}$ to be the set of n–gon linkages with side-lengths $\mathbf{r}$. We will define the moduli space $M_\mathbf{r}$ of n-gon linkages to be the quotient of the configuration space by $\mathrm{SO}(3, \mathbb{R})$. The space $M_\mathbf{r}$ is a complex analytic space (see [KM]) with isolated singularities at the degenerate n-gon linkages.

Recall that we have defined a reference convex planar n-gon P. Let u_i, u_j be an ordered pair of nonconsecutive vertices of P. For any n-gon $\mathbf{e} \in \mathbb{R}^3$ we have corresponding vertices v_i and v_j defined up to simultaneous translation. The vector in $\mathbb{R}^3$ pointing from v_i to v_j will be called a *diagonal* of $\mathbf{e}$. We let $d_{ij}(\mathbf{e})$ be the length of this diagonal. In [KM] and [Kly] the authors described the Hamiltonian flow corresponding to $d_{ij}(\mathbf{e})$ (see the introduction). In [KM] these flows were called *bending flows*. Furthermore it was proved in [KM] and [Kly] if two such diagonals do not intersect, then the corresponding bending flows commute. Since each triangulation $\mathcal{T}$ of P contains $n - 3 = \frac{1}{2}\dim(M_\mathbf{r})$ nonintersecting diagonals, it follows that each one has an integrable system on $M_\mathbf{r}$ for each such $\mathcal{T}$. Unfortunately these flows are not everywhere defined. The bending flow corresponding to d_{ij} is not well defined for those $\mathbf{e}$ where $d_{ij}(\mathbf{e})$ is zero and the Hamiltonian d_{ij} is not differentiable at such $\mathbf{e}$.

3 The Space of Imploded Spin-framed Euclidean n-gons and the Grassmannian of Two Planes in Complex n Space

In this section we will construct the space $P_n(\mathrm{SU}(2))$ of imploded spin-framed n-gons in $\mathbb{R}^3$ modulo $\mathrm{SU}(2)$ and prove that this space is isomorphic to $\mathrm{AffGr}_2(\mathbb{C}^n)$ as a symplectic manifold and as a complex projective variety. We will first construct $P_n(\mathrm{SU}(2))$ using the extension and implosion technique of [HJ] without reference to n-gons in $\mathbb{R}^3$, then relate the result to $\mathrm{Gr}_2(\mathbb{C}^n)$. Then we will show that a point in $P_n(\mathrm{SU}(2))$ can be interpreted as a Euclidean n-gon equipped with an imploded spin-frame.

3.1 The Imploded Extended Moduli Spaces $P_n(G)$ of n-gons and n-gon Linkages

In this subsection we will define the imploded extended moduli space $P_n(G)$ of n-gons in $\mathfrak{g}^*$ for a general semisimple Lie group G. We have included this subsection to make the connection with [HJ]. Throughout we assume that G is semisimple.

3.1.1 The Moduli Spaces of n-gons and n-gon Linkages

To motivate the definition of the next subsection we briefly recall two definitions.

Definition 3.1 An n-gon in $\mathfrak{g}^*$ is an n-tuple of vectors $e_i, 1 \leq i \leq n, \in \mathfrak{g}^*$ satisfying the closing condition

$$e_1 + \cdots + e_n = 0.$$

We define the moduli space of n-gons to be the set of all n-gons modulo the diagonal coadjoint action of G.

Now choose n coadjoint orbits $\mathcal{O}_i, 1 \leq i \leq n$.

Definition 3.2 We define an n-gon *linkage* to be an n-gon such that $e_i \in \mathcal{O}_i, 1 \leq i \leq n$. We define the moduli space of n-gon linkages to be the set of all n-gon linkages modulo the diagonal coadjoint action of G.

We leave the proof of the following lemma to the reader.

Lemma 3.3 *The moduli space of n-gon linkages is the symplectic quotient*

$$G\backslash\backslash(\prod_{i=1}^{n} \mathcal{O}_i).$$

Remark 3.4 The moduli space of n-gons is in fact a moduli space of flat connections modulo gauge transformations (equivalently a character variety). The moduli space of n-gon linkages is a moduli space of flat connections with the conjugacy classes of holonomies fixed in advance (a relative character variety). In this case the flat connections are over an n-fold punctured 2-sphere, and the structure group is the cotangent bundle $T^*(G) = G \ltimes \mathfrak{g}^*$. The holonomy around each puncture is a "translation", *i.e.,* a group element of the form $(1, v), v \in \mathfrak{g}^*$. To see this connection in more detail the reader is referred to [KM, §5]. We will not need this connection in what follows.

3.1.2 The Extended Moduli Spaces of n-gons and n-gon Linkages

The following definition is motivated by the definition of the extended moduli spaces of flat connections of [J].

Definition 3.5 We define the extended moduli space $M_n(G)$ to be the symplectic quotient of $T^*(G)^n$ by the left diagonal action of G:

$$M_n(G) = G\backslash\backslash T^*(G)^n.$$

The space $M_n(G)$ has a G^n action coming from right multiplication on $T^*(G)$. We take each $T^*(G)$ to be identified with $G \times \mathfrak{g}^*$ using the left-invariant trivialization.

Lemma 3.6 $M_n(G) = G\backslash\{((g_1,\alpha_1),\ldots,(g_n,\alpha_n)) : \sum_{i=1}^n Ad_{g_i}(\alpha_i) = 0\}$.

Proof The momentum mapping associated with the diagonal left action on $T^*(G)^n$ (identified with $(G \times \mathfrak{g}^*)^n$ using the left-invariant trivialization) is

$$\mu_L\big((g_1,\alpha_1),\ldots,(g_n,\alpha_n)\big) = -\sum_{i=1}^{n} Ad_{g_i}(\alpha_i).$$

The expression on the right above clearly corresponds to the 0-momentum level of μ_L, which is $M_n(G)$. ■

Remark 3.7 The reason for the term *extended* moduli space of n-gons is that this space is obtained from the moduli space of n-gons by adding the frames $g_1, g_2, \ldots, g_n$. Note that the moduli space of n-gons is embedded in the extended moduli space as the subspace corresponding to $g_1 = g_2 = \cdots = g_n = e$. If we wish to fix the conjugacy classes of the second components, we will call the above the extended moduli space of n-gon *linkages*.

3.1.3 The Imploded Extended Moduli Spaces of n-gons and n-gon Linkages

Choose a maximal torus $T_G \subset G$ and a (closed) Weyl chamber Δ contained in the Lie algebra $\mathfrak{t}$ of T_G. Note that the action of G^n on $M_n(G)$ by right multiplication induces an action by the torus T_G^n on $M_n(G)$. We now obtain the imploded extended moduli space $P_n(G)$ by *imploding*, following [GJS], the extended moduli space $M_n(G)$.

Definition 3.8 $P_n(G) = M_n(G)_{\mathrm{impl}}$.

Following [HJ] we will use $\mathcal{E}T^*(G)$ to denote the imploded cotangent bundle

$$\mathcal{E}T^*(G) = T^*(G)_{\mathrm{impl}}.$$

For the benefit of the reader we will recall the definition of $\mathcal{E}T^*(G)$. We have

$$\mathcal{E}T^*(G) = \mu_G^{-1}(\Delta)/\sim$$

Here μ_G is the momentum map for the action of G by right translation and we have identified $\mathfrak{t}$ (resp. Δ) with $\mathfrak{t}^*$ (resp. a dual Weyl chamber) using the Killing form. The equivalence relation $\sim$ is described as follows. Let F be an open face of Δ and let h_F be a generic element of F. Let G_F be the subgroup of G that is the commutator subgroup of the stabilizer of h_F under the adjoint representation. We define x and y in $\mu_T^{-1}(\Delta)$ to be equivalent if $\mu_T(x)$ and $\mu(y)$ lie in the same face F of Δ and x and y are in the same orbit under G_F. Thus we divide out the inverse images of faces by different subgroups of G. In what follows we will use the symbol $\sim$ to denote this equivalence relation (assuming the group G, the torus T, and the chamber Δ are understood).

We define the space $E_n(G)$ by $E_n(G) = \mathcal{E}T^*(G)^n$. Noting that right implosion commutes with left symplectic quotient we have

$$P_n(G) = G\backslash E_n(G) =$$
$$G\backslash\big\{\, \big([g_1,\alpha_1],\ldots,[g_n,\alpha_n]\big) \;\big|\; \alpha_1,\ldots,\alpha_n \in \Delta, \sum_{i=1}^{n} Ad_{g_i}(\alpha_i) = 0 \big\},$$

where $[g_i,\alpha_i]$ denotes the equivalence class in $T^*(G)$ relative to the equivalence relation $\sim$ above. We will sometimes use $E_n(G)$ to denote the product $\mathcal{E}T^*(G)^n = \mathcal{E}T^*(G^n)$. In what follows we let $\mathbf{t}(\lambda)$ denote the element of the complexified maximal torus $\underline{T}_{\mathrm{SU}(2)^n}$.

3.2 The Isomorphism of $Q_n(\mathrm{SU}(2))$ and $\mathrm{Gr}_2(\mathbb{C}^n)$

In this subsection we will prove Theorem 1.2 by calculating $P_n(\mathrm{SU}(2))$, the imploded extended moduli space for the group SU(2) and its quotient $Q_n(\mathrm{SU}(2))$. We will in fact need a slightly more precise version than that stated in the Introduction.

Theorem 3.9 (i) *There exists a homeomorphism $\psi\colon \mathrm{AffGr}_2(\mathbb{C}^n) \to P_n(\mathrm{SU}(2))$.*

(ii) *The homeomorphism ψ intertwines the natural action of the maximal torus $T_{\mathrm{U}(n)}$ of $\mathrm{U}(n)$ with (the inverse of) that of $T^n_{\mathrm{SU}(2)}$ acting on imploded frames.*

(iii) *The homeomorphism ψ intertwines the grading circle (resp. $\mathbb{C}^*$) actions on $\mathrm{AffGr}_2(\mathbb{C}^n)$ with the actions of $\mathbf{t}((\exp i\theta)^{-1/2})$ (resp. $\mathbf{t}((\lambda)^{-1/2})$)*

(iv) *The homeomorphism ψ induces a homeomorphism between $\mathrm{Gr}_2(\mathbb{C}^n)$ and $Q_n(\mathrm{SU}(2))$.*

In order to prove the theorem we first need to compute $\mathcal{E}T^*(\mathrm{SU}(2))$.

3.2.1 The Imploded Cotangent Bundle of SU(2)

In this section we will review the formula of [GJS] for the (right) imploded cotangent bundle $\mathcal{E}T^*(\mathrm{SU}(2))$. Let $T_{\mathrm{SU}(2)}$ be the maximal torus of SU(2) consisting of the diagonal matrices. Let $\mathfrak{t}$ be the Lie algebra of $T_{\mathrm{SU}(2)}$ and let Δ be the positive Weyl chamber in $\mathfrak{t}$ (so Δ is a ray in the one-dimensional vector space $\mathfrak{t}$).

We will take as basis for $\mathfrak{t}$ the coroot $\alpha^\vee$ (multiplied by i), that is

$$\alpha^\vee = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then $\alpha^\vee$ may be identified with a basis vector over $\mathbb{Z}$ for the cocharacter lattice $X_*(T_{\mathrm{SU}(2)})$. It is tangent at the identity to a unique cocharacter. Let ϖ_1 be the fundamental weight of $\mathrm{SL}(2,\mathbb{C})$ thus $\varpi_1(\alpha^\vee) = 1$. Then ϖ_1 may be identifed with a character of T (it is the derivative at the identity of a unique character), which is a basis for the character lattice of $T_{\mathrm{SU}(2)}$.

Let μ_R be the momentum map for the action of SU(2) on $T^*\mathrm{SU}(2)$ induced by right multiplication. Specializing the definition of the imploded cotangent bundle to

$G = \mathrm{SU}(2)$, we find that the (right) implosion of $T^*\mathrm{SU}(2)$ is the set of equivalence classes

$$\mathcal{E}T^*(\mathrm{SU}(2)) = \mu_R^{-1}(\Delta)/\sim,$$

where two points $x, y \in \mu_R^{-1}(\Delta)$ are equivalent if

$$\mu_R(x) = \mu_R(y) = 0 \text{ and } x = y \cdot g \text{ for some } g \in \mathrm{SU}(2).$$

It follows from the general theory of [GJS] that $\mathcal{E}T^*(\mathrm{SU}(2))$ has an induced structure of a stratified symplectic space in the sense of [SjL] with induced isometric actions of SU(2) induced by the left action of SU(2) on $T^*\mathrm{SU}(2)$ and $T_{\mathrm{SU}(2)}$ induced by the right action of $T_{\mathrm{SU}(2)}$ on $T^*(\mathrm{SU}(2))$. However the exceptional feature of this special case is that $\mathcal{E}T^*(\mathrm{SU}(2))$ is a *manifold*. The multiplicative group $\mathbb{R}_+$ acts on $\mathcal{E}T^*(\mathrm{SU}(2))$ by the formula

$$\mu \cdot [g, \lambda\varpi_1] = [g, \mu\lambda\varpi_1],$$

whereas the actions of $g_0 \in \mathrm{SU}(2)$ and $t \in T_{\mathrm{SU}(2)}$ described above are given by

$$g_0 \cdot [g, \lambda\varpi_1] = [g_0 g, \lambda\varpi_1] \text{ and } t \cdot [g, \lambda\varpi_1] = [gt, \lambda\varpi_1].$$

The product group $\mathrm{SU}(2) \times \mathbb{R}_+$ fixes the point $[e, 0]$ and acts transitively on the complement of $[e, 0]$ in $\mathcal{E}T^*(\mathrm{SU}(2))$. The reader will verify, see [GJS, Example 4.7], that the symplectic form ω on $\mathcal{E}T^*(\mathrm{SU}(2))$ is homogeneous of degree one under $\mathbb{R}_+$.

The following lemma is extracted from [GJS, Example 4.7]. Since this lemma is central to what follows we will go into more detail for its proof than is in [GJS].

Lemma 3.10 *The map $\phi\colon \mathcal{E}T^*(\mathrm{SU}(2)) \to \mathbb{C}^2$ given by*

$$\phi([g, \lambda\varpi_1]) = \sqrt{2\lambda}\, g \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi([e, 0]) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

induces a symplectomorphism of stratified symplectic spaces onto $\mathbb{C}^2$, where $\mathbb{C}^2$ is given its standard symplectic structure. Under this isomorphism the action of SU(2) *on $\mathcal{E}T^*(\mathrm{SU}(2))$ goes to the standard action on $\mathbb{C}^2$, and the action of T on $\mathcal{E}T^*(\mathrm{SU}(2))$ goes to the action of T given by $t \cdot (z, w) = (t^{-1}z, t^{-1}w)$.*

Proof The inverse to ϕ is given (on nonzero elements of $\mathbb{C}^2$) by

$$\psi((z, w)) = \left[g, \frac{|z|^2 + |w|^2}{2}\varpi_1\right],$$

where

$$g = \frac{1}{\sqrt{|z|^2 + |w|^2}} \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}.$$

It follows that ϕ is a homeomorphism. It remains to check that it is a symplectomorphism. Noting that ϕ is homogeneous under $\mathbb{R}_+$ of degree one-half and the standard symplectic form $\omega_{\mathbb{C}^2}$ is homogeneous of degree two, it follows (from the transitivity of the action of $\mathrm{SU}(2) \times \mathbb{R}_+$) that the symplectic form $\phi^*\omega_{\mathbb{C}^2}$ is a constant multiple of ω. Thus it suffices to prove that the two above forms both take value 1 on the ordered pair of tangent vectors $(\partial/\partial\lambda, \alpha^\vee)$. We leave this to the reader. ∎

Remark 3.11 The fact that we have inverted the usual action of the circle on the complex plane will have the effect of changing the signs of the torus Hamiltonians that occur in the rest of this paper (from *minus* one-half a sum of squares of norms of complex numbers to *plus* one-half of the corresponding sum).

Taking into account the above remark we obtain a formula for the momentum map for the action of $T_{\mathrm{SU}(2)}$ on $\mathcal{E}T^*(\mathrm{SU}(2))$ that will be critical in what follows. Now we have

Lemma 3.12 *The momentum map μ_T for the action of $T_{\mathrm{SU}(2)}$ on $\mathcal{E}T^*(\mathrm{SU}(2)) = \mathbb{C}^2$ is given by*

$$\mu_T((z,w)) = (1/2)(|z|^2 + |w|^2)\varpi_1.$$

Consequently, the Hamiltonian $f_{\alpha^\vee}(z,w)$ for the fundamental vector field on $\mathcal{E}T^*(\mathrm{SU}(2))$ induced by $\alpha^\vee$ is given by

$$f_{\alpha^\vee}(z,w) = (1/2)(|z|^2 + |w|^2).$$

3.2.2 The Computation of $P_n(\mathrm{SU}(2))$ and $Q_n(\mathrm{SU}(2))$

We can now prove Theorem 3.9. Recall we have introduced the abbreviation $E_n(\mathrm{SU}(2))$ for the product $\mathcal{E}T^*(\mathrm{SU}(2))^n$. We will use $\mathbf{E}$ to denote an element of $E_n(\mathrm{SU}(2))$. It will be convenient to regard $\mathbf{E}$ as a function from the edges of the reference polygon P into $\mathcal{E}T^*(\mathrm{SU}(2))$. We note that there is a map from $E_n(\mathrm{SU}(2))$ to not necessarily closed n-gons in $\mathbb{R}^3$, the map that scales the first element of the frame by λ and forgets the other two elements of the frame this sends each equivalence class $[g_i, \lambda\varpi_1]$ to $Ad^* g_i(\lambda_i \varpi_1)$ for $1 \le i \le n$.

By the previous lemmas we may represent an element of $E_n(\mathrm{SU}(2))$ by the 2 by n matrix

$$A = \begin{pmatrix} z_1 & z_2 & \cdots & z_{n-1} & z_n \\ w_1 & w_2 & \cdots & w_{n-1} & w_n \end{pmatrix}.$$

The left diagonal action of the group $\mathrm{SU}(2)$ on $E_n(\mathrm{SU}(2))$ is then represented by the diagonal action of $\mathrm{SU}(2)$ on the columns of A. We let $[A]$ denote the orbit equivalence class of A for this action. We let $T_{\mathrm{U}(n)}$ be the compact n torus of diagonal elements in $\mathrm{U}(n)$). Then $T_{\mathrm{U}(n)}$ acts by scaling the columns of the matrices A. This action coincides with the inverse of the action of the n-torus $T_{\mathrm{SU}(2)^n}$ coming from the theory of the imploded cotangent bundle. By this we mean that the imploded action of an element $\mathbf{t} \in T_{U(n)}$ upon a matrix A is given by $\mathbf{t} \cdot A = A\mathbf{t}^{-1}$.

The first statement of Theorem 3.9 will be a consequence of the following two lemmas. Let $M_{2,n}(\mathbb{C})$ be the space of 2 by n complex matrices A as above. In what follows we will let Z denote the first row and W the second row of the above matrix A. We give $M_{2,n}(\mathbb{C})$ the Hermitian structure given by $(A,B) = Tr(AB^*)$. We will use μ_G to denote the momentum map for the action of $\mathrm{SU}(2)$ on the left of A.

Lemma 3.13

$$\mu_G(A) = (1/2)\begin{pmatrix} (\|Z\|^2 - \|W\|^2)/2 & Z \cdot W \\ W \cdot Z & (\|W\|^2 - \|Z\|^2)/2 \end{pmatrix}$$

Hence $\mu_G(A) = 0 \iff$ *the two rows of* A *have the same length and are orthogonal.*

Proof First note that the momentum map μ for the action of SU(2) on $\mathbb{C}^2$ is given by

$$\mu(z,w) = (1/2)\begin{pmatrix} (|z|^2 - |w|^2)/2 & z\overline{w} \\ w\overline{z} & (|w|^2 - |z|^2)/2 \end{pmatrix}.$$

Now add the momentum maps for each of the columns to get the lemma. ■

We find that

$$\mathrm{SU}(2)\backslash\backslash M_{2,n}(\mathbb{C}) = \mathrm{SU}(2)\backslash\{(Z,W) : \|Z\|^2 = \|W\|^2, Z\cdot W = 0\}.$$

Lemma 3.14 $\mathrm{SU}(2)\backslash\backslash M_{n,2}(\mathbb{C}) \cong \mathrm{AffGr}_2(\mathbb{C}^n)$.

Proof The map $\varphi\colon \mu^{-1}(0) \to \bigwedge^2(\mathbb{C}^n)$ given by $F(A) = Z \wedge W$ maps onto the decomposable vectors, descends to the quotient by SU(2), and induces the required isomorphism. ■

In what follows it will be important to understand the above result in terms of the GIT quotient of $M_{2,n}(\mathbb{C})$ by $\mathrm{SL}(2,\mathbb{C})$ acting on the left. Since we are taking a quotient of affine space by a reductive group the Geometric Invariant Theory quotient of $M_{2,n}(\mathbb{C})$ by $\mathrm{SL}(2,\mathbb{C})$ coincides with the symplectic quotient by SU(2) and consequently is $\mathrm{AffGr}_2(\mathbb{C}^n)$, [KN], see also [S, Theorem 4.2]. Note that since SU(2) is simple, the question of normalizing the momentum map does not arise. It will be important to understand this in terms of the subring $\mathbb{C}[M_{2,n}(\mathbb{C})]^{\mathrm{SL}(2,\mathbb{C})}$ of invariant polynomials on $M_{2,n}(\mathbb{C})$. In what follows let $Z_{ij} = Z_{ij}(A)$ or $[i,j]$ be the determinant of the 2 by 2 submatrix of A given by taking columns i and j of A (the ij-th Plücker coordinate or bracket). The following result is one of the basic results in invariant theory, see [Do, Chapter 2].

Proposition 3.15 *$\mathbb{C}[M_{2,n}(\mathbb{C})]^{\mathrm{SL}(2,\mathbb{C})}$ is generated by the Z_{ij} subject to the Plücker relations.*

But the above ring is the homogeneous coordinate ring of $\mathrm{Gr}_2(\mathbb{C}^n)$. This gives the required invariant-theoretic proof of Theorem 1.2.

Remark 3.16 In fact we could do *all* of the previous analysis in terms of invariant theory by using [GJS, Example 6.12] to replace the imploded cotangent bundle $\mathcal{E}T^*\mathrm{SU}(2) \cong \mathbb{C}^2$ by the quotient $\mathrm{SL}(2,\mathbb{C})//N \cong \mathbb{C}^2$, where N is the subgroup of $\mathrm{SL}(2,\mathbb{C})$ of strictly upper-triangular matrices. We leave the details to the reader.

We have now proved the first statement in Theorem 3.9 for both symplectic and GIT quotients. We note the consequence (since every element of $\bigwedge^2(\mathbb{C}^3)$ is decomposable):

$$P_3(\mathrm{SU}(2)) \cong \textstyle\bigwedge^2(\mathbb{C}^3). \tag{3.1}$$

It is clear that ψ is equivariant as claimed, and the second statement follows.

Since the grading action of $\lambda \in \mathbb{C}^*$ on $\mathrm{AffGr}_2(\mathbb{C}^n)$ is the action that scales each Plücker coordinate by λ the third statement is also clear. It remains to prove the fourth statement of Theorem 3.9. First recall the standard $\mathrm{U}(n)$ invariant positive definite Hermitian form $(\,\cdot\,,\,\cdot\,)$ on $\bigwedge^2(\mathbb{C}^n)$ is given on the bivector $a \wedge b$ by

$$(a \wedge b, a \wedge b) = det \begin{pmatrix} (a,a) & (a,b) \\ (b,a) & (b,b) \end{pmatrix}.$$

We can obtain $\mathrm{Gr}_2(\mathbb{C}^n)$ as a symplectic manifold the cone $\mathrm{AffGr}_2(\mathbb{C}^n)$ of decomposable bivectors by taking the subset of decomposable bivectors of norm squared R then dividing by the action of the circle by scalar multiplication (symplectic quotient by S^1 of level $\frac{R}{2}$). Thus we can find a manifold diffeomorphic to $\mathrm{Gr}_2(\mathbb{C}^n)$ by computing the pull-backs of the function $f(a \wedge b) = (a \wedge b, a \wedge b)$ and scalar multiplication of bivectors by the circle under the map φ. We represent elements of $P_n(\mathrm{SU}(2))$ by left $\mathrm{SU}(2)$ orbit-equivalence classes $[A]$ of 2 by n matrices A such that the rows Z and W of A are orthogonal and of the same length.

Lemma 3.17

$$\varphi^* f(A) = \left(\frac{\|Z\|^2 + \|W\|^2}{2} \right)^2.$$

Proof We have

$$\varphi^* f(A) = det \begin{pmatrix} (Z,Z) & (Z,W) \\ (W,Z) & (W,W) \end{pmatrix}.$$

But $(Z,Z) = (W,W)$ and $(Z,W) = (W,Z) = 0$. Hence

$$\varphi^* f(A) = \|Z\|^2 \|W\|^2 = \left(\frac{\|Z\|^2 + \|W\|^2}{2} \right) \left(\frac{\|Z\|^2 + \|W\|^2}{2} \right). \qquad \blacksquare$$

Now we ask the reader to look ahead and see that in Subsection 3.4 we define a map F_n from $P_n(\mathrm{SU}(2))$ to the space of n-gons in $\mathbb{R}^3$. Furthermore, $F_n([A])$ is the n-gon underlying the imploded spin-framed n-gon represented by $[A]$. Accordingly, we obtain the following corollary of the above lemma.

Corollary 3.18 *$\varphi^* f(A)$ is four times the square of the perimeter of the n-gon in $\mathbb{R}^3$ underlying the imploded spin-framed n-gon represented by $[A]$.*

Now observe that scaling the bivector $Z \wedge W$ by $\exp i\theta$ corresponds to scaling the class of the matrix $[A]$ by a square-root of $\exp i\theta$. Note that the choice of square-root is irrelevant, since $[-A] = [A]$. But scaling the columns of A corresponds to rotating the corresponding imploded spin-frames. Thus fixing the perimeter of the underlying n-gon to be equal to 1 corresponds to taking the symplectic quotient of the cone of decomposable bivectors by S^1 at level 2. We have now completed the proof of the second part of Theorem 1.2.

3.3 The Momentum Polytope for the $T_{\mathrm{U}(n)} = T_{\mathrm{SU}(2)^n}$ Action on $P_n(\mathrm{SU}(2))$

We have identified the momentum map for the action of SU(2) on $M_{2,n}(\mathbb{C})$. The torus $T_{\mathrm{U}(n)}$ acts on $M_{2,n}(\mathbb{C})$ by scaling the columns. We leave the proof of the following lemma to the reader. Let μ_{T_n} be the momentum map for the above action of $T_{\mathrm{U}(n)}$.

Lemma 3.19

$$\mu_{T_n}(A) = \left(\frac{\|C_1\|^2}{2}, \ldots, \frac{\|C_n\|^2}{2} \right).$$

The action of $T_{\mathrm{U}(n)}$ descends to the quotient $P_n(\mathrm{SU}(2))$ with the same momentum map (only now A must satisfy $\mu_G(A) = 0$). Thus we have a formula for the momentum map for the action of $T_{\mathrm{U}(n)}$ on the Grassmannian in terms of special representative matrices A (of SU(2)-momentum level zero). It will be important to extend this formula to all $A \in M_{2,n}(\mathbb{C})$ of rank two. We do this by relating the above formula to the usual formula for the momentum map of the action of $T_{\mathrm{U}(n)}$ on a general two plane in $\mathbb{C}^n$ represented by a general rank 2 matrix $A \in M_{2,n}(\mathbb{C})$. This momentum map is described in [GGMS, Proposition 2.1]. Let A be a 2 by n matrix of rank two and $[A]$ denote the two plane in $\mathbb{C}^n$ spanned by its columns. Then the i-th component of the momentum map of $T_{\mathrm{U}(n)}$ is given by

$$\mu_i([A]) = \frac{1}{\|A\|^2} \sum_{j, j \neq i} \|Z_{ij}(A)\|^2.$$

Here $\|A\|^2$ denotes the sum of the squares of the norms of the Plücker coordinates of $[A]$. The required extension formula is then implied by the following.

Lemma 3.20 *Suppose that $\mu_G(A) = 0$. Let C_i be the i-th column of A. Then*

$$\frac{\|C_i\|^2}{2} = \frac{1}{\|A\|^2} \sum_{j, j \neq i} \|Z_{ij}(A)\|^2.$$

Proof It suffices to prove the formula in the special case that $i = 1$. If the first row of A is zero, then both sides of the equation are zero. So assume the first row is not zero. Apply g to A so that the first row of Ag is of the form $(r, 0)$, where $r = \|R_1\|$. Let z_i', w_i' be the i-th row of Ag. Note that $\mu_G(Ag)$ is still zero, hence the length of the second column of Ag is equal to $(1/2)\|Ag\|^2 = (1/2)\|A\|^2$. Also $Z_{ij}(Ag) = Z_{ij}(A)$. Now compute the left-hand side of the equation. We have $Z_{1j}(A) = Z_{1j}(Ag) = rw_j', 2 \leq j \leq n$, and hence (noting that $w_1' = 0$) we have

$$\sum_{j, j > 1} \|Z_{1j}(A)\|^2 = r^2 \sum_{j=1}^{n} |w_j'|^2.$$

Note that the sum on the right-hand side is the length squared of the second column of Ag. The lemma follows. ∎

Let $D_n \subset \Delta^n$ be the cone defined by

$$D_n = \left\{ \mathbf{r} \in \Delta^n \;\middle|\; 2r_i \leq \sum_{j=1}^{n} r_j, 1 \leq i \leq n \right\}.$$

Then D_n is the cone on the hypersimplex Δ^2_{n-2} of [GGMS, §2.2]. Note that D_3 is the set of nonnegative real numbers satisfying the usual triangle inequalities. We can now apply Lemma 3.20 and the results of [GGMS, §2.2] to deduce the following.

Proposition 3.21 $\mu_{T_n}(P_n(\mathrm{SU}(2)) = D_n$.

We give another proof of this theorem in terms of the side-lengths inequalities for Euclidean n-gons in the next subsection.

3.4 The Relation Between Points of $P_n(\mathrm{SU}(2))$ and Imploded Spin-framed n-gons

The point of this subsection is to show that a point of $P_n(\mathrm{SU}(2))$ may be interpreted as an imploded spin-framed n-gon. Recall the definition of imploded spin-framed vector (and n-gon) from Section 1.1. The critical role in this interpretation is played by a map $F\colon \mathbb{C}^2 \to \mathbb{R}^3$. The map F is the Hopf map.

3.4.1 The Equivariant Map F from $\mathcal{E}T^*(\mathrm{SU}(2))$ to $\mathbb{R}^3$

In this section we define a map F from the imploded cotangent bundle $\mathcal{E}T^*(\mathrm{SU}(2)) \cong \mathbb{C}^2$ to $\mathbb{R}^3$. This map will give the critical connection between the previous constructions and polygonal linkages in $\mathbb{R}^3$.

We first define $p\colon \mathcal{E}T^*(\mathrm{SU}(2)) \to \mathfrak{su}^*(2)$ by $p([g, \lambda\varpi_1]) = Ad^*(g)(\lambda\varpi_1)$, where ϖ_1 denotes the fundamental weight of SU(2). We leave the following lemma to the reader.

Lemma 3.22 *The map p factors through the action of $T_{\mathrm{SU}(2)}$. The map p restricted to the complement of the point $[e, 0]$ is a principal $T_{\mathrm{SU}(2)}$ bundle (the Hopf bundle).*

We have that $\mathcal{E}T^*(\mathrm{SU}(2))$ is symplectomorphic to $\mathbb{C}^2$ and $\mathfrak{su}^*(2)$ is isomorphic as a Lie algebra to $\mathbb{R}^3$, and so the map p induces a map from $\mathbb{C}^2$ to $\mathbb{R}^3$. The induced map is the map F. We now give details.

Identify $\mathfrak{su}^*(2)$ with the traceless Hermitian matrices $\mathcal{H}^0_2$ via the pairing $\mathcal{H}^0_2 \times \mathfrak{su}(2) \to \mathbb{R}$ given by $\langle X, Y\rangle = \Im(\mathrm{tr}(XY))$. In this identification we have

$$\varpi_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

We recall there is a Lie algebra isomomorphism g from $\mathbb{R}^3$ to the traceless skew-Hermitian matrices given by

$$g(x_1, x_2, x_3) = \frac{1}{2}\begin{pmatrix} ix_1 & ix_2 - x_3 \\ ix_2 + x_3 & -ix_1 \end{pmatrix}.$$

The isomorphism f dual to g from the traceless Hermitian matrices $\mathcal{H}_2^0$ to $\mathbb{R}^3$ is given by

$$f\begin{pmatrix} x_1 & x_2+\sqrt{-1}x_3 \\ x_2-\sqrt{-1}x_3 & -x_1 \end{pmatrix} = (x_1, x_2, x_3).$$

Note that in particular, $f(\varpi_1) = (1/2, 0, 0)$.

In order to prove Theorem 3.31 we will need the following lemma. In what follows we identify the coadjoint action of SU(2) with its action (through its quotient $SO(3,\mathbb{R})$) on $\mathbb{R}^3$. Let $\mathcal{V}$ denote the space of imploded spin-framed vectors. Recall that

$$\mathcal{V} = \{(g,e) \in SU(2)\times\mathbb{R}^3 \mid (\exists t \geq 0)(e = t\pi(g)(\epsilon_1))\}/\sim\ ,$$

where $\epsilon_1 = (1,0,0)$, and $\pi\colon SU(2) \to SO(3,\mathbb{R})$ is the double cover. The equivalence relation $\sim$ is given by $(g_1, 0) \sim (g_2, 0)$ for all $g_1, g_2 \in SU(2)$. There is a natural projection map $\rho\colon \mathcal{V} \to \mathbb{R}^3$ given by $[g,e] \mapsto e$. The fiber over any nonzero $e \in \mathbb{R}^3$ is a circle, and we think of a point of the fiber $\rho^{-1}(e)$ is a *spin-frame* on e. The *frame* on e is the triple $(\pi(g)(1,0,0), \pi(g)(0,1,0), \pi(g)(0,0,1))$. This is a properly oriented triple of orthogonal unit vectors, where the first vector points in the direction of e. The map $[g,e] \mapsto (e, \pi(g)(1,0,0), \pi(g)(0,1,0), \pi(g)(0,0,1))$ is two to one, the extra data is "spin", and we call a point in the fiber over e a *spin-frame* on e. Now ρ restricted over $\mathbb{R}^3 - 0$ is the *spin-frame bundle* of $\mathbb{R}^3 - 0$.

Lemma 3.23 (i) *The map p restricted to the subset of $\mathcal{E}T^*(SU(2))$ with $\lambda \neq 0$ is the spin-frame bundle of $\mathbb{R}^3 - 0$.*

(ii) *The action of $T_{SU(2)}$ on $\mathcal{E}T^*(SU(2))$ preserves the set where $\lambda \neq 0$ and induces the circle action on the normal spin-frames.*

Proof We have a commutative diagram,

$$\begin{array}{ccc} \mathcal{E}T^*(SU(2)) & \xrightarrow{p} & \mathfrak{su}(2)^* \cong \mathcal{H}_2^0 \\ \big\downarrow & & \big\downarrow f \\ \mathcal{V} & \xrightarrow{\rho} & \mathbb{R}^3, \end{array}$$

where the left vertical arrow is defined by $[g, \lambda\varpi_1] \mapsto [g, \pi(g)(\lambda\epsilon_1/2)]$. This map is clearly an isomorphism. Part (ii) is trivial. ■

We now define the map F. Define $h\colon \mathbb{C}^2 \to \mathcal{H}_2$ as follows. Recall

$$\mu_{U(2)}(z,w) = 1/2\begin{pmatrix} z\bar{z} & w\bar{z} \\ z\bar{w} & w\bar{w} \end{pmatrix}.$$

Then define $h(z,w) = \mu_{U(2)}(z,w)^0$, where the superscript zero denotes traceless projection. Hence

$$h(z,w) = (1/4)\begin{pmatrix} z\bar{z}-w\bar{w} & 2w\bar{z} \\ 2z\bar{w} & w\bar{w}-z\bar{z} \end{pmatrix}.$$

We define F by $F = f \circ h$.

Remark 3.24 The map h is the momentum map for the action of SU(2) on $\mathbb{C}^2$ after we identify $\mathcal{H}_2^0$ with the dual of the Lie algebra of SU(2) using the imaginary part of the trace. Accordingly we will use the notations h and $\mu_{\mathrm{SU}(2)}$ interchangeably.

In the next lemma we note an important equivariance property of F. Recall $\pi\colon \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ is the double cover. We leave its proof to the reader.

Lemma 3.25 *Let $g \in \mathrm{SU}(2)$. Then $F \circ g = \pi(g) \circ F$.*

We next have the following.

Lemma 3.26 *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{E}T^*(\mathrm{SU}(2)) & \xrightarrow{p} & \mathfrak{su}(2)^* \\ \phi\big\downarrow & & \big\downarrow f \\ \mathbb{C}^2 & \xrightarrow{F} & \mathbb{R}^3. \end{array}$$

Proof First observe that $f \circ p$ and $F \circ \phi$ are homogeneous of degree one with respect to the $\mathbb{R}_+$ actions on their domain and range (note that ϕ is homogeneous of degree 1/2 and F is homogeneous of degree two). Also, both intertwine the SU(2) action on $\mathcal{E}T^*(\mathrm{SU}(2))$ with its action through the double cover on $\mathbb{R}^3$. Since the action of $\mathrm{SU}(2) \times \mathbb{R}_+$ on $\mathcal{E}T^*(\mathrm{SU}(2))$ is transitive away from $[e, 0]$, it suffices to prove the two above maps coincide at the point $[e, \varpi_1]$. But it is immediate that both maps take the value $(1/2, 0, 0)$ at this point. ■

Remark 3.27 We will need the following calculation:

$$F \circ \phi([g, \lambda\varpi_1]) = \lambda f(Ad_g^*(\varpi_1)).$$

In particular this implies that $\|F \circ \phi([g, \lambda\varpi_1])\| = \frac{1}{2}\lambda$.

The following lemma is a direct calculation.

Lemma 3.28 *The formula for $F\colon \mathbb{C}^2 \to \mathbb{R}^3$ in the usual coordinates is*

$$F(z, w) = \tfrac{1}{4}\left(z\bar{z} - w\bar{w},\ 2\Re(w\bar{z}),\ 2\Im(w\bar{z})\right).$$

Consequently, the Euclidean length of the vector $F(z, w) \in \mathbb{R}^3$ is given by

$$\|F(z, w)\| = \tfrac{1}{4}(|z|^2 + |w|^2).$$

Corollary 3.29 *Note that the length of F is related to the Hamiltonians $f_{\alpha^\vee}(z, w)$ for the infinitesimal action of $\mathfrak{t}$ by*

$$\|F(z, w)\| = (1/2)f_{\alpha^\vee}(z, w).$$

Also, by the above remark if $\phi([g, \lambda\varpi_1]) = (z, w)$, then $\lambda = f_{\alpha^\vee}(z, w)$.

Later we will need the following determination of the fibers of F.

Lemma 3.30

$$F(z_1, w_1) = F(z_2, w_2) \iff z_1 = cz_2 \textit{ and } w_1 = cw_2 \textit{ with } |c| = 1.$$

Proof The implication $\Leftarrow$ is immediate. We prove the reverse implication. Thus we are assuming the equations

$$|z_1|^2 - |w_1|^2 = |z_2|^2 - |w_2|^2, \quad z_1\bar{w}_1 = z_2\bar{w}_2$$

Square each side of the first equation. Take four times the norm squared of each side of the second equation and adding the resulting equation to the new first equation to obtain

$$(|z_1|^2 + |w_1|^2)^2 = (|z_2|^2 + |w_2|^2)^2.$$

Hence $|z_1|^2 + |w_1|^2 = |z_2|^2 + |w_2|^2$ and consequently

$$|z_1|^2 = |z_2|^2 \text{ and } |w_1|^2 = |w_2|^2.$$

Now it is an elementary version of the first fundamental theorem of invariant theory that if we are given two ordered pairs of vectors in the plane so that the lengths of corresponding vectors are equal and the symplectic inner products between the two vectors in each pair coincide, then there is an element in SO(2) that carries one ordered pair to the other (this is one definition of the oriented angle between two vectors). ∎

We now construct the required map to Euclidean n-gons. We define a map

$$F_n \colon P_n(\mathrm{SU}(2)) \to (\mathbb{R}^3)^n/\mathrm{SO}(3, \mathbb{R})$$

by defining $F_n(A)$ to be the orbit of $(F(C_1), \dots, F(C_n))$ under the diagonal action of $\mathrm{SO}(3, \mathbb{R})$. Here C_i is the i-th column of A. Let $\mathrm{Pol}_n(\mathbb{R}^3)$ denote the space of closed n-gons in $\mathbb{R}^3$.

Theorem 3.31

(i) *F_n induces a homeomorphism from $P_n(\mathrm{SU}(2))/T_{\mathrm{SU}(2)^n}$ onto* $\mathrm{Pol}_n(\mathbb{R}^3)/\mathrm{SO}(3, \mathbb{R})$.
(ii) *The fiber of F_n over a Euclidean n-gon is naturally homeomorphic to the set of imploded spin framings of the edges of that n-gon.*
(iii) *The side-lengths of $F_n(A)$ are related to the norm squared of the columns of A by*

$$\|e_i(F_n(A))\| = \frac{\|C_i\|^2}{4} = \frac{|z_i|^2 + |w_i|^2}{4}.$$

Here $e_i(F_n(A)) = F_n(C_i)$ is the i-th edge of any n-gon in the congruence class represented by $F_n(A)$.

Proof We first prove that if A is in the zero level set of μ_G, then $F_n(A)$ is a *closed* n-gon. Recall C_i, $1 \leq i \leq n$ is the i-th column of A. Then we have

$$F_n(A) = (f((C_1C_1^*)^0), \dots, f((C_nC_n^*)^0)).$$

Hence the sum s of the edges $F_n(A)$ is given by

$$s = f((C_1C_1^* + \cdots + C_nC_n^*)^0).$$

Now it is a formula in elementary matrix multiplication that we have

$$AA^* = C_1C_1^* + \cdots + C_nC_n^*.$$

But by Lemma 3.13 we find that AA^* is scalar, whence $(AA^*)^0 = 0$ and

$$s = f((AA^*)^0) = f(0) = 0.$$

We next prove that F_n is onto. It is clear that F_n maps $M_{n,2}(\mathbb{C})$ onto $(\mathbb{R}^3)^n$ (this may be proved one column at a time). Let $\mathbf{e} = (e_1, \dots, e_n)$ be the edges of a closed n-gon in $\mathbb{R}^3$. Choose $A \in M_{2,n}(\mathbb{C})$ such that $F_n(A) = \mathbf{e}$. But by the above

$$f((AA^*)^0) = e_1 + \cdots + e_n = 0.$$

Hence $(AA^*)^0 = 0$, whence AA^* is scalar and $\mu_G(A) = 0$. We now prove that F_n is injective. Suppose there exists $g \in \mathrm{SU}(2)$ such that $F_n(A) = \rho(g)F_n(A')$. Then $F(A) = F(gA')$. Hence by Lemma 3.30 we have $A = gA't$ for some $t \in T_{\mathrm{U}(n)}$.

The second statement follows because the action of $T_{\mathrm{U}(n)}$ corresponds to a transitive action on the set of imploded spin framings. Indeed using the identification above between $\mathbb{R}^3$ and $\mathfrak{su}(2)^*$, we may replace F_n by the map $\pi_n\colon \mathcal{E}T^*(\mathrm{SU}(2))^n \to (\mathfrak{su}(2)^*)^n$ given by

$$\pi_n([g_1, \lambda_1\varpi_1], \dots, [g_n, \lambda_n\varpi_i]) = (Ad^*g_1(\lambda_1\varpi_1), \dots, Ad^*g_n(\lambda_n\varpi_1)).$$

If no λ_i is zero by Lemma 3.23 the (classes of) the n-tuple $[g_1, \lambda_1\varpi_1], \dots, [g_n, \lambda_n\varpi_1]$ represent the imploded spin-frames over the n vectors in the image. Then by Lemmas 3.22 and 3.23 two such n-tuples correspond to frames over the same image n-gon if and only the two n-tuples are related by right multiplication by $t_1, \dots, t_n$ with t_i fixing ϖ_1 for all i. Hence if no λ_i is zero, then the two n-tuples are related as above if and only if they are in the same $T_{\mathrm{U}(n)}$ orbit by definition of the $T_{\mathrm{U}(n)}$ action. Since $T_{\mathrm{U}(n)}$ acts transitively on the spin-frames over a given n-gon, we see that in this case the fiber of π_n is the set of spin-frames over the image n-gon and the second statement is proved. If some subset of the λ_i's is zero, then we replace the corresponding g_i's by the identity (this does not change the imploded frame) and proceed as above with the remaining components.

The last statement in the theorem is Lemma 3.28. ∎

Let $\sigma\colon \mathrm{Pol}_n(\mathbb{R}^3) \to \mathbb{R}_+^n$ be the map that assigns to a closed n-gon the lengths of its sides. It is standard (see for example the introduction of [KM]) that the image of σ is the polyhedral cone D_n. We can now give another proof of Proposition 3.21 based on Euclidean geometry. We restate it for the convenience of the reader.

Proposition 3.32 $\mu_{T_n}(P_n(\mathrm{SU}(2))) = D_n$.

Proof The proposition follows from the commutative diagram

$$\begin{array}{ccc} P_n(\mathrm{SU}(2)) & \xrightarrow{\mu_{T_n}} & \mathbb{R}_+^n \\ {\scriptstyle F_n}\downarrow & & \downarrow{\scriptstyle I} \\ \mathrm{Pol}_n(\mathbb{R}^3) & \xrightarrow{\sigma} & \mathbb{R}_+^n. \end{array}$$

■

3.5 The Space $P_n(\mathrm{SO}(3,\mathbb{R})$

We now compute $P_n(\mathrm{SO}(3,\mathbb{R}))$ using the fact that $\mathrm{SO}(3,\mathbb{R})$ is covered by $\mathrm{SU}(2)$. Let the semisimple Lie Group $\bar{G}$ be a quotient of the semisimple Lie Group G by a finite group Γ

$$\Gamma \to G \xrightarrow{\pi} \bar{G}.$$

In [GJS, Example 4.7], Guillemin, Jeffrey, and Sjamaar show that this quotient gives a description of the homeomorphism $\phi\colon \mathcal{E}(T^*G)/\Gamma \to \mathcal{E}(T^*\overline{G}))$. The following lemma is left to the reader to prove.

Lemma 3.33 *The homeomorphism ϕ induces a homeomorphism from $\Gamma\backslash\backslash P_n(G)$ to $P_n(\overline{G})$.*

Theorem 3.34 *The double cover $\phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ induces a homeomorphism from $H\backslash\backslash P_n(\mathrm{SU}(2))$ to $P_n(\mathrm{SO}(3,\mathbb{R}))$. Here H is the finite 2-group $Z(\mathrm{SU}(2))^n$, and $Z(\mathrm{SU}(2)) \cong \mathbb{Z}/2$ is the center of $\mathrm{SU}(2)$.*

We have the following analogue of Theorem 3.31. We leave its proof to the reader. We now have orthogonal (imploded) framings instead of imploded spin framings. We note that the map F_n above descends to a map

$$\overline{F}_n\colon P_n(\mathrm{SO}(3,\mathbb{R})) \to \mathrm{Pol}_n(\mathbb{R}^3)/\mathrm{SO}(3,\mathbb{R}).$$

Theorem 3.35 (i) *$\overline{F}_n$ induces a homeomorphism from $P_n(\mathrm{SO}(3,\mathbb{R}))/T_{\mathrm{SO}(3,\mathbb{R})^n}$ onto the polygon space $\mathrm{Pol}_n(\mathbb{R}^3)/\mathrm{SO}(3,\mathbb{R})$.*

(ii) *The fiber of $\overline{F}_n$ over a Euclidean n-gon of the induced map from $P_n(\mathrm{SU}(2))$ onto $\mathrm{Pol}_n(\mathbb{R}^3)/\mathrm{SO}(3,\mathbb{R})$ is naturally homeomorphic to the set of imploded orthogonal framings of the edges of that n-gon.*

4 Toric Degenerations Associated with Trivalent Trees

Suppose one has a planar regular n-gon subdivided into triangles. We call this a triangulation of the n-gon. The dual graph is a tree with n leaves and $n-2$ internal trivalent nodes (see Figure 4.1). We will see below how the tree $\mathcal{T}$ determines a Gröbner degeneration of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ to a toric variety. These toric degenerations first appeared in [SpSt].

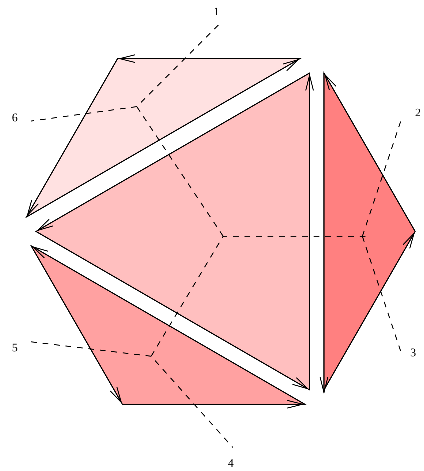

Figure 4.1: A triangulated hexagon with associated dual tree.

4.1 Toric Degenerations of $\mathrm{Gr}_2(\mathbb{C}^n)$

Recall that the standard coordinate ring $R = \mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)]$ (for the Plücker embedding) of the Grassmannian is generated by $Z_{i,j}$, for $1 \leq i < j \leq n$, subject to the quadric relations, $Z_{i,j}Z_{k,l} - Z_{i,k}Z_{j,l} + Z_{i,l}Z_{j,k} = 0$ for $1 \leq i < j < k < l \leq n$. These relations generate the Plücker ideal $I_{2,n}$. For each pair of indices i, j let $w^{\mathcal{T}}_{i,j}$ denote the length of the unique path in $\mathcal{T}$ joining leaf i to leaf j. For example, in Figure 4.1 we have $w^{\mathcal{T}}_{1,4} = 4$. To any monomial $m = \prod_k Z_{i_k,j_k}$ we assign a weight $w^{\mathcal{T}}(m) = \sum_k w^{\mathcal{T}}_{i_k,j_k}$. Let $I^{\mathcal{T}}_{2,n}$ denote the initial ideal with respect to the weighting $w^{\mathcal{T}}$.

It is a standard result in the theory of Gröbner degenerations that one has a flat degeneration of $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)] = \mathbb{C}[\{Z_{i,j}\}_{i<j}]/I_{2,n}$ to $\mathbb{C}[\{Z_{i,j}\}_{i<j}]/I^{\mathcal{T}}_{2,n}$. Below we outline how this works using the Rees algebra.

The weight $w^{\mathcal{T}}$ induces an (increasing) filtration on the ring of the Grassmannian. Let $F^{\mathcal{T}}_m$ be the vector subspace of R spanned by monomials of weight at most m. Then, for any elements $x \in F^{\mathcal{T}}_{m_1}$ and $y \in F^{\mathcal{T}}_{m_2}$, the product xy belongs to $F^{\mathcal{T}}_{m_1+m_2}$. Let $R^{\mathcal{T}}$ denote the associated graded ring $R^{\mathcal{T}} = \bigoplus_{m=0}^{\infty} F^{\mathcal{T}}_m / F^{\mathcal{T}}_{m-1}$, where $F^{\mathcal{T}}_{-1} := 0$. The Rees algebra $\mathcal{R}^{\mathcal{T}}$ is given by, $\mathcal{R}^{\mathcal{T}} = \bigoplus_{m=0}^{\infty} t^m F^{\mathcal{T}}_m$, where t is an indeterminate. Then (cf. [AB]) $\mathcal{R}^{\mathcal{T}}$ is flat over $\mathbb{C}[t]$,

$$\mathcal{R}^{\mathcal{T}} \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong R[t, t^{-1}], \quad \text{and} \quad \mathcal{R}^{\mathcal{T}} \otimes_{\mathbb{C}[t]} (\mathbb{C}[t]/(t)) \cong R^{\mathcal{T}}.$$

Since the ring R is already graded, there is a natural grading of the Rees algebra $\mathcal{R}^{\mathcal{T}}$, where the degree of the indeterminate variable t is defined to be zero, and in general, the degree of a product $t^m x$, where $x \in R^{(k)} \cap F^{\mathcal{T}}_m$, is equal to $\deg(x) = k$. Now, the specializations $\mathcal{R}^{\mathcal{T}} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t-a)$ (at $t = a$) are all graded algebras, where $\deg(y \otimes \overline{p(t)}) = \deg(y)$ for $y \in \mathcal{R}^{\mathcal{T}}$. In particular the associated graded algebra $R^{\mathcal{T}}$ is bigraded; one grading comes from the grading of R, and the other from the filtration of R. We have that $\mathrm{Proj}(\mathcal{R}^{\mathcal{T}})$ has a flat morphism to the affine line

$\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[t])$ with all fibers projective varieties, with general fiber isomorphic to $\mathrm{Gr}_2(\mathbb{C}^n)$ and special fiber $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ at $t = 0$.

Definition 4.1 We shall reserve a special name for the case where the triangulation is such that all diagonals contain the initial vertex, forming a fan. The associated tree has the appearance of a caterpillar and was called such in [SpSt]. We shall call this triangulation the "fan" and the associated degeneration the LG-degeneration, since the special fiber in this case agrees with the special fiber of the degeneration of $\mathrm{Gr}_2(\mathbb{C}^n)$ given by Lakshmibai and Gonciulea in [LG].

Now we will see why the special fiber $\mathcal{R} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t)$ is toric. This was also proved in [SpSt] where these degenerations were first discovered. First we establish a basis for $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)]$ as a complex vector space. Suppose that the vertices $1, 2, \ldots, n$ of a multigraph (multiple edges allowed between two vertices) are drawn on the unit circle, in cyclic clockwise order, and all edges are drawn as straight line segments (a chord of the circle). Then, if no two edges cross (it is okay for an edge to have multiplicity greater than one), we call the graph a Kempe graph, in honor of the work of A. Kempe in 1894 (see [Ke]). For an example, see Figure 4.2. In fact two edges ij and kl cross exactly when $i < k < j < l$, assuming that $i < k$.

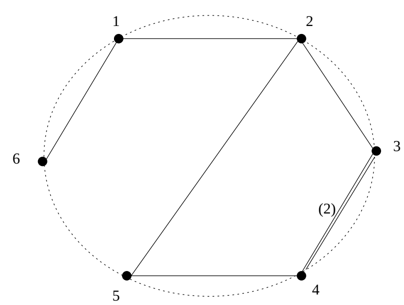

Figure 4.2: A Kempe graph for $n = 6$.

We assign a monomial m_G to each graph G, given by

$$m_G = \prod_{ij \in E(G)} Z_{i,j},$$

where $E(G)$ is the multi-set of edges of G. For a graph G, let $\deg(G)$ be the number of edges in G, which is one half of the total sum of valencies of the all the vertices. Note that $\deg(G) = \deg(m_G)$. The proof of the proposition and theorem below appears in [HMSV].

Proposition 4.2 *The monomials m_G as G runs over the set of all Kempe graphs with vertex set $V(G) = \{1, 2, \ldots, n\}$ form a $\mathbb{C}$-basis for $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)]$. Furthermore, the m_G for which* $\deg(G) = k$ *and G is a Kempe graph form a basis for the k-th graded piece of R.*

Theorem 4.3 *Suppose that G_1 and G_2 are Kempe graphs. Suppose that*

$$m_{G_1} m_{G_2} = \sum_{G \text{ is Kempe}} c_G m_G.$$

Then there is a unique Kempe graph G^* such that

- $c_{G^*} = 1$,
- $w^{\mathcal{T}}(m_{G^*}) = w^{\mathcal{T}}(m_{G_1}) + w^{\mathcal{T}}(m_{G_2})$,
- *for all G, if $c_G \neq 0$ and $G \neq G^*$, then $w^{\mathcal{T}}(m_G) < w^{\mathcal{T}}(m_{G^*})$.*

Proof Suppose that $i_1 j_1$ is an edge of G_1 and $i_2 j_2$ is an edge of G_2 such that $i_1 j_1$ and $i_2 j_2$ cross. Without loss of generality we can assume that $i_1 < i_2 < j_1 < j_2$. We have the Plücker relation $Z_{i_1,j_1} Z_{i_2,j_2} = Z_{i_1,i_2} Z_{j_1,j_2} + Z_{i_1,j_2} Z_{i_2,j_1}$, and the latter two terms (viewed as graphs having two edges) are Kempe graphs, since neither pair of edges cross. Denote $G_1 \cdot G_2$ as the graph with edge set $E(G_1) \coprod E(G_2)$. Let $G_0 = (G_1 \cdot G_2)$ with the two edges $i_1 j_1$ and $i_2 j_2$ removed. We have

$$m_{G_1} m_{G_2} = m_{G_0 \cdot i_1 j_1 \cdot i_2 j_2} = m_{G_0 \cdot i_1 i_2 \cdot j_1 j_2} + m_{G_0 \cdot i_1 j_2 \cdot i_2 j_1}.$$

We will show that one of the two monomials on the right-hand side of the above equation has the same $\mathcal{T}$-weight ($w^{\mathcal{T}}$) as does $m_{G_1} m_{G_2}$ and the other term has strictly smaller weight. After sufficiently many Plücker relations as above are applied we get $m_{G_1} m_{G_2}$ expressed as a unique integral combination of Kempe graphs with a unique term of maximal weight equal to the weight of $m_{G_1} m_{G_2}$.

Let $\gamma(i, j)$ denote the shortest path in $\mathcal{T}$ joining i to j. Note that the paths $\gamma(i_1, j_1)$ and $\gamma(i_2, j_2)$ must intersect one another, since they cross when drawn as straight line segments in the graph $G_1 \cdot G_2$. Now consider the two pairs of paths in $\mathcal{T}$:

- $\gamma(i_1, i_2)$ and $\gamma(j_1, j_2)$.
- $\gamma(i_1, j_2)$ and $\gamma(i_2, j_1)$.

Exactly one of the above two above pairs of paths cover precisely the same set of edges in $\mathcal{T}$ as does the crossing pair $(\gamma(i_1, j_1),\gamma(i_2, j_2))$. Say for example it is the pair $(\gamma(i_1, i_2), \gamma(j_1, j_2))$. Then $w^{\mathcal{T}}(Z_{i_1,i_2} Z_{j_1,j_2}) = w^{\mathcal{T}}(Z_{i_1,j_1} Z_{i_2,j_2})$. Furthermore, the two paths $\gamma(i_1, i_2)$ and $\gamma(j_1, j_2)$ must meet one another within the tree $\mathcal{T}$ along some internal edges $e_1, \ldots, e_k$ (although they are non-crossing when drawn as straight line segments). The edges $e_1, \ldots, e_k$ are also the edges common to the two paths $\gamma(i_1, j_1)$ and $\gamma(i_2, j_2)$. The third pair of paths $(\gamma(i_1, j_2), \gamma(i_2, j_1))$ covers the same set of edges as do the first two pairs of paths, excepting the edges $e_1, \ldots, e_k$ above. In particular, we have that

$$w^{\mathcal{T}}(Z_{i_1,i_2} Z_{j_1,j_2}) = w^{\mathcal{T}}(Z_{i_1,j_1} Z_{i_2,j_2}) = w^{\mathcal{T}}(Z_{i_1,j_2} Z_{i_2,j_1}) + 2k. \qquad \blacksquare$$

Definition 4.4 Let $G_1 *_{\mathcal{T}} G_2$ denote the Kempe graph G^* from the above theorem. Let $\mathcal{S}_n^{\mathcal{T}}$ denote the graded commutative semigroup of Kempe graphs G with binary operation $(G_1, G_2) \mapsto G_1 *_{\mathcal{T}} G_2$. The grading is given by $\deg(G) = \deg(m_G)$.

Corollary 4.5 (to Theorem 4.3) *The special fiber at $t = 0$ of the degeneration, $\mathcal{R} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t)$, is isomorphic to the semigroup algebra $\mathbb{C}[\mathcal{S}_n^{\mathcal{T}}]$.*

Given any graph G, there is an associated weighting w_G of the tree $\mathcal{T}$ where the weight assigned to an edge e of $\mathcal{T}$ is equal to the number of edges ij in G such that the path $\gamma(i, j)$ in $\mathcal{T}$ joining i to j passes through e. We denote this weight by $w_G(e)$. We now determine the image of the map $G \mapsto w_G$.

Proposition 4.6 *Given a weighting w of nonnegative integers to the edges of $\mathcal{T}$, then $w = w_G$ for some graph G if and only if for each triple e_1, e_2, e_3 of edges meeting at a common (internal) vertex of $\mathcal{T}$ the sum $w(e_1)+w(e_2)+w(e_3)$ is even, and $w(e_1), w(e_2), w(e_3)$ satisfy the triangle inequalities.*

Proof Let a G-path mean a path in the tree $\mathcal{T}$ joining vertices i to j, where ij is an edge of G. Thus G-paths are in bijection with edges of G, and are meant to be counted with multiplicity. Now fix an internal vertex v_0 of $\mathcal{T}$ with neighboring vertices v_1, v_2, v_3 with connecting respective edges e_1, e_2, e_3. For each i, j with $1 \leq i < j \leq 3$, let x_{ij} be the number of G-paths passing through each of v_i and v_j. Thus

- $w_G(e_1) = x_{12} + x_{13}$.
- $w_G(e_2) = x_{12} + x_{23}$.
- $w_G(e_3) = x_{13} + x_{23}$.

Now solving for the x_{ij} we have:

- $x_{12} = \frac{1}{2}(w_G(e_1) + w_G(e_2) - w_G(e_3))$.
- $x_{13} = \frac{1}{2}(w_G(e_1) + w_G(e_3) - w_G(e_2))$.
- $x_{23} = \frac{1}{2}(w_G(e_2) + w_G(e_3) - w_G(e_1))$.

Since the x_{ij} are non-negative we have that $w_G(e_1)$, $w_G(e_2)$, and $w_G(e_3)$ satisfy the triangle inequalities. Since the x_{ij} are integers we have that $w_G(e_1) + w_G(e_2) + w_G(e_3)$ is even.

Conversely, given a weighting of $\mathcal{T}$ satisfying the above conditions, we will get non-negative integers $x_{12}(v), x_{13}(v), x_{23}(v)$ at each internal vertex v. We may thus define a graph G_v on the three neighboring vertices v_1, v_2, v_3 by setting the multiplicity of edge ij to be $x_{ij}(v)$. If v' is a neighboring internal vertex, say for example sharing edge e_1 with v, then we have $x_{12}(v) + x_{13}(v) = x_{12}(v') + x_{13}(v') = w(e_1)$. Hence we may glue together G_v and $G_{v'}$ to form a Kempe graph $G_{v,v'}$ on the neighboring vertices of v and v' by adjoining the edges whose respective paths go through the edge vv' of $\mathcal{T}$. Note that the weighting of the edges of the tree pertaining to $G_{v,v'}$ is equal to w. Continuing in this way, we may form a Kempe graph G on the leaves of $\mathcal{T}$ such that $w = w_G$, by gluing together all the G_v for v and internal vertex of $\mathcal{T}$. ∎

Definition 4.7 We shall call a weighting w or $\mathcal{T}$ an admissible weighting if $w = w_G$ for some graph G.

Proposition 4.8 *For each admissible weighting w of $\mathcal{T}$, there exists a unique Kempe graph G such that $w = w_G$. Furthermore, if G_1 and G_2 are Kempe graphs, then $w_{G_1} + w_{G_2} = w_{G_1 *_{\mathcal{T}} G_2}$.*

Proof Certainly the paths in the tripods of the proof of Proposition 4.6 may be drawn so that they are non-crossing. Such a non-crossing tripod graph is unique.

In the process of gluing all these tripod graphs together to form a graph G such that $w_G = w$, it is clear there is only one way to join the paths so that no two paths are non-crossing. Now by straightening these non-crossing paths into line segments, we see that they remain non-crossing and so form a Kempe graph.

A quick inspection of the proof of Theorem 4.3 reveals that the graph $G_1 *_{\mathcal{T}} G_2$ has weight equal to the sum $w_{G_1} + w_{G_2}$. ■

Note that $w^{\mathcal{T}}(G) = \sum_{ij \in E(G)} w^{\mathcal{T}}_{i,j} = \sum_{e \in E(\mathcal{T})} w_G(e)$. We define the degree of an admissible weighting $w^{\mathcal{T}}$ to be one -half the sum of the weights of the leaf edges (a leaf edge is an edge incident to a leaf). Since each edge of the graph G contributes a weight of one to two leaf edges we have $\deg(m_G) = \deg(G) = \deg(w^{\mathcal{T}})$.

Definition 4.9 Define $\mathcal{W}^{\mathcal{T}}_n$ to be the graded semigroup of admissible weightings of the edges of $\mathcal{T}$.

Proposition 4.10 *The map that associates the induced weighting of the edges of $\mathcal{T}$ with an element of $\mathcal{S}^{\mathcal{T}}_n$ induces an isomorphism $\Omega_n \colon \mathcal{S}^{\mathcal{T}}_n \to \mathcal{W}^{\mathcal{T}}_n$.*

4.2 Induced Toric Degeneration of the Space $M_{\mathbf{r}}$ of Polygonal Linkages

Here we briefly review the construction of the GIT quotients of the Grassmannian by the maximal torus T in $\mathrm{SL}(n, \mathbb{C})$. The GIT quotient (or technically, the subspace of closed points in the GIT quotient) is homeomorphic to a configuration space $M_{\mathbf{r}}$ of polygonal linkages in $\mathbb{R}^3$ with prescribed integral side lengths $\mathbf{r} = (r_1, \ldots, r_n)$. See [KM, Theorem 2.3] or [Kly] for more details.

Let T be the torus of diagonal matrices in $\mathrm{SL}(n, \mathbb{C})$. There is a natural action of T on the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$. The GIT quotient $\mathrm{Gr}_2(\mathbb{C}^n)//T$ depends upon the choice of a T-linearized line bundle of $\mathrm{Gr}_2(\mathbb{C}^n)$. Let L be the line bundle associated to the Plücker embedding of $\mathrm{Gr}_2(\mathbb{C}^n)$. Let $P \subset \mathrm{SL}(n, \mathbb{C})$ be the parabolic subgroup fixing the point $e_1 \wedge e_2 \in \mathrm{Gr}_2(\mathbb{C}^n)$. It is the subgroup of matrices $[a_{ij}]_{1 \le i,j \le n}$ of determinant one, such that $a_{ij} = 0$ for $i = 1, 2$ and $j > 2$. We identify $\mathrm{Gr}_2(\mathbb{C}^n)$ with the homogeneous space $\mathrm{SL}(n, \mathbb{C})/P$, by identifying $gP \in \mathrm{SL}(n, \mathbb{C})/P$ with $g \cdot (e_1 \wedge e_2) \in \mathrm{Gr}_2(\mathbb{C}^n)$. Let $\varpi_2 = (1, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ be the second fundamental weight of $\mathrm{SL}(n, C)$. Associated to ϖ_2 is a character $\chi \colon P \to \mathbb{C}^*$ given by

$$\chi([a_{ij}]_{1 \le i,j \le n}) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Following the Borel-Weil construction, we may take the total space of L to be the product $\mathrm{SL}(n, \mathbb{C}) \times \mathbb{C}$ modulo the equivalence relation $(g, z) \sim (gp, \chi(p)z)$, for all $g \in \mathrm{SL}(n, \mathbb{C})$, $p \in P$, and $z \in \mathbb{C}$. We denote the equivalence class of (g, z) as above by $[g, z]$. The bundle map $\pi \colon L \to \mathrm{Gr}_2(\mathbb{C}^n)$ is given by $[g, z] \mapsto gP$. The Plucker coordinate ring of $\mathrm{Gr}_2(\mathbb{C}^n)$ is now $\bigoplus_{N=0}^{\infty} \Gamma(\mathrm{Gr}_2(\mathbb{C}^n), L^{\otimes N})$, and it is generated in degree one by the brackets $[i, j]$ which are associated to global sections s_{ij} of L, by $s_{ij}(gP) := [g, g_{i1}g_{j2} - g_{i2}g_{j1}]$, where $g = [g_{kl}]_{1 \le k,l \le n}$.

We suppose that $|\mathbf{r}| = r_1 + \cdots + r_n$ is an even integer. The line bundle $L^{\otimes |\mathbf{r}|/2}$ of L may be identified with the product $\mathrm{SL}(n, \mathbb{C}) \times \mathbb{C}$ modulo the relation $(g, z) \sim$

$(gp, \chi(p)^{\otimes|\mathbf{r}|/2} z)$. We define an action of T on $L^{\otimes|\mathbf{r}|/2}$ via the character $\chi_{\mathbf{r}}$ of T, given by $\chi_{\mathbf{r}}(t_1, \ldots, t_n) := \prod_{i=1}^n t_i^{r_i}$. We define the action of T on $L^{\otimes|\mathbf{r}|/2}$ by

$$(t_1, \ldots, t_n) \cdot [g, z] = [\mathbf{t} \cdot g, \chi_{\mathbf{r}}(\mathbf{t}) z],$$

where $\mathbf{t} = (t_1, \ldots, t_n) \in T$. We call this the $\mathbf{r}$–linearization of $L^{\otimes|\mathbf{r}|/2}$.

The space $M_{\mathbf{r}}$ is homeomorphic to

$$\mathrm{Gr}_2(\mathbb{C}^n)//_{\mathbf{r}} T = \mathrm{Projm}\Big(\bigoplus_{N=0}^{\infty} \Gamma\big(\mathrm{Gr}_2(\mathbb{C}^n), L^{\otimes N|\mathbf{r}|/2} \big)^T \Big),$$

where $\Gamma(\mathrm{Gr}_2(\mathbb{C}^n), L^{\otimes N|\mathbf{r}|/2})^T$ indicates the T-invariant global sections of $L^{\otimes N|\mathbf{r}|/2}$. The ring

$$R_{\mathbf{r}} = \bigoplus_{N=0}^{\infty} \Gamma(\mathrm{Gr}_2(\mathbb{C}^n), L^{\otimes N|\mathbf{r}|/2})^T$$

is naturally graded by N. The invariant sections of $L^{\otimes N|\mathbf{r}|/2}$ are spanned by monomials $m = m_G$ over all graphs G having multi-degree $N\mathbf{r}$, *i.e.,* the valency of vertex i is r_i for each i. The degree of m_G is then N, if G has multidegree $N\mathbf{r}$.

Restricting to torus invariants is an exact functor so the flat degenerations of the Grassmannian described above restrict to flat degenerations of $\mathrm{Gr}_2(\mathbb{C}^n)//_{\mathbf{r}} T$. Furthermore, the special fiber of this restricted degeneration is toric, since it is the T-quotient of a toric variety. Therefore we obtain a flat degeneration of $\mathrm{Gr}_2(\mathbb{C}^n)//_{\mathbf{r}} T$ to a toric variety $(\mathrm{Gr}_2(\mathbb{C}^n)//_{\mathbf{r}} T)^{\mathcal{T}}$ for each triangulation $\mathcal{T}$ of the model n-gon. The associated semigroup $\mathcal{S}_{\mathbf{r}}^{\mathcal{T}}$ is the set of Kempe graphs having valency a multiple of $\mathbf{r}$; it is a sub-semigroup of $\mathcal{S}_n^{\mathcal{T}}$, however the grading of $\mathcal{S}_{\mathbf{r}}^{\mathcal{T}}$ is not the same as that of $\mathcal{S}_n^{\mathcal{T}}$. Instead the degree of a Kempe graph $G \in \mathcal{S}_{\mathbf{r}}^{\mathcal{T}}$ of multidegree $N\mathbf{r}$ is N, rather than $N|\mathbf{r}|/2$ as it would have been in $\mathcal{S}_n^{\mathcal{T}}$. We have that

$$(\mathrm{Gr}_2(\mathbb{C}^n)//_{\mathbf{r}} T)^{\mathcal{T}} = \mathrm{Projm}(\mathbb{C}[S_{\mathbf{r}}^{\mathcal{T}}]),$$

with $\mathcal{S}_{\mathbf{r}}^{\mathcal{T}}$ graded as described above.

Definition 4.11 Define the graded semigroup $\mathcal{W}_{\mathbf{r}}^{\mathcal{T}}$ to be the graded subsemigroup of $\mathcal{W}_n^{\mathcal{T}}$ with leaf-edge weights that are integral multiples of $\mathbf{r}$.

The admissible weightings of the tree $\mathcal{T}$ relating to the sub-semigroup $\mathcal{S}_{\mathbf{r}}^{\mathcal{T}}$ must satisfy that the weighting of the outer edges $e_1, \ldots, e_n$ (the edge e_i is adjacent to leaf i) is some multiple N of $\mathbf{r}$, meaning that $w(e_i) = N r_i$ for each i, and the multiple N is the degree of the weighting. Thus we have the following.

Proposition 4.12 *The isomorphism Ω_n of Proposition* 4.10 *induces an isomorphism $\Omega_{\mathbf{r}} \colon \mathcal{S}_{\mathbf{r}}^{\mathcal{T}} \to \mathcal{W}_{\mathbf{r}}^{\mathcal{T}}$.*

5 $\mathcal{T}$-Congruence of Polygons and Polygonal Linkages

In this section we will define an equivalence relation on polygons and polygonal linkages that depends on the choice of a trivalent tree $\mathcal{T}$. First we will collect some results about trivalent trees that will be useful in what follows.

5.1 Trivalent Trees and their Decompositions into Forests

Let $\mathcal{T}$ be a trivalent tree with n leaves that we assume is dual to a triangulation of P. We will say a vertex is *internal* if it is not a leaf. The triangles in the triangulation of P correspond to the internal vertices of $\mathcal{T}$. We will say an edge is a *leaf edge* or an *outer edge* if it is incident to a leaf. Thus the leaf edges are dual to the edges of P. An edge of $\mathcal{T}$ that does not border a leaf will be called an *inner edge*. Thus the inner edges of $\mathcal{T}$ are dual to the diagonals of the triangulation of P.

Definition 5.1 We say two leaves of $\mathcal{T}$ are a matched pair of leaves if they have a common neighbor.

The following technical lemma will be very useful for giving inductive proofs.

Lemma 5.2 *For any trivalent tree $\mathcal{T}$ it is possible to find a sequence of subtrees*

$$\mathcal{T}_0 \subset \cdots \subset \mathcal{T}_{n-3} = \mathcal{T}$$

such that

- *the tree $\mathcal{T}_0$ is a tripod;*
- *the tree $\mathcal{T}_i$ can be identified with $\mathcal{T}_{i-1}$ joined with a tripod along some e_i;*
- *each internal edge of $\mathcal{T}$ appears as some e_i.*

Proof Let $\mathcal{T}$ be a trivalent tree. Let $\mathcal{T}'$ be the (not necessarily trivalent) subtree of $\mathcal{T}$ with vertex set the internal vertices of $\mathcal{T}$ and edge set equal to all edges of $\mathcal{T}$ not connected to leaves. It is easy to see that $\mathcal{T}'$ is also a tree and therefore has a leaf. Let $n \in \mathcal{T}$ be the vertex corresponding to this leaf, then n is trivalent by definition and therefore must be connected to two leaves in $\mathcal{T}$. This shows that $\mathcal{T}$ must have a vertex v connected to two leaf edges; v is connected to a third edge, e. By splitting $\mathcal{T}$ along e we obtain a tripod and a new trivalent tree. The three items above now follow by induction. ∎

For each tree $\mathcal{T}$ we choose once and for all an ordering on the trivalent vertices of $\mathcal{T}$ as follows. Pick any matched pair of leaves and label their trivalent vertex τ_{n-2} with n the number of leaves of $\mathcal{T}$. Then, repeat this proceedure for the tree given by removing τ_{n-2} and its matched pair. Note that this ordering is not unique.

We now discuss the decompositions of $\mathcal{T}$ obtained by splitting certain internal edges. We first describe the decompositions of P that will induce the required decompositions of $\mathcal{T}$. For a diagonal d in the triangulation $\mathcal{T}$, we define the operation of splitting P along d by removing a small tubular neighborhood of d in P. This results in a disjoint union of two triangulated polygons $P' \cup P'' = P^{\{d\}}$. We may generalize this operation by performing the same operation for any subset S of the set of diagonals of P to obtain a union of triangulated polygons P^S. The dual graph to P^S is a forest $\mathcal{T}^S$ that may be obtained by removing a small open interval from each edge corresponding to a diagonal in S. If we choose S to be the entire set of diagonals, we obtain the decomposition P^D of P into triangles and $\mathcal{T}$ into the forest $\mathcal{T}^D$ of trinodes.

It will be important in what follows that we have the quotient map $\pi^S\colon \mathcal{T}^S \to \mathcal{T}$ that glues together the pairs of vertices that are the boundaries of the open intervals removed from $\mathcal{T}$. We will say two vertices of $\mathcal{T}^S$ are equivalent if they have the same image under π^S, and two edges of $\mathcal{T}^S$ are equivalent if they contain equivalent vertices.

There are two types of leaf edges of $\mathcal{T}^S$. The first type corresponds to edges of P (leaf edges of $\mathcal{T}$), and the second corresponds to (split) diagonals of P. The n edges of $\mathcal{T}^S$ corresponding to leaf edges of $\mathcal{T}$ will from now on be referred to as distinguished edges; we denote this set $\mathrm{dist}(\mathcal{T}^S)$. For a connected component C of $\mathcal{T}^S$ we let $\mathrm{dist}(C) = \mathrm{dist}(\mathcal{T}^S) \cap C$. Note that these components uniquely partition the set $\mathrm{dist}(\mathcal{T}^S) = \bigcup_{C_i \subset \mathcal{T}^S} \mathrm{dist}(C_i)$.

5.2 $\mathcal{T}$-Congruence of Polygonal Linkages

Fix a triangulation of the model n-gon with dual tree $\mathcal{T}$. Recall $\mathrm{Pol}_n(\mathbb{R}^3)$, the space of n-gons in $\mathbb{R}^3$. A polygon $\mathbf{e} \in \mathrm{Pol}_n(\mathbb{R}^3)$ comes with a fixed ordering on its edges. These ordered edges are in bijection with the leaves of the tree $\mathcal{T}$. A set of diagonals S corresponds to a set of internal edges of $\mathcal{T}$. We know that such a set defines a unique partition of the edges of $\mathbf{e}$, $E(\mathbf{e}) = E_1(\mathbf{e}) \cup \cdots \cup E_m(\mathbf{e})$ given by grouping together the distinguished edges of $\mathcal{T}^S$ that lie in a component tree of the forest $\mathcal{T}^S$.

Proposition 5.3 *For $\mathbf{e} \in \mathrm{Pol}_n(\mathbb{R}^3)$, if all diagonals in the set S are zero, then the edge sets $E_i(\mathbf{e})$ define closed polygons.*

Proof Split the polygon P along S to obtain a union of polygons. Choose a polygon P_i in the union. Then P_i is dual to a component C_i of the forest $\mathcal{T}^S$. The edges of the polygon P_i are either edges of the original polygon (so distinguished leaves of the tree C_i) and hence edges of the Euclidean polygon $\mathbf{e}$ or diagonals of S and hence zero diagonals of the Euclidean polygon $\mathbf{e}$. We assume that we have chosen i consistently with the division of the distinguished edges above whence the set of distinguished edges of C_i equals the set $E_i(\mathbf{e})$. Now since P_i closes up, the sum of all the vectors in $\mathbb{R}^3$ associated with the edges of P_i is zero. But this sum is the sum of the vectors in $\mathbb{R}^3$ associated with the distinguished edges of C_i (that is the elements in $E_i(\mathbf{e})$) and a set of vectors all of which are zero. ∎

The following groups will be useful in defining structures on spaces of polygons.

Definition 5.4 Let $G^{\mathrm{dist}(\mathcal{T}^S)}$ be the group of maps from the set $\mathrm{dist}(\mathcal{T}^S)$ into a group G. Define $G^{D(S)}$ to be the subgroup of $G^{\mathrm{dist}(\mathcal{T}^S)}$ of maps that are constant along the distinguished edges of each component C of $\mathcal{T}^S$

Notice that $G^{D(S)}$ naturally splits as a product of G over the components of $\mathcal{T}^S$. Let $G^{D(C)}$ be the component corresponding to the component C. The sets S of zero diagonals define a filtration of the space $\mathrm{Pol}_n(\mathbb{R}^3)$ where the subspace $\mathrm{Pol}_n(\mathbb{R}^3)^S$ in the filtration is defined to be the set of all points $\mathbf{e}$ such that the diagonals in S have zero length. This in turn defines a decomposition of $\mathrm{Pol}_n(\mathbb{R}^3)$ into subspaces

$$\mathrm{Pol}_n(\mathbb{R}^3)^{[S]} = \mathrm{Pol}_n(\mathbb{R}^3)^S \setminus \cup_{S \subset J} \mathrm{Pol}_n(\mathbb{R}^3)^J.$$

The subspace $\mathrm{Pol}_n(\mathbb{R}^3)^{[S]}$ is the collection of all points in $\mathrm{Pol}_n(\mathbb{R}^3)$ such that exactly the diagonals in S have zero length. This also induces a filtration on $M_{\mathbf{r}}$.

Definition 5.5 Define an action of $SO(3,\mathbb{R})^{D(S)}$ on $\mathrm{Pol}_n(\mathbb{R}^3)^{[S]}$ by letting $SO(3,\mathbb{R})^{D(C_i)}$ act diagonally on the edges in $E_i(\mathbf{e})$. The equivalence relation given by $SO(3,\mathbb{R})^{D(S)}$ orbit type on $\mathrm{Pol}_n(\mathbb{R}^3)^{[S]}$ fit together to give an equivalence relation $\sim_{\mathcal{T}}$, which we call $\mathcal{T}$-congruence, on Pol_n.

We may describe the above equivalence relation geometrically as follows. A polygon $\mathbf{e} \in \mathrm{Pol}_n(\mathbb{R}^3)^{[S]}$ is a wedge of $|S|+1$ closed polygons $\mathbf{e}_i$ wedged together at certain vertices of the polygon $\mathbf{e}$. Although each $\mathbf{e}_i$ may contain several wedge points, since the vertices of $\mathbf{e}$ are ordered, there will be a first wedge point v_i. Then we apply a rotation g_i about v_i to each $\mathbf{e}_i$ for $1 \le i \le |S|+1$ and identify points in the resulting orbit of $SO(3,\mathbb{R})^{D(S)}$.

$\mathcal{T}$-congruence for $\mathrm{Pol}_n(\mathbb{R}^3)$ induces an equivalence relation on the space of n-gon linkages $\widetilde{M}_{\mathbf{r}} \subset \mathrm{Pol}_n(\mathbb{R}^3)$, which we also call $\mathcal{T}$-congruence.

Definition 5.6 Define $V_{\mathbf{r}}^{\mathcal{T}} = \widetilde{M}_{\mathbf{r}}/\sim_{\mathcal{T}}$.

Kamiyama and Yoshida studied the space $V_{\mathbf{r}}^{\mathcal{T}}$ for the special case when $\mathcal{T}$ was the *fan* tree. Note that $V_{\mathbf{r}}^{\mathcal{T}}$ inherits a filtration by the subspaces $(V_{\mathbf{r}}^{\mathcal{T}})^S$. Let $(V_{\mathbf{r}}^{\mathcal{T}})^{[S]} = (V_{\mathbf{r}}^{\mathcal{T}})^S \setminus \cup_{S\subset J}(V_{\mathbf{r}}^{\mathcal{T}})^J$. The spaces $(V_{\mathbf{r}}^{\mathcal{T}})^{[S]}$ define a decomposition of $V_{\mathbf{r}}^{\mathcal{T}}$. We can say more about the pieces of this decomposition.

Theorem 5.7 *$(V_{\mathbf{r}}^{\mathcal{T}})^{[S]}$ is canonically homeomorphic to $\prod_{C_i\subset\mathcal{T}^S} M_{\mathbf{r}_i}^o$, where $\mathbf{r}_i$ is the subvector of linkage lengths corresponding to the elements in* $\mathrm{dist}(C_i)$, *and $M_{\mathbf{r}_i}^o$ is the dense open subset of $M_{\mathbf{r}_i}$ corresponding to polygons with no zero diagonals.*

Proof Let us first describe a map $\tilde{F}\colon M_{\mathbf{r}}^S \to V_{\mathbf{r}_1} \times \cdots \times V_{\mathbf{r}_k}$. The diagonals S define a partition of edges sets $E_i(\mathbf{e})$ for each $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}^S$. By the above proposition, $E_i(\mathbf{e})$ corresponds to a closed polygon. So we may send a member of the equivalence class of $\mathbf{e}$ to the product of the equivalence classes defining these closed polygons in $V_{\mathbf{r}_1} \times \cdots \times V_{\mathbf{r}_k}$, with the appropriate $\mathbf{r}_i$. This map is certainly onto, and by the definition of $V_{\mathbf{r}_i}$, it factors through the relation $\sim \mathcal{T}$. Hence, we get a 1-1 and onto continuous function $F\colon (V_{\mathbf{r}}^{\mathcal{T}})^S \to V_{\mathbf{r}_1}\times\cdots\times V_{\mathbf{r}_k}$, which is a homeomorphism by the compactness of $V_{\mathbf{r}_1} \times \cdots \times V_{\mathbf{r}_k}$.

We may restrict this map to the spaces $(V_{\mathbf{r}}^{\mathcal{T}})^{[S]}$ which define the induced decomposition of $V_{\mathbf{r}}^{\mathcal{T}}$. Recall that these are the polygons with exactly the S diagonals zero. We therefore obtain that $(V_{\mathbf{r}}^{\mathcal{T}})^{[S]}$ is homeomorphic to $M_{\mathbf{r}_1}^o \times \cdots \times M_{\mathbf{r}_k}^o$. ■

Remark 5.8 Since the fibers of $\pi\colon M_{\mathbf{r}} \to V_{\mathbf{r}}^{\mathcal{T}}$ are sometimes odd-dimensional, the quotient map π cannot be algebraic even when $\mathbf{r}$ is integral.

5.3 $\mathcal{T}$-Congruence of Imploded Spin-framed Polygons

In this section we introduce a generalization of the $\mathcal{T}$-congruence relation by lifting $\mathcal{T}$-congruence from $\mathrm{Pol}_n(\mathbb{R}^3)$ to $P_n(SU(2))$. In Section 3 we constructed a map $F_n\colon P_n(\mathrm{SU}(2)) \to \mathrm{Pol}_n(\mathbb{R}^3)/\mathrm{SO}(3,\mathbb{R})$. Pulling back the decomposition of $\mathrm{Pol}_n(\mathbb{R}^3)/\mathrm{SO}(3,\mathbb{R})$ into $\mathcal{T}$-congruence classes by F_n produces a decomposition of $P_n(\mathrm{SU}(2))$ by the spaces we will denote $P_n(\mathrm{SU}(2))^{[S]} = F_n^{-1}(\mathrm{Pol}_n(\mathbb{R}^3)^{[S]})$.

Lemma 5.9 *Elements of $P_n(\mathrm{SU}(2))^{[S]}$ are the spin-framed polygons in $P_n(\mathrm{SU}(2))$ such that the following equation holds,*

$$\sum \lambda_i Ad^*_{g_i}(\varpi_1) = 0$$

where the sum is over all $[g_i, \lambda_i\varpi] \in \mathbf{E}_j(\mathbf{e})$, the edges in the j-th partition defined by S, for each j.

Proof This follows from Theorem 3.31 and Remark 3.27. ∎

By the previous lemma we have a decomposition of $\mathbf{e} \in P_n(\mathrm{SU}(2))^{[S]}$ into imploded spin-framed polygons

$$\mathbf{E} = \mathbf{E}_1(\mathbf{e}) \cup \cdots \cup \mathbf{E}_{|S|+1}(\mathbf{e}).$$

We define an action of $\mathrm{SU}(2)^{D(S)}$ on $P_n(\mathrm{SU}(2))^{[S]}$ by letting $\mathrm{SU}(2)^{D(C)}$ act diagonally on the framed edges associated with the component C of $\mathcal{T}^S$.

Definition 5.10 The $\mathrm{SU}(2)^{D(S)}$-orbits on $P_n(\mathrm{SU}(2))^{[S]}$ fit together to give an equivalence relation $\sim_{\mathcal{T}}$, which we again call $\mathcal{T}$-congruence, on $P_n(\mathrm{SU}(2))$.

Definition 5.11 $V_n^{\mathcal{T}} = P_n(\mathrm{SU}(2))/\sim_{\mathcal{T}}$

Note that this defines a decomposition on $V_n^{\mathcal{T}}$ into subspaces

$$(V_n^{\mathcal{T}})^{[S]} = P_n(\mathrm{SU}(2))^{[S]}/\mathrm{SU}(2)^{D(S)}.$$

We have seen that $P_n(\mathrm{SU}(2))$ carries an action of $T_{\mathrm{SU}(2)^n}$ given by the following formula. Let $\mathbf{t} = (t_1, \ldots, t_n) \in T_{\mathrm{SU}(2)^n}$,

$$\mathbf{t} \circ ([g_1, \lambda_1\varpi_1, \ldots, [g_n, \lambda_n\varpi_1) = ([g_1t_1, \lambda_1\varpi_1], \ldots, [g_nt_n, \lambda_n\varpi_1).$$

In particular, note that the action of the diagonal (-1) element in $T_{\mathrm{SU}(2)^n}$ is trivial, because this corresponds to acting on the left by the nontrivial central $\mathrm{SU}(2)$ element. Since t_i fixes ϖ_1 for $1 \le i \le n$, this action does not change the image of a point under F_n, hence it fixes each piece of the decomposition. Also this action commutes with the $\mathrm{SU}(2)^{D(S)}$-action on the piece $P_n(\mathrm{SU}(2))^{[S]}$, so the action of $T_{\mathrm{SU}(2)^n}$ must descend to $V_n^{\mathcal{T}}$.

6 The Toric Varieties $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ and $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$

In this section we will construct the affine toric variety $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ of imploded triangulated $\mathrm{SU}(2)$-framed n-gons (without imposing the condition that the side-lengths are $\mathbf{r}$) and its projective quotient $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$.

We have tried to follow the notation of [HJ] when possible. In [HJ] the superscript D on P^D refers to a "pair of pants" decomposition D of the surface. For us superscript $\mathcal{T}$ on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ refers to the triangulation of the n-gon. The connection is the following. Under the correspondence between moduli spaces of n-gons and character varieties briefly explained in Remark 3.4 (and explained in detail in [KM, §5]) the standard triangulation corresponds to the following "pair of pants" decomposition of the n times punctured two-sphere. Represent the sphere as the complex plane with a point at infinity. Take the punctures to be the points $1, 2, \ldots, n$ on the real line. Draw small circles around the punctures. Now draw n-3 more circles with centers on the x-axis, so that the first circle contains the small circles around 1 and 2; the next circle contains the circle just drawn and the small circle around 3, and the last circle contains all the previous circles except the small circles around $n-1$ and n. The graph dual to the pair of pants decomposition is the tree $\mathcal{T}$ that is dual to the triangulation. See Figure 6.1. Furthermore, the decomposed tree $\mathcal{T}^D$ is dual to the pair-of-pants decomposition of the n-punctured sphere obtained by cutting the sphere apart along the above $2n-3$ circles. We might say that $\mathcal{T}^D$ is the pair-of-pants decomposition of $\mathcal{T}$. Using this correspondence the reader should be able to relate what follows with [HJ] for the case of the n-fold punctured sphere.

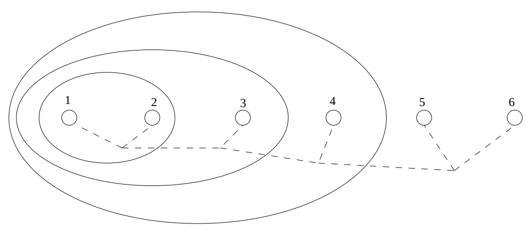

Figure 6.1: The pair of pants decomposition dual to the standard triangulation, for $n = 6$.

It will be important in what follows that we have earlier defined the quotient map $\pi^D\colon \mathcal{T}^D \to \mathcal{T}$ that glues together the pairs of vertices that are the boundaries of the open intervals removed from $\mathcal{T}$.

6.1 The Space $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ of Imploded Framed Edges

We define $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ to be the product of $\mathcal{E}T^*(\mathrm{SU}(2))$ over the edges of $\mathcal{T}^D$, hence an element $\mathbf{T} \in E_n^{\mathcal{T}}(\mathrm{SU}(2))$ is a map from the $3(n-2)$ edges of $\mathcal{T}^D$ into $\mathcal{E}T^*(\mathrm{SU}(2))$ or equivalently an assignment of an element of $\mathcal{E}T^*(\mathrm{SU}(2))$ to each each of the $3(n-2)$ edges of the triangles τ_i, $1 \leq n-2$ in the triangulation of P. It will be important later to note that there is a forgetful map that restricts $\mathbf{T}$ to the distinguished edges of $\mathcal{T}^D$ to obtain an element $\mathbf{E}$ of $E_n(\mathrm{SU}(2))$.

It will be convenient to represent the resulting product $\mathcal{E}T^*\mathrm{SU}(2)^{3n-6} \cong (\mathbb{C}^2)^{3n-6}$ by a 2 by $3n-6$ matrix. To do this we will linearly order the $3n-6$ edges by first ordering the $n-2$ triangles (tripods) and then ordering the 3 edges for each triangle (tripod). Now use the lexicographic ordering on pairs $(triangle, edge)$ $((tripod, edge))$. The point is that in the matrix $A^{\mathcal{T}}$ below the $(\mathbb{C}^2)$'s belonging to the same triangle (tripod) appear consecutively. Let (τ_i, j) denote the j-th edge of the i-th tripod.

We will use the following notation for the entries in the matrix $A^{\mathcal{T}}$ to indicate that each successive group of 3 columns of $A^{\mathcal{T}}$ belongs to a triangle in the triangulation.

$$A^{\mathcal{T}} = \begin{pmatrix} z_1(\tau_1) & z_2(\tau_1) & z_3(\tau_1) & z_1(\tau_2) & z_2(\tau_2) & z_3(\tau_2) & \cdots \\ w_1(\tau_1) & w_2(\tau_1) & w_3(\tau_1) & w_1(\tau_2) & w_2(\tau_2) & w_3(\tau_2) & \cdots \end{pmatrix.$$
$$\begin{matrix} \cdots & z_1(\tau_{n-2}) & z_2(\tau_{n-2}) & z_3(\tau_{n-2}) \\ \cdots & w_1(\tau_{n-2}) & w_2(\tau_{n-2}) & w_3(\tau_{n-2}) \end{pmatrix}.$$

In what follows we will abbreviate $\mathcal{E}T^*(\mathrm{SU}(2))^{3n-6}$ to $E_n^{\mathcal{T}}(\mathrm{SU}(2))$.

6.2 The Space $X_n^{\mathcal{T}}(\mathrm{SU}(2))$ of Imploded Framed Triangles

The action of the group $\mathrm{SU}(2)^{3n-6}$ on $\mathcal{E}T^*(\mathrm{SU}(2))^{3n-6}$ is then represented by acting on the columns of $A^{\mathcal{T}}$. We will represent elements $\mathbf{g}$ of $\mathrm{SU}(2)^{3n-6}$ as $3n-6$-tuples

$$\mathbf{g} = \left(f_1(\tau_1), f_2(\tau_1), f_3(\tau_1) | \cdots | f_1(\tau_{n-2}), , f_2(\tau_{n-2}), f_3(\tau_{n-2}) \right).$$

We let $\mathrm{SU}(2)^{n-2}$ denote the "diagonal" subgroup of $\mathrm{SU}(2)^{3n-6}$ defined by the condition that for each triangle T_i (tripod τ_i) we have $f_1(\tau_i) = f_2(\tau_i) = f_3(\tau_i)$. We will regard $\mathrm{SU}(2)^{n-2}$ as the space of mappings $\mathbf{f}$ from the tripods in $\mathcal{T}^D$ to $\mathrm{SU}(2)$.

Definition 6.1 We then define the space $X_n^{\mathcal{T}}(\mathrm{SU}(2))$ as the symplectic quotient

$$X_n^{\mathcal{T}}(\mathrm{SU}(2)) = \mathrm{SU}(2)^{n-2} \backslash\backslash (\mathcal{E}T^*\mathrm{SU}(2))^{3n-6}.$$

Thus $X_n^{\mathcal{T}}(\mathrm{SU}(2))$ is obtained by taking the symplectic quotient of each triple of edges belonging to one of the triangles T_i, $1 \le i \le n-2$, by the group $\mathrm{SU}(2)^{n-2}$. The resulting space $X_n(\mathrm{SU}(2))$ is clearly the product of $n-2$ copies of $(P_3(\mathrm{SU}(2)))^{n-2} \cong (\bigwedge^2(\mathbb{C}^3))^{n-2}$.

We will often denote an element of $X_n^{\mathcal{T}}(\mathrm{SU}(2))$ by $\mathbf{T} = ([T_1], [T_2], \dots, [T_{n-2}])$, where $[T_i]$, the i-th triangle denotes the equivalence class of the matrix

$$A_{\tau_i} = \begin{pmatrix} z_1(\tau_i) & z_2(\tau_i) & z_3(\tau_i) \\ w_1(\tau_i) & w_2(\tau_i) & w_3(\tau_i) \end{pmatrix}$$

such that the momentum image of A_i under the momentum map for $\mathrm{SU}(2)$ is zero (equivalently the rows are orthogonal with the same length).

6.3 The Affine Toric Variety $P_n^{\mathcal{T}}(\mathrm{SU}(2))$

In equation (3.1) we saw that $P_3(\mathrm{SU}(2)) \cong \bigwedge^2(\mathbb{C}^3)$. Since the space of imploded triangles is the product of $n-2$ copies of $P_3(\mathrm{SU}(2))$, we see that $X_n^{\mathcal{T}}(\mathrm{SU}(2))$ is the affine space obtained by taking the product of $n-2$ copies of $\bigwedge^2(\mathbb{C}^3)$. It has an action of a $3n-6$ torus $\mathbb{T}$ induced by the right actions of the torus $\mathbb{T} = T^{3n-6}_{\mathrm{SU}(2)}$ on the $3n-6$ copies of $\mathcal{E}T^*(\mathrm{SU}(2))$. In terms of our matrices $A^{\mathcal{T}}$ this amounts to scaling the columns of $A^{\mathcal{T}}$ (right multiplication of $A^{\mathcal{T}}$ by $\mathbb{T}$). However, note that the entry of an element of $\mathbb{T}$ corresponding to the edge e of $\mathcal{T}^D$ scales the column of $A^{\mathcal{T}}$ corresponding to e by its *inverse*. We will use the notation $\underline{\mathbb{T}}$ to indicate the complexification of $\mathbb{T}$. Similarly for any compact group G that appears in this paper, the notation $\underline{G}$ indicates the complexification of G.

Let $\mathbb{T}_e$, $\mathbb{T}_d$, $\mathbb{T}_d^-$ be as in the introduction. Now we glue the diagonals of P back together by taking the symplectic quotient by $\mathbb{T}_d^-$ at level 0.

Definition 6.2 $P_n^{\mathcal{T}}(\mathrm{SU}(2)) = X_n^{\mathcal{T}}(\mathrm{SU}(2)//\mathbb{T}_d^- = \left(\bigwedge^2(\mathbb{C}^3)\right)^{n-2}//\mathbb{T}_d^-$.

6.4 The Space $P_n^{\mathcal{T}}(\mathrm{SO}(3,\mathbb{R}))$

The spaces $E_n^{\mathcal{T}}(\mathrm{SU}(2))$, $X_n^{\mathcal{T}}(\mathrm{SU}(2))$, and $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ all have analogues when SU(2)) is replaced by $\mathrm{SO}(3,\mathbb{R})$ that are quotients by a finite group of the corresponding spaces for SU(2). There are also analogues of the tori $\mathbb{T}$, $\mathbb{T}_e$, and $\mathbb{T}_d^-$ for $\mathrm{SO}(3,\mathbb{R})$ that we will denote $\mathbb{T}(\mathrm{SO}(3,\mathbb{R}))$, $\mathbb{T}_e(\mathrm{SO}(3,\mathbb{R}))$, and $\mathbb{T}_d^-(\mathrm{SO}(3,\mathbb{R}))$ respectively that are quotients of the corresponding tori for SU(2). We leave the details to the reader.

6.5 $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ as a GIT Quotient

Since affine GIT quotients coincide with symplectic quotients (see [KN] and [S, Theorem 4.2]), we may also obtain $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ as the GIT quotient of $(\bigwedge^2(\mathbb{C}^3))^{n-2}$ by the complex torus $\underline{\mathbb{T}}^d$. For each triangle T_k (tripod τ_k) we have a corresponding $\bigwedge^2(\mathbb{C}^3)$ with Plücker coordinates $Z_{ij}(\tau_k)$, $1 \leq i, j \leq 3$. The coordinate ring of the affine variety $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ will be the ring of invariants $\mathbb{C}[(Z_{ij}(\tau_k))]^{\underline{\mathbb{T}}_d^-}$. This ring of invariants will be spanned by the ring of invariant *monomials* that we now determine. There is a technical problem here. We need to know that we have chosen the correctly normalized momentum map for the action $\mathbb{T}_d^-$. But by Theorem A.7 the correct normalization is the one that is homogeneous linear in the squares of the norms of the coordinates, which is the one we have used here.

6.6 The Semigroup $\mathcal{P}_n^{\mathcal{T}}$

As a geometric quotient of affine space by a torus, the space $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ is an affine toric variety. We now analyze the affine coordinate ring of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$. In what follows we label the leaf of the tripod τ_i incident to the edge (τ_i, k) by k. We will do this only when the tripod of which k is a vertex is clearly indicated. We leave the proof of the following lemma to the reader.

Lemma 6.3 *The monomial*

$$f(Z) = \prod_{i=1}^{n-2} Z_{12}(\tau_i)^{x_{12}(\tau_i)} Z_{13}(\tau_i)^{x_{13}(\tau_i)} Z_{23}(\tau_i)^{x_{23}(\tau_i)}$$

is $\mathbb{T}_d^-$-invariant if and only if the exponents $\mathbf{x} = (x_{jk}(\tau_i))$ satisfy the system of equations

$$x_{k,m}(\tau_i) + x_{k,l}(\tau_i) = x_{l,k}(\tau_j) + x_{l,m}(\tau_j), \quad \textit{for } (\tau_i, k) \textit{ identified to } (\tau_j, l) \textit{ in } \mathcal{T}.$$

Before stating a corollary of the lemma we need a definition.

Definition 6.4 Let $\mathcal{P}_n^{\mathcal{T}}$ be the subset of $\mathbf{x} \in (\mathbb{N}^3)^{(n-2)}$ satisfying the equations in the above lemma. Clearly $\mathcal{P}_n^{\mathcal{T}}$ is a semigroup under addition. Let $\mathbb{C}[\mathcal{P}_n^{\mathcal{T}}]$ denote the associated semigroup algebra.

Corollary 6.5 *The affine coordinate ring of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ is isomorphic to the semigroup ring $\mathbb{C}[\mathcal{P}_n^{\mathcal{T}}]$.*

Now we will relate the semigroup $\mathcal{P}_n^{\mathcal{T}}$ and the monomials of the lemma to graphs on vertices of the decomposed tree $\mathcal{T}^D$. Monomials in the Plücker coordinates $Z_{ij}(\tau)$ for τ fixed correspond to graphs on the vertices i, j, k of the tripod τ as follows. We associate with the exponent $x_{ij}(\tau)$ of $Z_{ij}(\tau)$ the graph consisting of $x_{ij}(\tau)$ arcs joining the vertex i of τ to the vertex j. Each triple $\{x_{ij}(\tau)\}$ for τ fixed determines a graph on the leaves of the tripod τ. Thus each element $\mathbf{x} \in \mathcal{P}_n^{\mathcal{T}}$ corresponds to a collection of $n-2$ graphs, each on 3 vertices, one for each tripod in $\mathcal{T}^D$. Also each such element $\mathbf{x}$ corresponds to a product of monomials attached to tripods. This leads to a bijective correspondence between monomials and graphs.

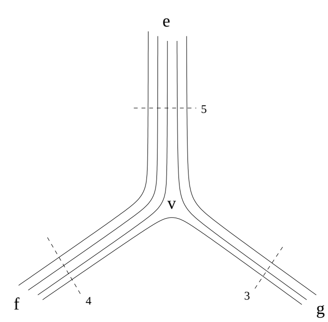

Figure 6.2: A single tripod τ with vertex v, with $x_{ef}(\tau) = 3$, $x_{eg}(\tau) = 2$, $x_{fg}(\tau) = 1$. Hence $x_{ef}(\tau) + x_{eg}(\tau) = 5$, $x_{ef}(\tau) + x_{fg}(\tau) = 4$, $x_{eg}(\tau) + x_{fg}(\tau) = 3$.

The condition

$$x_{km}(\tau_i) + x_{k\ell}(\tau_i) = x_{\ell,k}(\tau_j) + x_{\ell,m}(\tau_j)$$

specifies that the number of arcs associated with the identified edges (τ_i, k) of the tripod τ_i, and (τ_j, ℓ) of the tripod τ_j must agree. Later, this will allow us to glue these "local" graphs to obtain a "global" graph on n vertices. In the next section we will associate a weighting of $\mathcal{T}$ with the $\{x_{ij}(\tau_k)\}$.

6.7 The Projective Toric Variety $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$

We define $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$, a projectivization of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$, which will be isomorphic to $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$. The coordinate ring of the affine toric variety $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ is the semigroup algebra $\mathbb{C}[\mathcal{P}_n^{\mathcal{T}}]$. Hence to define the $\mathbb{C}^*$ action required for projectivization, it suffices to define a grading on $\mathcal{P}_n^{\mathcal{T}}$. First, for $\mathbf{x} \in P^{\mathcal{T}}$ we define the associated weight of the edge (τ, i) of the tripod τ in $\mathcal{T}^D$. let $w_{(\tau,i)}(\mathbf{x}) = x_{ij}(\tau) + x_{ik}(\tau)$. We define the degree of an element $\mathbf{x} \in \mathcal{P}_n^{\mathcal{T}}$ by

$$\mathrm{degree}(\mathbf{x}) = \frac{1}{2} \sum_{(\tau,i)\in \mathrm{dist}(\mathcal{T}^D)} w_{(\tau,i)}(\mathbf{x})$$

where the sum is over all (τ, i), which represent leaf edges of $\mathcal{T}$.

Remark 6.6 We will see below that under the isomorphism from the semigroup $\mathcal{P}_n^{\mathcal{T}}$ to the semigroups of admissible weightings $\mathcal{W}_n^{\mathcal{T}}$ the number $w_{(\tau,i)}(\mathbf{x})$ will be the weight assigned to the edge (τ, i) of the tree $\mathcal{T}$. Thus the degree of $\mathbf{x}$ is then half the sum of the weights of the leaf edges.

Definition 6.7 Give $\mathcal{P}_n^{\mathcal{T}}$ the grading defined above, then we define

$$Q_n^{\mathcal{T}}(\mathrm{SU}(2)) = \mathrm{Projm}(\mathbb{C}[\mathcal{P}_n^{\mathcal{T}}])$$

Note that as defined $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$ is a GIT $\mathbb{C}^*$-quotient of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$.

7 The Toric Varieties $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$ and $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ are Isomorphic

In this section we will prove the first statement of Theorem 1.8, namely that the toric variety $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$ constructed in the previous section is the isomorphic to $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ by proving Proposition 7.1.

Recall that in Section 4 it was shown that the two semigroups $\mathcal{S}_n^{\mathcal{T}}$ and $\mathcal{W}_n^{\mathcal{T}}$ are isomorphic. $\mathcal{S}_n^{\mathcal{T}}$ is the semigroup of Kempe Graphs on $n = |\,\mathrm{edge}(\mathcal{T})|$ vertices, with the product $*_{\mathcal{T}}$. $\mathcal{W}_n^{\mathcal{T}}$ is the semigroup of admissible weightings on $\mathcal{T}$ under addition. Recall that an admissible weighting is an integer weight satisfying the triangle inequalities about each internal vertex of $\mathcal{T}$, along with the condition that the sum of the weights about each internal node be even. It was also shown in Section 4 that the semigroup algebra of $\mathcal{S}_n^{\mathcal{T}} \cong \mathcal{W}_n^{\mathcal{T}}$ is isomorphic to the coordinate ring of $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$.

Proposition 7.1 *The graded semigroups $\mathcal{W}_n^{\mathcal{T}}$ and $\mathcal{P}_n^{\mathcal{T}}$ are isomorphic.*

Proof In all that follows we deal with tripods, so we give the unique leaf edge incident to the leaf labelled i the label i, and vice-versa. With this in mind, X_{ij} can be thought of as giving the number of arcs in a graph on the leaves of a tripod Y between the i-th and j-th leaves. The number N_i is a natural number assigned to the i-th edge of a tripod Y obtained by counting the number of arcs that have unique path in Y containing the i-th edge. The elements of both $\mathcal{P}_n^{\mathcal{T}}$ and $\mathcal{W}_n^{\mathcal{T}}$ both associate triples of integers with each tripod of $\mathcal{T}^D$ with certain “gluing conditions”. Let us consider a single tripod $Y \in \mathcal{T}^D$. A triple of numbers N_1, N_2, N_3 is an admissible weighting of Y if and only if there are integers X_{ij}, $i, j \in \{1, 2, 3\}$ such that $X_{ij} + X_{ik} = N_i$. To see this simply note that the equations

$$X_{ij} = \frac{N_i + N_j - N_k}{2}$$

have natural solutions if and only if (N_1, N_2, N_3) is admissible. Therefore, we may define a map from $\mathcal{W}_n^{\mathcal{T}}$ to $\mathcal{P}_n^{\mathcal{T}}$ by solving for $x_{ij}(\tau)$, with an obvious inverse given by solving for the weighting on the edge (τ, i). To see that these maps are well defined, note that the gluing condition

$$x_{km}(\tau_i) + x_{k\ell}(\tau_i) = x_{\ell,k}(\tau_j) + x_{\ell,m}(\tau_j), \quad \text{for } (\tau_i, k) \text{ identified to } (\tau_j, \ell) \in \mathcal{T},$$

is exactly equivalent to the weights on (τ_i, n) and (τ_j, l) being equal when these tuples represent the same diagonal in $\mathcal{T}$. Since both semigroup operations are defined by adding integers, and since both Φ and its inverse are linear functions over each trinode, these maps are semigroup isomorphisms. Finally, note that the grading on $\mathcal{P}_n^{\mathcal{T}}$ was chosen specifically to match the grading on $\mathcal{S}_n^{\mathcal{T}}$, we leave direct verification of this to the reader. ■

7.0.1 An Explicit Description of the Ring Isomorphism

The semigroup $\mathcal{S}_n^{\mathcal{T}}$ is generated as a graded semigroup by the elements corresponding to graphs with exactly one edge. The element corresponding to the graph with one edge, between the i and j vertices corresponds to the Plücker coordinate Z_{ij}. Recall that the unique path in the tree $\mathcal{T}$ joining the leafs i and j has been denoted $\gamma(ij)$. The path $\gamma(ij)$ gives rise to a sequence of edges of the forest $\mathcal{T}^D$ (where we pass from one tripod to the next by passing from an edge (τ_i, k) to an equivalent edge (τ_j, ℓ). We let $Z_{\gamma(ij)}$ be the corresponding product of Plücker coordinates $Z_{st}(\tau_k)$. Thus

$$Z_{\gamma(ij)} = \prod Z_{st}(\tau),$$

where (τ, s) and (τ, t) are the edges in the unique path defined by the path $\gamma(i, j)$ corresponding to Z_{ij} in $\mathcal{T}$. Note that $Z_{\gamma(ij)}$ is a $\underline{\mathbb{T}}_d^-$-invariant monomial in the homogeneous coordinate ring of $(\bigwedge^2(\mathbb{C}^3))^{n-2}$. Moreover, it is a generator of the ring of invariants. The isomorphism of toric rings Φ from the homogeneous coordinate ring of $\mathrm{Gr}_2(\mathbb{C}^n)_0^{\mathcal{T}}$ to $P_n^{\mathcal{T}}(\mathrm{SU}(2)$ is then given on generators by $\Phi(Z_{ij}) = Z_{\gamma(ij)}$.

Remark 7.2 It is important to see that the degree assigned to $Z_{\gamma(ij)} = \prod Z_{st}(\tau)$ by the isomorphism Φ (namely *one*) is different from that given by counting the $Z_{st}(\tau)$ in the product formula for $Z_{\gamma(i,j)}$. The latter count is in fact the Speyer–Sturmfels weight $w_{i,j}^{\mathcal{T}}$ of the Plücker coordinate Z_{ij}.

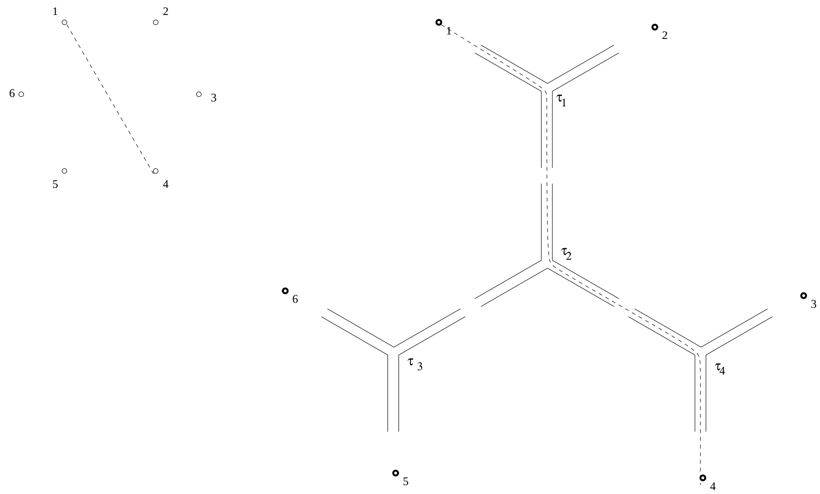

Figure 7.1: This illustrates $\Phi(Z_{14}) = Z_{\gamma(14)} = Z_{13}(\tau_1)Z_{12}(\tau_2)Z_{13}(\tau_4)$ in the case $n = 6$ with the symmetric tree.

7.0.2 The Grading Circle Action on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$

We verify the second statement of Theorem 1.8. We have seen that the action of $\lambda \in \mathbb{C}^*$ gives the grading scales each $Z_{\gamma(i,j)}$ by λ. Clearly this action is induced by the action on the matrix $A^{\mathcal{T}}$ that scales each row corresponding to a leaf edge by $\sqrt{\lambda}$, in other words by the action of the element $\mathbf{t}_e((\sqrt{\lambda})^{-1})$ as claimed in the second statement of Theorem 1.8. We conclude by explaining why the actions of $\mathbf{t}_e(\sqrt{\lambda})$ and $\mathbf{t}_e(-\sqrt{\lambda})$ coincide. It suffices to prove that $\mathbf{t}_e(-1)$ acts trivially. But since the operation that scales all rows of $A^{\mathcal{T}}$ by -1 is induced by an element of $\mathrm{SU}(2)^{n-2}$, it acts trivially on $X_n^{\mathcal{T}}(\mathrm{SU}(2))$ and hence on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$. But also the operation of scaling all the rows of $A^{\mathcal{T}}$ that belong to nonleaf edges of $\mathcal{T}^D$ is induced by an element of $\underline{\mathbb{T}}_d^-$, hence this operation too is trivial on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$. But $\mathbf{t}_e(-1)$ is the composition of the two operations just proved to be trivial.

8 The Spaces $W_n^{\mathcal{T}}$ and $Q_n^{\mathcal{T}}(\mathrm{SU}(2))$ are Homeomorphic.

We now give the proof of Theorem 1.4. We first note that we have identified an ordered subset of edges of $\mathcal{T}^D$ with the leaf edges of $\mathcal{T}$ (equivalently the edges of P). This identification gives an isomorphism $\rho\colon T_{\mathrm{SU}(2)^n} \to \mathbb{T}_e$.

8.1 A Homeomorphism of Affine Varieties

We will prove the following.

Theorem 8.1 *The spaces $V_n^{\mathcal{T}}$ and $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ are equivariantly homeomorphic with respect to ρ.*

We will use the symbol $\Psi_n^{\mathcal{T}}$ to denote the above equivariant homeomorphism and $\Phi_n^{\mathcal{T}}$ to denote its inverse. By the results of Subsection 6.5 it suffices to construct a ρ-equivariant homeomorphism (and its inverse) from $V_n^{\mathcal{T}}$ to the above symplectic quotient that we will continue to denote $P_n^{\mathcal{T}}(\mathrm{SU}(2))$. We first construct the ρ-equivariant map $\Phi_n^{\mathcal{T}}\colon P_n^{\mathcal{T}}(\mathrm{SU}(2)) \to V_n^{\mathcal{T}}$. There is a simple idea behind this map. We have an inclusion of the leaf edges of $\mathcal{T}$ into the edges of $\mathcal{T}^D$. Now an element of $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ is a map $\mathbf{T}$ from the edges of $\mathcal{T}^D$ into $\mathbb{C}^2$. The map $\Phi_n^{\mathcal{T}}$ is induced by the map $\widetilde{\Phi}_n^{\mathcal{T}}\colon E_n^{\mathcal{T}}(SU(2)) \to E_n(\mathrm{SU}(2))$ that restricts $\mathbf{T}$ to the leaf edges of $\mathcal{T}$. However we need to verify that the image of an element of momentum level zero for $\mathrm{SU}(2)^{n-2} \times \mathbb{T}_d^-$ has $\mathrm{SU}(2)$-momentum level zero, and that the induced map of zero momentum levels descends to the required quotients. For the rest of this discussion, let $\mathbf{T} = (F_1(\tau_1), \ldots, F_{n-2}(\tau_{n-2}))$ be an element of $E_n^{\mathcal{T}}(\mathrm{SU}(2))$. Each F_i is an imploded spin framing of the edges of the triangle τ_i. This means that

$$F_i = \left[\, \big(g_1(i), \lambda_1(i)\varpi_1\big)\,, \big(g_2(i), \lambda_2(i)\varpi_1\big)\,, \big(g_3(i), \lambda_3(i)\varpi_1\big) \,\right]$$

such that

$$\lambda_1(i)g_1(i)\varpi_1 + \lambda_2(i)g_2(i)\varpi_1 + \lambda_3(i)g_3(i)\varpi_1 = 0$$

We will henceforth denote $T_i = F_i(\tau_i)$, thus the symbol T_i stands for a triangle together with an imploded spin-frame on its edges.

8.1.1 ϱ Flips and Normalized Framings

Let $\varrho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that if $t \in \mathrm{SU}(2)$ fixes ϖ_1 by conjugation, then t is diagonal and $\varrho t \varrho^{-1} = t^{-1}$. Furthermore, $\varrho\varpi_1\varrho^{-1} = -\varpi_1$. Let $[[T_1], [T_2], \ldots, [T_{n-2}]$ be an element of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$.

Lemma 8.2 *Suppose that the k-th edge of τ_i is identified with the ℓ-th edge of τ_j in $\mathcal{T}$ and that the edge (τ_j, ℓ) comes after the edge (τ_i, k) in the above ordering of edges of $\mathcal{T}^D$. Then using the left actions of* SU(2) *we may arrange that $g_{(\tau_j,\ell)} = g_{(\tau_i,k)}\varrho$.*

Proof Suppose we have

$$[T_i] = [(a_1, \ell_1\varpi_1), (a_2, \ell_2\varpi_1), (h, d\varpi_1)].$$

If the diagonal defined by (τ_i, ℓ) and (τ_j, k) has length 0, there is nothing to prove since any two frames are equivalent. Suppose then that this diagonal is not 0. Then by applying $g = h\varrho(h')^{-1}$ to T_j, we get

$$g \cdot T_j = [(h\varrho, d\varpi_1), (ga_3, \ell_3\varpi_1), (ga_4, \ell_4\varpi_1)],$$

which is equivalent to T_j. Hence the consecutive diagonal frames (the third frame of T_i and the first frame of $g \cdot T_{i+1}$) are now related by right multiplication by ϱ. ∎

From now on, we shall choose representatives in $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ for elements of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ so that nonzero frames associated with equivalent edges satisfy this "ϱ flip condition". If the frame associated with one edge of a pair of equivalent edges is zero then we require that the frame associated with the other edge of the pair is also zero. We say such elements of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ are *normalized*. Note that the definition of normalized depends on the ordering of the triangles T_i.

It is important to note that if an element $A = ([T_1], [T_2], \ldots, [T_{n-2}])$ is normalized, then so is $t \cdot A$ for any $t \in \mathbb{T}_d^-$. This is because $h\varrho t^{-1} = ht\varrho$. We leave the details to the reader. Thus we may speak of a $\mathbb{T}_d^-$-equivalence class as being *normalized*. We will define $\Phi_n^{\mathcal{T}}$ on such normalized elements.

Lemma 8.3 *For any $\boldsymbol{T} \in E_n^{\mathcal{T}}(\mathrm{SU}(2))$ there exist $\mathbf{f} \in \mathrm{SU}(2)^{n-2}$ and a normalized $\boldsymbol{T}' \in E_n^{\mathcal{T}}(\mathrm{SU}(2))$ such that $\mathbf{fT} = \mathbf{T}'$.*

Proof Let $\mathcal{T}' \subset \mathcal{T}$ be connected, and let $Y \subset \mathcal{T}$ be a tripod that shares exactly one edge, d, with $\mathcal{T}'$. Let (τ, i) and (τ', j) be the edges of $\mathcal{T}^D$ that map to d. Suppose $\mathbf{T} \in E_n^{\mathcal{T}}(\mathrm{SU}(2))$ is normalized at all diagonals corresponding to internal edges of $\mathcal{T}'$. Let $\mathbf{f} \in \mathrm{SU}(2)^{n-2}$ be the element that is $g_{(\tau,i)}\varrho g_{(\tau',j)}^{-1}$ on the τ'-th factor, and the identity elsewhere. Then $\mathbf{T}' = \mathbf{fT}$ is normalized at all diagonals corresponding to internal edges of $\mathcal{T}' \cup Y$. By Lemma 5.2 we can always find a sequence of connected trees $\mathcal{T}_i \subset \mathcal{T}_{i+1}$ with $Y_i = \mathcal{T}_{i+1}^D \setminus \mathcal{T}_i^D$ a tripod such that each internal edge of $\mathcal{T}$ appears as an edge shared by the images of Y_i and $\mathcal{T}_i$ in $\mathcal{T}$ for some i. The lemma now follows by induction. ∎

8.1.2 Definition of the Maps $\Phi_n^{\mathcal{T}}$ and $\Psi_n^{\mathcal{T}}$

Given a normalized element $\mathbf{T} \in E_n^{\mathcal{T}}(\mathrm{SU}(2))$ define $\widetilde{\Phi}_n^{\mathcal{T}}(\mathbf{T})$ to be the element in $E_n(\mathrm{SU}(2))$ given by projecting on the components $(g_{(\tau_i,j)}, \lambda_{(\tau_i,j)}\varpi_1) \in \mathbf{T}$ such that $(\tau_i, j) \in \mathcal{T}^D$ maps to a leaf edge in $\mathcal{T}$ under $\pi_{\mathcal{T}}$.

Lemma 8.4 *If*

$$\boldsymbol{T} \in \mu_{\mathrm{SU}(2)^{n-2}}^{-1}(0) \cap \mu_{\mathbb{T}_d^-}^{-1}(0)$$

and is normalized, then $\widetilde{\Phi}_n^{\mathcal{T}}(\boldsymbol{T}) \in \widetilde{P}_n(\mathrm{SU}(2))$, that is, the polygon in $\mathbb{R}^3$ associated with $\widetilde{\Phi}_n^{\mathcal{T}}(\boldsymbol{T})$ closes up.

Proof First observe that $\lambda g\varpi_1 = -\lambda g\varrho\varpi_1$. Now because each triangle closes up, the sum of $g\lambda\varpi_1$ over all edges of $\mathcal{T}^D$ is zero. But by the observation just above the sum over pairs of equivalent edges of $\mathcal{T}^D$ cancel. Hence the sum over leaf edges of $\mathcal{T}$ is zero as required. ∎

Lemma 8.5 *The element $\widetilde{\Phi}_n^{\mathcal{T}}$ induces a well-defined map on the quotient*

$$\widehat{\Phi}_n^{\mathcal{T}} : \mathrm{SU}(2)^{n-2} \backslash\backslash (\mu_{\mathrm{SU}(2)^{n-2}}^{-1}(0) \cap \mu_{\mathbb{T}_d^-}^{-1}(0)) \to V_n^{\mathcal{T}}$$

and a well-defined map $\Phi_n^{\mathcal{T}} : P_n^{\mathcal{T}}(\mathrm{SU}(2)) \to V_n^{\mathcal{T}}$.

Proof To define $\widehat{\Phi}_n^{\mathcal{T}}$ we need only show that for any pair of normalized $\mathbf{T}$ and $\mathbf{T}'$ in $\mu_{\mathrm{SU}(2)^{n-2}}^{-1}(0) \cap \mu_{\mathbb{T}_d^-}^{-1}(0)$ and $\mathbf{f} = (f_1, \ldots, f_{n-2}) \in G$ such that $\mathbf{fT} = \mathbf{T}'$, we have $\widetilde{\Phi}_n^{\mathcal{T}}(\mathbf{T}) = \widetilde{\Phi}_n^{\mathcal{T}}(\mathbf{T}')$ in $V_n^{\mathcal{T}}$.

The assumption $\mathbf{fT} = \mathbf{T}'$ is equivalent to

$$(f_i g_{(\tau_i,j)}, \lambda_{(\tau_i,j)}\varpi_1) = (g'_{(\tau_i,j)}, \lambda'_{(\tau_i,j)}\varpi_1)$$

for each $j \in \{1,2,3\}$ and $1 \le i \le n-2$. Hence $\lambda_{(\tau_i,j)} = \lambda'_{(\tau_i,j)}$, and in particular a (pair of) diagonals of $\mathbf{T}$ vanishes if and only if the same diagonals vanish for $\mathbf{T}'$ that is $S(\mathbf{T}) = S(\mathbf{T}')$. Hence the forests $\mathcal{T}^{S(\mathbf{T})}$ and $\mathcal{T}^{S(\mathbf{T}')}$ coincide. We must show that the images of $\widetilde{\Phi}_n^{\mathcal{T}}(\mathbf{T})$ and $\widetilde{\Phi}_n^{\mathcal{T}}(\mathbf{T}')$ in $V_n^{\mathcal{T}}$ coincide, which amounts to proving that the $\mathbf{f}$ is constant on any connected component in the forest $\mathcal{T}^{S(\mathbf{T})}$ (recall that we think of $\mathbf{f}$ as a function from the trivalent vertices of $\mathcal{T}$ to $\mathrm{SU}(2)$). Clearly it suffices to prove that if (τ_i, j) and (τ_h, k) are equivalent, then $f_h = f_i$. Suppose the imploded spin framings on the two edges are respectively $(g_{(\tau_i,j)}, \lambda_{(\tau_i,j)}\varpi_1)$ and $(g_{(\tau_h,k)}, \lambda_{(\tau_h,k)}\varpi_1)$. We have equations

$$f_i g_{(\tau_i,j)} = g'_{(\tau_i,j)}, \quad f_h g_{(\tau_h,k)} = g'_{(\tau_h,k)},$$

and because we have assumed both $\mathbf{T}$ and $\mathbf{T}'$ are normalized we have

$$g_{(\tau_h,k)} = g_{(\tau_i,j)}\varrho, \quad g'_{(\tau_h,k)} = g'_{(\tau_i,j)}\varrho.$$

We can rearrange these equations to get $f_i g_{(\tau_h,k)}\varrho^{-1} = g'_{(\tau_h,k)}\varrho^{-1}$. We cancel ϱ^{-1} to obtain $f_i g_{(\tau_h,k)} = g'_{(\tau_h,k)}$. But from above, $f_h g_{(\tau_h,k)} = g'_{(\tau_h,k)}$. This proves the first statement of the lemma. It is clear that $\widehat{\Phi}_n^{\mathcal{T}}$ is constant on $\mathbb{T}_d^-$ orbits, so it descends to give $\Phi_n^{\mathcal{T}}$. ■

This shows that $\Phi_n^{\mathcal{T}}$ is well defined and continuous because of the continuity of $\widetilde{\Phi}_n^{\mathcal{T}}$.

Lemma 8.6 *$\Phi_n^{\mathcal{T}}$ is proper (hence closed).*

Proof Suppose K is a compact subset of $P_n(\mathrm{SU}(2))$. Then the lengths of the columns of any matrix A (so the edge-lengths of the corresponding n-gon) representing an element of K are uniformly bounded by a constant C. We may reinterpret the above uniform bound as a bound on all edge lengths of all triangles in the given triangulation of P that are also edges of P. But we may assume all our matrices A are in the zero level sets of the momentum maps for $\mathrm{SU}(2)^{n-2}$ and $\mathbb{T}_d^-$. It follows by Lemma 5.2 and the triangle inequalities that the lengths of all columns of any matrix $A^{\mathcal{T}}$ representing any element in $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ in the inverse image of K are bounded by $N(\mathcal{T})C$, where $N(\mathcal{T})$ is a positive integer depending only on the tree $\mathcal{T}$. Then $(\Phi_n^{\mathcal{T}})^{-1}(K)$ is contained in the image of a subset of $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ homeomorphic to the product of $3(n-2)$ copies of the ball of radius $N(\mathcal{T})C$ in $\mathbb{C}^2$ ■

We now construct the map $\Psi_n^{\mathcal{T}}\colon V_n^{\mathcal{T}} \to P_n^{\mathcal{T}}(\mathrm{SU}(2))$ inverse to $\Phi_n^{\mathcal{T}}$. We first define $\widetilde{\Psi}_n^{\mathcal{T}}$ on $E_n(\mathrm{SU}(2))$ then verify that the resulting map descends to $V_n^{\mathcal{T}}$. We will need to be able to add a diagonal frame on a triangle with two already framed edges.

Lemma 8.7 *Suppose that two sides of an imploded spin-framed triangle $(g_1, \lambda_1\varpi_1)$, $(g_2, \lambda_2\varpi_1)$ are given. Then we can find $g_3 \in \mathrm{SU}(2)$ and $\lambda_3 \in \mathbb{R}$ so that $(g_1, \lambda_1\varpi_1)$, $(g_2, \lambda_2\varpi_1)$, $(g_3, \lambda_3\varpi_1)$ is an imploded spin-framed triangle. Precisely, we may solve*

$$\lambda_1 g_1\varpi_1 + \lambda_2 g_2\varpi_1 + \lambda_3 g_3\varpi_1 = 0.$$

Moreover, λ_3 is uniquely determined by λ_1 and λ_2, and if $\lambda_3 \neq 0$, any two choices of g_3 for given g_1 and g_2 are related by right multiplication of $T_{\mathrm{SU}(2)}$.

Proof If both λ_1 and λ_2 are zero, then we take λ_3 to be zero and $g_3 = 1$. Suppose then that exactly one of λ_1 and λ_2 is nonzero. Without loss of generality, assume λ_1 is nonzero. Put $\lambda_3 = \lambda_1$ and choose $g_3 = g_1 \varrho$. Hence we have

$$\lambda_1 g_1 \varpi_1 = -\lambda_3 g_3 \varpi_1.$$

Note that the map $S^3 \to S^2$ given by $g \to g\varpi_1$ is the Hopf fibration, and consequently we may solve the equation $g\varpi_1 = -g_1\varpi_1$ locally in a neighborhood of g_1, in particular, we can solve this equation in a neighborhood of $(g_1, 1)$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The number $\lambda_3 \geq 0$ is uniquely determined, but g_3 is defined only up to right multiplication by an element of $\mathbb{T}$. Now assume that both λ_1 and λ_2 are nonzero. First assume that the sum $\lambda_1 g_1 \varpi_1 + \lambda_2 g_2 \varpi_1$ is zero, then we are forced to take $\lambda_3 = 0$, and we may choose any framing data we wish, so we choose $g_3 = 1$. Assume then that $\lambda_1 g_1 \varpi_1 + \lambda_2 g_2 \varpi_1$ is nonzero. We choose $g_3 \in \mathrm{SU}(2)$ and λ_3 so that

$$\lambda_1 g_1 \varpi_1 + \lambda_2 g_2 \varpi_1 = -\lambda_3 g_3 \varpi_1.$$

Again, $\lambda_3 \neq 0$ is uniquely determined, and g_3 is defined up to right multiplication by an element of $\mathbb{T}$. We define the required framed triangle T by

$$T = (g_1, \lambda_1 \varpi_1), (g_2, \lambda_2 \varpi_1), (g_3, \lambda_3 \varpi_1). \qquad \blacksquare$$

We make use of Lemma 8.7 to extend any framing of the edges of P to an element of $E_n^{\mathcal{T}}(\mathrm{SU}(2))$. The resulting object will be a normalized element

$$\mathbf{T} \in \mu^{-1}_{\mathrm{SU}(2)^{n-2}}(0) \cap \mu^{-1}_{\mathbb{T}_d^-}(0) \subset E_n^{\mathcal{T}}(\mathrm{SU}(2)).$$

Lemma 8.8 *Suppose we are given an imploded spin framing* $\mathbf{E}$ *of the edges of the model convex n-gon P that is of momentum level zero for the action of* $\mathrm{SU}(2)$*. Then we may extend the framing by choosing imploded spin-frames on the diagonals and an enumeration of the triangles of P so that the resulting element* $\mathbf{T} \in E_n^{\mathcal{T}}(\mathrm{SU}(2))$ *is*

(i) *normalized,*
(ii) *of momentum level zero for* $\mathrm{SU}(2)^{n-2}$,
(iii) *of momentum level zero for* $\mathbb{T}_d^-$.

Moreover, any two such extensions are equivalent under the action of the torus $\mathbb{T}_d^-$.

Proof We prove the existence of the framing by induction on n. We take $n - 3$ as base case. here there is nothing to prove.

Let P be a model convex n-gon with an imploded spin framing on its edges. Define $P_{i_1} = P$. Choose a triangle T in the triangulation of P that shares two edges with P. Hence two sides of T are framed. Let e be the remaining side (a diagonal of P). Suppose e has length λ. Define $T_{i_1} = T$. Apply Lemma 8.7 to frame the third side e such that the resulting framing is of momentum zero level for $\mathrm{SU}(2)$. Split T off from

P to obtain an $n-1$-gon P_{i_2}. Suppose the framing on e is $[g, \lambda\varpi_1]$. Give the edge e' of P_{i_2} that is not yet framed the framing $[g\varrho, \lambda\varpi_1]$. We obtain the existence part of the lemma by induction.

Now we prove uniqueness. Suppose that $\mathbf{T}'$ is another extension of $\mathbf{E}$, and that $\mathbf{T}'$ assigns the frame $[h, \lambda\varpi_1]$ to e with $h \neq g$. Then again by Lemma 8.7 either $\lambda = 0$ or there exists an element $t \in T_{\mathrm{SU}(2)}$ such that $h = gt$. In case $\lambda \neq 0$ the frame on e' is necessarily $[gt\varrho, \lambda\varpi_1] = [g\varrho t^{-1}, \lambda\varpi_1]$. Thus the new frames on the pair of equivalent edges of $\mathcal{T}^D$ are $[gt, \lambda\varpi_1]$ followed by $[g\varrho t^{-1}, \lambda\varpi_1]$. Note that the framing of P_{i_2} obtained by restriction of $\mathbf{T}'$ satisfies the three properties in the statement above, and it is also an extension of the framing of its boundary induced by $\mathbf{T}'$. Hence for the case $\lambda \neq 0$ the induction step of the uniqueness part of the lemma is completed. In the case $\lambda = 0$ we may then represent both frames by $[I, \lambda\varpi_1]$. But since $\lambda = 0$ we have (by definition of implosion) for any $t \in T_{\mathrm{SU}(2)}$

$$[I, \lambda\varpi_1] = [t, \lambda\varpi_1] = [t^{-1}, \lambda\varpi_1].$$

This completes the induction step in the uniqueness part of the lemma. ■

By the above lemma we obtain a well-defined map

$$\widetilde{\Psi}_n^{\mathcal{T}} \colon E_n(\mathrm{SU}(2)) \to (\mu^{-1}_{\mathrm{SU}(2)^{n-2}}(0) \cap \mu^{-1}_{\mathbf{T}_d^-}(0))//\mathbb{T}_d^-.$$

We prove this map descends to $V_n^{\mathcal{T}}$. We let $\widehat{\Psi}_n^{\mathcal{T}}$ denote the map obtained from $\widetilde{\Psi}_n^{\mathcal{T}}$ by postcomposing it with projection to $P_n^{\mathcal{T}}(\mathrm{SU}(2))$. Let us note the following obvious refinement of Lemma 8.8. Suppose P_1 is a subpolygon of P that meets P along diagonals of length zero in the realization of P given by the framing of the edges. Then given any extension of the framing of P to all diagonals of P_1 we may find an extension of the framing of P to all diagonals of P agreeing with the given one on P_1.

Lemma 8.9 *The map $\widehat{\Psi}_n^{\mathcal{T}}$ descends to $V_n^{\mathcal{T}}$.*

Proof We first verify that $\widehat{\Psi}_n^{\mathcal{T}}$ descends to $P_n(\mathrm{SU}(2))$. In fact we will show it is equivariant under $\mathrm{SU}(2)$, where the action of $\mathrm{SU}(2)$ on $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ is by the diagonal in $\mathrm{SU}(2)^{n-2}$. Let $f \in \mathrm{SU}(2)$. Let $\mathbf{E}$ be a framing of the edges of P, and $\mathbf{T}$ be the extension of Lemma 8.8. Apply Lemma 8.8 to extend $f\mathbf{F}$ to an element $\mathbf{T}'$ of $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ so that this extension also satisfies the three properties in the statement of Lemma 8.8. We wish to prove that the resulting element of $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ is in the $\mathbb{T}_d^-$-orbit of $f\mathbf{T}$. But since any two extensions satisfying the three properties of Lemma 8.8 are equivalent under $\mathbb{T}_d^-$, it suffices to prove that the image of *any one* of the above extensions is equal to $f\mathbf{T}$. But there is an extension that is obviously equivalent to $f\mathbf{T}$, namely $f\mathbf{T}$ itself.

It remains to check that $\widehat{\Psi}_n^{\mathcal{T}}$ descends to $V_n^{\mathcal{T}}$. Let $\mathbf{E}$ and $\mathbf{T}$ be as above. Let $\mathbf{E}'$ be another framing of the edges of P in the same $\mathcal{T}$-congruence class as $\mathbf{E}$. We have to prove that $\widehat{\Psi}_n^{\mathcal{T}}(\mathbf{E}) = \widehat{\Psi}_n^{\mathcal{T}}(\mathbf{E}')$. By transitivity of the $\mathcal{T}$-congruence relation it suffices to prove this for $\mathbf{E}'$.

Let C_1 be a component of $\mathcal{T}^{S(\mathbf{T})}$. We may assume that C_1 contains leaf edges. Apply an element $f \in \mathrm{SU}(2)$ to the frames of the leaf edges of $\mathcal{T}_1$ to obtain a new framing

$\mathbf{E}'$ of the edges of P. We now prove $\widehat{\Psi}_n^{\mathcal{T}}(\mathbf{E}) = \widehat{\Psi}_n^{\mathcal{T}}(\mathbf{E}')$ for this choice of $\mathbf{E}'$. Apply Lemma 8.8 to $\mathbf{E}'$ to obtain an element $\mathbf{T}'$ of $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ such that the three properties in Lemma 8.8 are satisfied. We must prove that $\mathbf{T}$ and $\mathbf{T}'$ have the same image in $P_n^{\mathcal{T}}(\mathrm{SU}(2))$. As in the previous paragraph, it suffices to find one extension $\mathbf{T}'$ of $\mathbf{E}'$ that has the same image as $\mathbf{T}$. We now construct such an extension $\mathbf{T}''$. The subtree C_1 corresponds to a subpolygon P' of P such that the edges of P' that correspond to diagonals of P have zero length. Define $\mathbf{T}''$ by defining the frame on an edge e of $\mathcal{T}^D$ that is in C_1 to be f applied to the value of $\mathbf{T}$ on e, and if e is not in C_1 then define the value of $\mathbf{T}''$ on that edge to be the value of $\mathbf{T}$ on that edge. We leave to the reader that $\mathbf{T}''$ satisfies the three properties of Lemma 8.8. Now we construct an element $\tilde{f} \in \mathrm{SU}(2)^{n-2}$ such that $\mathbf{T}'' = \tilde{f}\mathbf{T}$. Recall that an element of $\mathrm{SU}(2)^{n-2}$ may be regarded as a function from the trivalent nodes of $\mathcal{T}$ to $\mathrm{SU}(2)$. We define $\tilde{f}$ to be the function that is I if the trivalent node is not in C_1, and is f if the trinode is in C_1. Since an edge e of $\mathcal{T}$ is in C_1 if and only if the trivalent node incident to e, we do indeed have $\mathbf{T}'' = \tilde{f}\mathbf{T}$. ■

Because $\Psi_n^{\mathcal{T}}$ is the inverse of a closed map we have the following.

Lemma 8.10 *The map $\Psi_n^{\mathcal{T}}$ is continuous.*

Finally we have the following.

Lemma 8.11 *The maps $\Phi_n^{\mathcal{T}}$ and $\Psi_n^{\mathcal{T}}$ are inverses of each other.*

Proof It is obvious that $\Phi_n^{\mathcal{T}} \circ \Psi_n^{\mathcal{T}} = I_{P_n(\mathrm{SU}(2))}$. This is because neither of the two maps changes the framing of the edges of P. Now let $\mathbf{T}$ be a normalized element of $\mathrm{SU}(2)^{n-2} \times \mathbb{T}_d^-$-momentum level zero in $E_n^{\mathcal{T}}(\mathrm{SU}(2))$. Let $\mathbf{F}$ be the restriction of $\mathbf{T}$ to the edges of P. Thus $\mathbf{T}$ is an extension of $\mathbf{F}$ satisfying the three properties of Lemma 8.8. But $\Psi_n^{\mathcal{T}} \circ \Phi_n^{\mathcal{T}}(\mathbf{T})$ also has restriction to P given by $\mathbf{F}$ and also satisfies the three properties of Lemma 8.8. Hence $\mathbf{T}$ and $\Psi_n^{\mathcal{T}} \circ \Phi_n^{\mathcal{T}}(\mathbf{T})$ are in the same $\mathbb{T}_d^-$-orbit, and hence their images in $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ coincide. ■

It is clear from the definition of $\Phi_n^{\mathcal{T}}$ that this map intertwines the actions of $T_{\mathrm{SU}(2)^n}$ and $\mathbb{T}_e$. This proves Theorem 8.1.

8.2 Pulling Back Hamiltonian Functions from $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ to $V_n^{\mathcal{T}}$

In the previous subsection we constructed a homeomorphism

$$\Psi_n^{\mathcal{T}} \colon V_n^{\mathcal{T}} \to P_n^{\mathcal{T}}(\mathrm{SU}(2))$$

with inverse $\Phi_n^{\mathcal{T}}$. The torus $\mathbb{T}$ acts on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ such that the Hamiltonian for the circle factor corresponding to the edge (τ_i, j) of $\mathcal{T}^D$ is given by

$$f_{\tau_i,j}(A) = (1/2)|z_j(\tau_i)|^2 + |w_j(\tau_i)|^2.$$

In spin-framed coordinates this is $f_{\tau_i,j}(A) = \lambda_{(\tau_i,j)}$. We will compute the pullback of these functions to $V_n^{\mathcal{T}}$ via $\Psi_n^{\mathcal{T}}$.

Definition 8.12 Let $\nu^{\mathcal{T}}_{(\tau,i)} \colon E^{\mathcal{T}}_n(\mathrm{SU}(2)) \to \mathfrak{su}(2)^*$ by the composition of the projection on the (τ, i)-th factor of $E^{\mathcal{T}}_n(\mathrm{SU}(2))$ with the function h from Subsection 3.4. Let $\nu_e \colon E_n(\mathrm{SU}(2)) \to \mathfrak{su}(2)^*$ be the composition of the projection on the e-th factor of $E_n(\mathrm{SU}(2))$ with the function h from Subsection 3.4.

For a distinguished edge (τ, i) of $\mathcal{T}^D$ let $e(\tau, i)$ be the associated edge of the model convex planar n-gon. Then we have the identity

$$(\tilde{\Phi}^{\mathcal{T}}_n)^*(\nu_{e(\tau,i)}) = \nu^{\mathcal{T}}_{(\tau,i)}.$$

8.2.1 Pulling Back Distinguished Edge Hamiltonian Functions

In what follows we will need to compute the pull-back of $f_{(\tau_i,j)}$ to $V^{\mathcal{T}}_n$ for those distinguished edges (τ_i, j). Let f be the map from Subsection 3.4, then we have

$$f_{\tau_i,j}(\mathbf{E}) = 2\|f \circ \nu^{\mathcal{T}}_{(\tau_i,j)}(\mathbf{T})\|$$

for any normalized $\mathbf{T} \in \mu^{-1}_{\mathrm{SU}(2)^{n-2}\times\mathbb{T}^-_d}(0)$ that maps to $\mathbf{E}$.

Proposition 8.13 *For $A \in V^{\mathcal{T}}_n$ we have*

$$(\Psi^{\mathcal{T}}_n)^* f_{\tau_i,j}(A) = 2\|f \circ \nu_{e(\tau_i,j)}(\tilde{A})\|.$$

where $\tilde{A} \in E_n(\mathrm{SU}(2))$ maps to A.

Proof The proposition follows from

$$(\Psi^{\mathcal{T}}_n)^*(f_{\tau_i,j})(A) = f_{\tau_i,j}(\Psi^{\mathcal{T}}_n(A)) = 2\|f \circ (\Psi^{\mathcal{T}}_n)^*(\nu^{\mathcal{T}}_{(\tau_i,j)})(\tilde{A})\| = 2\|f \circ \nu_{e(\tau_i,j)}(\tilde{A})\|. \quad \blacksquare$$

Remark 8.14 Note that $\|f \circ \nu_{e(\tau_i,j)}(\tilde{A})\|$ is the length of the $e(\tau_i, j)$ edge of $F_n(A)$.

8.2.2 Pullback of the Internal Edge Hamiltonian Functions

We also need to compute the $\Psi^{\mathcal{T}}_n$ pullbacks of the Hamiltonian functions $f_{(\tau_j,i)}$ for (τ_j, i) an internal edge of $\mathcal{T}$. This will be important when we wish to indentify the Hamiltonian flows of these pullbacks with the bending flows. Note that (τ_j, i) corresponds to a diagonal $d(\tau_j, i)$ in a model n-gon P.

Lemma 8.15 *Let $\mathcal{T}' \subset \mathcal{T}$ be a connected subtree such that every leaf of $\mathcal{T}'$ except one, say (τ, k), is a leaf of $\mathcal{T}$. Then for any element $\mathbf{E} \in P^{\mathcal{T}}_n(\mathrm{SU}(2))$ we can compute*

$$f_{\tau,k}(\mathbf{E}) = \left\| \sum_{(\tau_i,j)\in \mathrm{dist}(\mathcal{T}')} \nu^{\mathcal{T}}_{(\tau_i,j)}(\mathbf{T}) \right\|$$

For $\mathbf{T}$ a normalized element of $\mu^{-1}_{\mathrm{SU}(2)^{n-2}\times\mathbb{T}^-_d}(0)$ that maps to $\mathbf{E}$.

Proof First note that we compute $f_{\tau,k}$ by taking half the length of the (τ, k) edge for any normalized element $\mathbf{T} \in \mu^{-1}_{\mathrm{SU}(2)^{n-2}\times\mathbb{T}_d^-}(0)$ which maps to $\mathbf{E}$, this is equal to $\lambda_{(\tau,k)}$. By the normalized condition and the closing condition imposed on $\mathbf{T}$, the leaf edge of $\mathcal{T}'$ form a closed polygon, which implies that

$$\lambda_{(\tau,k)} = \left\| \sum_{(\tau_i,j)\in \mathrm{dist}(\mathcal{T}')} \lambda_{(\tau_i,j)}(\mathbf{T}) Ad^*_{g_{(\tau_i,j)}(\mathbf{T})}(\varpi_1) \right\|. \qquad \blacksquare$$

The previous proposition allows us to conclude the following theorem.

Theorem 8.16 *Let $d(\tau_j, i)(A)$ be the associated diagonal in $F_n(A)$:*

$$(\Psi_n^{\mathcal{T}})^*(f_{(\tau_j,i)})(A) = 2\|d(\tau_j, i)(A)\|.$$

Proof We have the following equation

$$(\Psi_n^{\mathcal{T}})^*(f_{(\tau_j,i)})(A) = \left\| \sum_{(\tau_i,j)\in \mathrm{dist}(\mathcal{T}')} \nu^{\mathcal{T}}_{(\tau_i,j)}(\tilde{\Psi}_n^{\mathcal{T}}(\tilde{A})) \right\|.$$

The right-hand side of this equation is equal to

$$2\left\| \sum_{(\tau_i,j)\in \mathrm{dist}(\mathcal{T}')} f \circ \nu_{e(\tau_i,j)}(\tilde{A}) \right\|.$$

This last expression is in turn equal to $2\|d(\tau_i, j)(A)\|$. $\blacksquare$

This, along with Proposition 8.13, proves Theorem 1.11.

8.3 The Homeomorphism of Projective Varieties

In this subsection we will descend the homeomorphisms of affine varieties $\Phi_n^{\mathcal{T}}$ and $\Psi_n^{\mathcal{T}}$ to their projective quotients completing the proof of Theorem 1.8 and hence the proof of Theorem 1.4. Recall that we defined elements $\mathbf{t}(\lambda) \in T_{\mathrm{SU}(2)^n}$ and $\mathbf{t}_e(\lambda) \in \mathbb{T}_e$ for $\lambda \in S^1$. We note that $\rho(\mathbf{t}(\lambda)) = \mathbf{t}_e(\lambda)$. Since $\Psi_n^{\mathcal{T}}$ is equivariant, we have

$$\Psi_n^{\mathcal{T}} \circ \mathbf{t}(\lambda) = \mathbf{t}_e(\lambda) \circ \Psi_n^{\mathcal{T}}. \tag{8.1}$$

Proposition 8.17 *The map $\Psi_n^{\mathcal{T}}$ induces a homeomorphism from $W_n^{\mathcal{T}}$ to the symplectic quotient of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ by the grading circle action.*

Proof It suffices to prove that the pull-back by $\Psi_n^{\mathcal{T}}$ of the grading circle action on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ is the grading circle action on $V_n^{\mathcal{T}}$. This follows by replacing λ by $(\sqrt{\lambda})^{-1}$ in equation (8.1). $\blacksquare$

8.4 The Symplectic and GIT Quotients Coincide

We complete the proof of Theorems 1.8 and 1.4 by proving the following.

Proposition 8.18 *The symplectic quotient of $P_n^{\mathcal{T}}(\mathrm{SU}(2)$ by the grading circle action coincides with the GIT quotient of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ by the grading $\mathbb{C}^*$-action linearized by acting as the identity on the fiber of the trivial complex line bundle over $P_n^{\mathcal{T}}(\mathrm{SU}(2))$.*

We are forced to give an indirect argument because $P_n^{\mathcal{T}}(\mathrm{SU}(2)$ is not smooth and there do not seem to be theorems asserting the isomorphism of symplectic and GIT quotients of nonsmooth spaces.

We first note that the above GIT quotient is canonically isomorphic to the GIT quotient of $(\bigwedge^2(\mathbb{C}^3))^{n-2}$ by the product group $\mathbb{C}^* \times \underline{\mathbb{T}}_d^-$, where the first factor acts by the grading action linearized by the identity action on the fiber $\mathbb{C}$ of the trivial complex line bundle over $(\bigwedge^2(\mathbb{C}^3))^{n-2}$. This follows because we can take the ring of invariants for a product group acting on a ring R by first taking the invariants of one factor and then taking the invariants of the resulting ring by the second factor. The corresponding quotient by stages for symplectic quotients is also true but slightly harder. It is proved in [SjL, Theorem 4.1].

Thus it remains to prove that the symplectic quotient of $(\bigwedge^2(\mathbb{C}^3))^{n-2}$ by the product of the grading S^1-action and $\mathbb{T}_d^-$ is isomorphic to the GIT quotient of $(\bigwedge^2(\mathbb{C}^3))^{n-2}$ by the product of the grading $\mathbb{C}^*$ action and the complexified torus $\underline{\mathbb{T}}_d^-$, where the first factor acts by the grading action, and the product is linearized by acting as the identity map applied to the projection on the first factor. This follows immediately from Theorem A.7 once we establish that the momentum map for the product of the grading S^1-action and $\mathbb{T}_d^-$ is proper. We now prove that the momentum map for the product of the grading circle action, and $\mathbb{T}_d^-$ is proper. In fact we prove a different result that turns out to be equivalent to the one we need here because we will need this different result below.

Proposition 8.19 *The momentum mapping $\mu\colon (\bigwedge^2 \mathbb{C}^3)^{n-2} \to \mathbb{R}^{2n-3}$ for the action of $\mathbb{T}_e \times \mathbb{T}_d^-$ is proper.*

Proof Earlier we identified $\bigwedge^2 \mathbb{C}^3$ with the space of framed triangles $P_3(\mathrm{SU}(2))$. Let $(\ldots, [g_{(\tau_i,j)}, \lambda_{(\tau_i,j)}\varpi_1], \ldots)$ be an element of $P_3(\mathrm{SU}(2))^{n-2}$. Under this identification we have that $\mu_{\mathbb{T}_e}^{-1}(\ldots, [g_{(\tau_i,j)}, \lambda_{(\tau_i,j)}\varpi_1], \ldots)$ is the vector of elements $\lambda_{(\tau_i,j)}$, where (τ_i, j) is a distinguished edge of $\mathcal{T}^D$. Similarly, $\mu_{\mathbb{T}_d^-}(\ldots, [g_{(\tau_i,j)}, \lambda_{(\tau_i,j)}\varpi_1], \ldots)$ is the vector of elements $\lambda_{(\tau_i,j)} - \lambda_{(\tau_k,\ell)}$, where (τ_i, j) and (τ_k, ℓ) represent the same internal edge of $\mathcal{T}$. In order to show that $\mu_{\mathbb{T}_e \times \mathbb{T}_d^-}$ is a proper map, we must show that if all $\lambda_{(\tau_i,j)}$ are bounded for (τ_i, j) distinguished, and all differences $\lambda_{(\tau_i,j)} - \lambda_{(\tau_k,\ell)}$ are bounded for all (τ_i, j) and (τ_k, ℓ) that represent the same internal edge of $\mathcal{T}$, then all $\lambda_{(\tau_i,j)}$ are bounded.

Let $([g_1, \lambda_1\varpi_1], [g_2, \lambda_2\varpi_1], [g_3, \lambda_3\varpi_1])$ be an element of $P_3(\mathrm{SU}(2))$. Since

$$\sum_{i=1}^{3} \lambda_i Ad_{g_i}^*(\varpi_1) = 0,$$

the λ_i are side-lengths of a closed triangle, hence if two of the three are bounded, so is the third. Furthermore, if (τ_i, j) and (τ_k, ℓ) represent the same internal edge of $\mathcal{T}$, and $\lambda_{(\tau_i,j)}$ and the difference $\lambda_{(\tau_i,j)} - \lambda_{(\tau_k,\ell)}$ are bounded, then $\lambda_{(\tau_k,\ell)}$ is bounded. The proposition now follows from trivalent induction; see Lemma 5.2. ■

Corollary 8.20 *The momentum mapping $\mu\colon (\bigwedge^2 \mathbb{C}^3)^{n-2} \to \mathbb{R}^{2n-3}$ for the action of $S^1 \times \mathbb{T}_d^-$ is proper. Here S^1 acts by the grading circle action.*

Proof The Hamiltonian for the grading circle action is half the sum of the Hamiltonians for the circle $\mathbb{T}_e$ factors of $\mathbb{T}_e$. But note that each of the summands is bounded if and only if the sum is (since they have the same sign). ■

There is one more technical point. We claim that the GIT quotient coincides with the symplectic quotient at momentum level $(1, 0)$. We will prove first that the level for $\mathbb{T}_d^{-1}$ is zero. This follows because the torus $\mathbb{T}_d^{-1}$ acts linearly on $\bigwedge^{n-2}(\mathbb{C}^3)$ and trivially on the fiber, whereas the grading circle action is twisted by the identity action on the fiber.

9 The Space $V_{\mathbf{r}}^{\mathcal{T}}$ is Homeomorphic to the Toric Variety $(M_{\mathbf{r}})_0^{\mathcal{T}}$

In this section we will prove Theorem 1.6 by proving Theorem 1.9.

9.1 The Toric Varieties $(M_{\mathbf{r}})_0^{\mathcal{T}}$ and $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ are Isomorphic

In this subsection we prove the first statement of Theorem 1.9. By definition the toric variety $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ is obtained as the GIT quotient of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ by $\underline{\mathbb{T}}_e$ using the linearization given by the character $\chi_{\mathbf{r}}(\mathbf{t}(\lambda)) = \lambda_1^{r_1} \cdots \lambda_n^{r_n}$. Here $\mathbf{t}(\lambda)$ denotes the element of $\underline{\mathbb{T}}_e$ corresponding to $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. We recall that the graded ring of $(M_{\mathbf{r}})_0^{\mathcal{T}}$ is the semigroup ring $\mathbb{C}[\mathcal{S}_{\mathbf{r}}^{\mathcal{T}}]$, where $\mathcal{S}_{\mathbf{r}}^{\mathcal{T}}$ is the graded subsemigroup of $\mathcal{S}_n^{\mathcal{T}}$ defined by taking graphs with valence $k\mathbf{r}$ for positive integers k. Define the subsemigroup $\mathcal{P}_{\mathbf{r}}^{\mathcal{T}}$ of $\mathcal{P}_n^{\mathcal{T}}$ to be the inverse image of $\mathcal{W}_{\mathbf{r}}^{\mathcal{T}}$ under the isomorphism from $\mathcal{P}_n^{\mathcal{T}}$ to $\mathcal{W}_n^{\mathcal{T}}$ under the isomorphism of Proposition 7.1.

Lemma 9.1 *The $\underline{\mathbb{T}}_d^-$-invariant monomial*

$$f(Z) = \prod_{i=1}^{n-2} Z_{12}(\tau_i)^{x_{12}(\tau_i)} Z_{13}(\tau_i)^{x_{13}(\tau_i)} Z_{23}(\tau_i)^{x_{23}(\tau_i)}$$

is $\underline{\mathbb{T}}_e$–invariant for the twist $\chi_{\mathbf{r}}^p$ if and only if the exponents $\{x_{jk}(\tau_i)\}$ satisfy the system of equations

$$x_{k,m}(\tau) + x_{k,\ell}(\tau_i) = pr_{(\tau_i,k)}, \quad \text{for all leaf edges } (\tau_i, k), \text{ where } k, \ell, m = 1, 2, 3.$$

By Proposition 7.1 we may rewrite the above equation as a condition on the leaf weights

$$w_{(\tau_i,k)}^{\mathcal{T}} = pr_{(\tau_i,k)},\ (\tau_i, k) \in \mathcal{L}.$$

Proof The lemma follows from the formula for $\mathbf{t}(\lambda)$ acting on f, namely

$$\mathbf{t}(\lambda)\circ f(Z)=\prod_{(\tau_i,k)\in\mathcal{L}}\lambda_{(\tau_i,k)}^{[pr_{(\tau_i,k)}-(x_{k,m}(\tau_i)+x_{k,\ell}(\tau_i))]}f(Z).$$

Here $\lambda_{(\tau_i,k)}$ is the coordinate of $\underline{\mathbb{T}}_e$ corresponding to the leaf edge (τ_i,k) and $k,\ell,m=1,2,3$. ■

Remark 9.2 The set of $\underline{\mathbb{T}}_e$-invariant monomials in $Z_{ij}(\tau)$ for the twist by $\chi_{\mathbf{r}}^p$ is the subset of invariants *of degree* p (the p-th graded piece of the associate semigroup of lattice points).

As an immediate consequence we have the following.

Corollary 9.3 *The toric variety* $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ *is the projective toric variety associated with the graded semigroup* $\mathcal{P}_{\mathbf{r}}^{\mathcal{T}}$.

Now by Proposition 4.12 we have an isomorphism of graded semigroups $\Omega_{\mathbf{r}}\colon \mathcal{S}_{\mathbf{r}}^{\mathcal{T}}\to\mathcal{W}_{\mathbf{r}}^{\mathcal{T}}$. Since by definition we have an isomorphism of graded semigroups $\mathcal{P}_{\mathbf{r}}^{\mathcal{T}}\cong\mathcal{W}_{\mathbf{r}}^{\mathcal{T}}$, we obtain the required isomorphism of graded semigroups $\mathcal{S}_{\mathbf{r}}^{\mathcal{T}}\cong\mathcal{P}_{\mathbf{r}}^{\mathcal{T}}$, and we have proved the desired isomorphism of projective toric varieties.

9.2 $V_{\mathbf{r}}^{\mathcal{T}}$ is Homeomorphic to $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$

In this section we will prove the second statement of Theorem 1.9.

Proposition 9.4 $V_{\mathbf{r}}^{\mathcal{T}}$ *is homeomorphic to the symplectic quotient of* $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ *by* $\mathbb{T}_e$ *at level* $\mathbf{r}$.

Proof Recall that in Proposition 8.13 we proved the following formula. Let $\mathbf{E}\in V_n^{\mathcal{T}}$, then

$$(\Psi_n^{\mathcal{T}})^* f_{\tau_i,j}(A)=2\|f\circ\nu_{e(\tau_i,j)}(\tilde{A})\|=\|\nu_{e(\tau_i,j)}(\tilde{A})\|,$$

where $e(\tau_i,j)$ is the leaf edge of $\mathcal{T}$ corresponding to the leaf edge (τ_i,j) of the decomposed tree $\mathcal{T}^D$. Thus $\Psi_n^{\mathcal{T}}$ induces a homeomorphism between the $\mathcal{T}$-congruence classes of imploded spin-framed n-gons with side-lengths $\mathbf{r}$ and the $\mathbf{r}$-th level set for the momentum map of $\mathbb{T}_e$. But also $\Psi_n^{\mathcal{T}}$ is ρ-equivariant, hence the above bijection descends to a homeomorphism from the $T_{\mathrm{SU}(2)^n}$-quotient of the imploded spin-framed n-gons to the $\mathbb{T}_e$ symplectic quotient at level $\mathbf{r}$. ■

It remains to prove that the symplectic quotient of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ at level $\mathbf{r}$ coincides with the GIT quotient for the action on the trivial bundle using the twist by $\chi_{\mathbf{r}}$.

9.2.1 The Symplectic Quotient Coincides with the GIT Quotient

Let L be the trivial complex line bundle over $\bigwedge^{n-2}(\mathbb{C}^3)$. Then L descends to the trivial complex line bundle $\overline{L}$ over $P_n^{\mathcal{T}}(\mathrm{SU}(2))$. The torus $\underline{\mathbb{T}}_e$ acts on $\overline{L}$ by twisting by the character $\chi_{\mathbf{r}}$. We need to prove that the GIT quotient of $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ by $\underline{\mathbb{T}}_e$ with linearization the above action on $\overline{L}$ is homeomorphic to the symplectic quotient by the maximal compact subgroup of $\underline{\mathbb{T}}_e$ at level $\mathbf{r}$. The argument is almost the same as that of Subsection 8.4. In particular we use reduction in stages to reduce to the corresponding problem for the quotients of $\bigwedge^{n-2}(\mathbb{C}^3)$ by the product torus $\underline{\mathbb{T}}_e \times \underline{\mathbb{T}}_d^-$. Here we twist the action by the character that is $\chi_{\mathbf{r}}$ on $\underline{\mathbb{T}}_e$ and trivial on $\underline{\mathbb{T}}_d^-$. Since the momentum map for the action by $\underline{\mathbb{T}}_e \times \underline{\mathbb{T}}_d^-$ is proper by Proposition 8.19, the equality of quotients follows by Theorem A.7. The momentum level for the product group is then $(\mathbf{r}, 0)$.

10 The Residual Action of $\mathbb{T}/(\mathbb{T}_e \times \mathbb{T}_d^-)$ and Bending Flows

Now we will relate the action of $\mathbb{T}/(\mathbb{T}_e \times \mathbb{T}_d^-)$ on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ and $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ to bending flows.

10.1 A Complement to $\mathbb{T}_d^-$ in $\mathbb{T}_d$

In this subsection we will define a complement $\mathbb{T}_d^+$ to $\mathbb{T}_d^-$ in $\mathbb{T}_d$. It will be more convenient to work with this complement rather than the quotient $\mathbb{T}_d/\mathbb{T}_d^-$.

We recall that an element of $\mathbb{T}$ corresponds to a function f from the edges of $\mathcal{T}^D$ to the circle. The subgroup $\mathbb{T}_d$ corresponds to the functions with value the identity on the distinguished edges of $\mathcal{T}^D$. The nondistinguished edges of $\mathcal{T}^d$ occur in equivalent pairs. The subtorus $\mathbb{T}_d^-$ of $\mathbb{T}_d$ consists of those f whose values on equivalent pairs of nondistinguished edges are inverse to each other.

In order to construct the complement $\mathbb{T}_d^+$ we need to choose an edge from each distinguished pair. To do this in a systematic way we use the ordering of the trinodes of $\mathcal{T}^D$ induced by the trivalent induction construction; see Lemma 5.2. Suppose the edges (τ, i) and (τ', j) are equivalent. The two edges occur in different trinodes. We relabel the edges with ϵ^+ and ϵ^- by labeling the edge that comes in the first trinode with (the superscript) plus and the edge that comes in a later trinode with minus. Thus every nondistinguished edge is either a plus edge or a minus edge. We now define the subtorus $\mathbb{T}_d^+$ as the subtorus of $\mathbb{T}_d$ consisting of those functions that take value the identity on all nondistinguished minus edges. It is clear that $\mathbb{T}_d^+$ is the required complement. It is also clear that $\mathbb{T}_d^+$ is a complement to $\mathbb{T}_e \times \mathbb{T}_d^-$ in $\mathbb{T}$. We point out here that the choice of where to place the identity in the definition of this complement is irrelevant with respect to the Hamiltonian functions; see below.

10.2 The Action of $\mathbb{T}_d^+$ Coincides with Bending

We now prove the following theorem.

Theorem 10.1 (i) *The homeomorphism $\Psi_n^{\mathcal{T}}$ intertwines the bending flows on $V_n^{\mathcal{T}}$ with the action of $\mathbb{T}_d^+$ on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$.*

(ii) *The homeomorphism $\Psi_{\mathbf{r}}^{\mathcal{T}}$ intertwines the bending flows on $V_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ with the action of $\mathbb{T}_d^+$ on $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$.*

Proof Let $\mathbf{T} \in P_n^{\mathcal{T}}(\mathrm{SU}(2))$ and $\mathbf{t} \in \mathbb{T}_d^+$. Let d be a diagonal of the triangulated n-gon P and suppose the triangles T_i and T_j share the diagonal d. Suppose d divides P into two polygons P' and P'' with $T \in P'$ and $T' \in P''$. Let τ and τ' be the trinodes associated with T and T' respectively, and let $[g_{(\tau,i)}, \lambda_{(\tau,i)}\varpi_1]$ and $[g_{(\tau',j)}, \lambda_{(\tau',j)}\varpi_1]$ be spin-framed representatives of the edges of $\mathbf{T}$ corresponding to the diagonal d. Suppose $\epsilon(d)^- = (\tau', j)$.

Since the representatives are normalized, we have $g_{(\tau,i)} = g_{(\tau',j)}\varrho$. Let t be the $\epsilon(d)^+$-th component of $\mathbf{t}$ (the $\epsilon(d)^-$-th component is 1). Under the action of $\mathbf{t}$ the edge $[g_{(\tau,i)}, \lambda_{(\tau,i)}\varpi_1]$ becomes $[g_{(\tau,i)}t, \lambda_{(\tau,i)}\varpi_1]$ and all other components are unchanged. Thus the resulting element of $E_n^{\mathcal{T}}(\mathrm{SU}(2))$ is no longer normalized.

To normalize this element we multiply all the frames on the edges and diagonals of P'' by $g_{(\tau',j)}tg_{(\tau',j)}^{-1}$. Note that the image of $h = g_{(\tau',j)}tg_{(\tau',j)}^{-1} \in \mathrm{SU}(2)$. is a rotation about the oriented line in $\mathbb{R}^3$ corresponding to $Ad^*h(\varpi_1) \in \mathfrak{su}(2)^* \cong \mathbb{R}^3$ which is the diagonal corresponding to d of the Euclidean n-gon $\mathbf{e}$ underlying the imploded framed n-gon $\mathbf{T}$. Thus the Euclidean n-gon underlying the imploded framed n-gon $\mathbf{T}$ is bent along the diagonal d. Furthermore, the frames of all the edges and diagonals of P'' are transformed by applying the element of the one-parameter group $g_{(\tau',j)}tg_{(\tau',j)}^{-1}$ in SU(2) that covers rotation along the diagonal. Applying $\Phi_n^{\mathcal{T}}$ we forget the frames along the diagonals, but the frames on the edges transform in the same way as before. This amounts to bending the framed n-gon $\Phi_n^{\mathcal{T}}(\mathbf{T})$ along the diagonal d.

The first statement implies the second statement because the bending flows on $V_{\mathbf{r}}^{\mathcal{T}}$ are descended from those on $V_n^{\mathcal{T}}$, and the action of $\mathbb{T}_d^+$ on $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ is descended from the action on $P_n^{\mathcal{T}}(\mathrm{SU}(2))$ ■

10.3 The Hamiltonians for the Residual Action

It remains to prove that the action of $\mathbb{T}_d^+$ is Hamiltonian with the given Hamiltonians.

We claim that $\mathbb{T}_d^+$ preserves the orbit-type stratification. Since $\mathbb{T} = (\mathbb{T}_e \times \mathbb{T}_d^-) \times \mathbb{T}_d^+$ is abelian, the isotropy subgroup $(\mathbb{T}_e \times \mathbb{T}_d^-)_x \subset \mathbb{T}_e \times \mathbb{T}_d^-$ of $x \in P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ is equal to the isotropy group $(\mathbb{T}_e \times \mathbb{T}_d^-)_{t\cdot x}$ of $t \cdot x$, for any $t \in \mathbb{T}$; in particular, this is true for $t \in \mathbb{T}_d^+$. Although the space $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ is possibly singular, we can work in a given symplectic stratum. Thus it makes sense to say that $\mathbb{T}_d^+$ acts in a Hamiltonian fashion (on each individual stratum), and we may identify the torus action on the whole space by the Hamiltonians of the S^1 factors. These Hamiltonians are smooth in the sense of [SjL], since these functions are obtained by restricting $\mathbb{T}_e \times \mathbb{T}_d^-$ invariant continuous functions on $(\bigwedge^2\mathbb{C}^3)^{n-2}$ to individual strata.

Each $\mathbb{C}^2 \subset E_n^{\mathcal{T}}(\mathrm{SU}(2))$ is indexed by an edge of $\mathcal{T}^D$. Given an internal edge ϵ, the factor $(S^1)_\epsilon$ of $\mathbb{T}_d^+$ has Hamiltonian function $\|(z,w)_{\epsilon^-}\|^2/2 = \|(z,w)_{\epsilon^+}\|^2/2$ on any given orbit-type stratum. The $(z,w)_{\epsilon^-}$ corresponds to the diagonal d_ϵ of the associated n-gon in $\mathbb{R}^3$. By Lemmas 8.16 and 3.29 we have $\|d_\epsilon\| = \frac{1}{4}\|(z,w)_{\epsilon^-}\|^2$, and so $\|d_\epsilon\|$ is *one-half* that of the Hamiltonian for the ϵ-th factor $(S^1)_\epsilon$ of $\mathbb{T}_d^+$. In what follows we will need the following elementary lemma.

Lemma 10.2 *Suppose $A\colon S^1 \times X \to X$ is a Hamiltonian action of the circle on a stratified symplectic space X. Suppose the action is generated by the Hamiltonian potential f and that the element $-1 \in S^1$ acts trivially. Let $\overline{A}$ be the induced action of the quotient circle $\overline{S^1} = S^1/\{\pm 1\}$. Then the Hamiltonian potential for the $\overline{A}$ action is $\frac{f}{2}$.*

Proof We have $\overline{A}(\exp\sqrt{-1}t, x) = A(\exp\sqrt{-1}t/2, x)$. ∎

The bending flow torus
$$T_{bend} = \prod_{d \text{ diagonal}} (S^1)_d,$$
acting on $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$, is such that the $d(\epsilon)$-th factor has Hamiltonian $\|d_\epsilon\|$. Therefore, the $\mathbb{T}_d^+$ action factors through the action of T_{bend} via a two to one cover on each component of $\mathbb{T}_d^+$. Thus we may conclude that T_{bend} and $\mathbb{T}_d^+(\mathrm{SO}(3,\mathbb{R}))$ coincide.

Theorem 10.3 *Let $\mathbb{T}_d^+(\mathrm{SO}(3,\mathbb{R})) = \mathbb{T}_d^+/(\mathbb{Z}/2\mathbb{Z})^{n-3}$ be the image of $\mathbb{T}_d^+$ in $\mathrm{SO}(3,\mathbb{R})^{n-3}$ under the surjection $\mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$. The action of $\mathbb{T}_d^+(\mathrm{SO}(3,\mathbb{R}))$ on $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ coincides with the bending flows. Precisely, the action is Hamiltonian, and the circle factor corresponding to $d(\epsilon)$ has Hamiltonian potential $\|d_\epsilon\|$.*

Proof Let $\pi\colon \mathbb{T}_d^+ \to \mathbb{T}_d^+(\mathrm{SO}(3,\mathbb{R}))$ be the quotient map. Let ϵ be any internal edge of $\mathcal{T}$. The action of $(S^1)_\epsilon \subset \mathbb{T}_d^+$ on $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$ factors through the action of $\pi((S^1)_\epsilon) \in \mathbb{T}_d^+(\mathrm{SO}(3,\mathbb{R}))$, through the map $t \mapsto t^2$. Consequently by the lemma above the Hamiltonian function of $(S^1)_\epsilon \subset \mathbb{T}_d^+$ is twice that of the Hamiltonian of $\pi((S^1)_\epsilon) \subset \mathbb{T}_d^+(\mathrm{SO}(3,\mathbb{R}))$. Hence the Hamiltonian for $\pi((S^1)_\epsilon) \subset \mathbb{T}_d^+(\mathrm{SO}(3,\mathbb{R}))$ is $\|d_\epsilon\|$, as it is for $(S^1)_{d_\epsilon} \subset T_{bend}$. Since the Hamiltonian functions of $T_d^+(\mathrm{SO}(3,\mathbb{R}))$ and T_{bend} coincide, their actions must coincide on $P_{\mathbf{r}}^{\mathcal{T}}(\mathrm{SU}(2))$. ∎

A Appendix : Symplectic and GIT Quotients of Affine Space

The results in this appendix were obtained with the aid of W. Goldman.

A.1 Fiber Twists and Normalizing the Momentum Map

The goal of this appendix is to prove Theorem A.7. We match the level for the symplectic quotient with the twist used to define the linearization in forming the GIT quotient. In four places in the paper, namely Subsections 6.5, 8.1, 8.3, and Subsubsection 9.2.1 we applied the results of [Sj] to deduce that for a torus acting on affine space linearized by acting on the trivial line bundle by a character χ, the GIT quotient is homeomorphic to the symplectic quotient at level the derivative of χ at the identity provided the momentum map was proper. However, there is a technical problem about the normalization of the momentum map chosen for the action of the torus (there is an indeterminancy of an additive constant vector). The correct normalization of the momentum map must depend on the action of the torus on the total space of the line bundle. It is given in [Sj, Formula (2.3), p. 116], which we now state for the convenience of the reader.

Let $p\colon E \to M$ be a Hermitian line bundle L. Let G be a compact group acting on E by automorphisms of the Hermitian structure. Let $\xi \in \mathfrak{g}, e \in E$ and $m = p(e)$. In what follows ξ_E denotes the vector field on E induced by ξ; ξ_M^{hor} denotes the horizontal lift of the vector field ξ_M induced on M, and ν_E denotes the canonical vertical vector field (induced by the U(1) action). The connection and curvature forms take values in the Lie algebra $\mathfrak{u}(1) = \sqrt{-1}\mathbb{R}$ of U(1). We can now state the formula from [Sj]:

$$\xi_E(e) = \xi_M^{hor}(e) + 2\pi\langle\mu(m), \xi\rangle\nu_E. \tag{A.1}$$

Definition A.1 We will say a momentum map satisfying equation (A.1) is *normalized* relative to the linearization (action of G on the total space of the bundle).

Remark A.2 We can check the conventions involved in equation (A.1) by applying the connection form θ to both sides to obtain

$$\theta(\xi_E(e)) = 2\pi\sqrt{-1}\langle\mu(m), \xi\rangle.$$

Applying d to each side and Cartan's formula we find that $\langle\mu(m), \xi\rangle$ is a Hamiltonian potential for ξ_M if and only if the symplectic form ω and the connection form θ are related by

$$\omega = -\frac{1}{2\pi\sqrt{-1}}d\theta.$$

Suppose now we twist the action of the torus G on the total space of the line bundle by scaling each fiber by a fixed character χ. This changes the invariant sections and hence changes the GIT quotient. Note that the differential of χ at the identity of G is an element $\dot{\chi}$ of $\mathfrak{g}^*$. Thus we could change the momentum map μ by adding $\dot{\chi}$ and obtain a new momentum map. The following lemma is an immediate consequence of equation (A.1).

Lemma A.3 *Suppose we twist the action of G by a character χ. Then the normalized momentum map for the new action is obtained by adding $\dot{\chi}$.*

We now restrict to the case of a torus T acting linearly on a symplectic vector space V, ω. We assume we have chosen a T-invariant complex structure J on V so that ω is of type $(1, 1)$ for J (this means J is an isometry of ω), and the symmetric form B given by $B(v, v) = \omega(v, Jv)$ is positive definite. We let W be the subspace of $V \otimes \mathbb{C}$ of type $(1, 0)$ vectors, that is, $W = \{v - \sqrt{-1}Jv : v \in V\}$. We define a positive-definite Hermitian form H on W by

$$H(v_1 - \sqrt{-1}Jv_1, v_2 - \sqrt{-1}Jv_2) = B(v_1, v_2) - \sqrt{-1}\omega(v_1, v_2).$$

We will abbreviate $\sqrt{H(v, v)}$ to $\|v\|$ in what follows. We define a symplectic form A on W by $A(w_1, w_2) = -\Im H(w_1, w_2)$. We note that the map $w \to \Re w$ is a symplectomorphism from W, A to V, ω.

Suppose that we have chosen an H-orthonormal basis for W, so we have identified $W \cong \mathbb{C}^n$. We let $T_0 \cong (S^1)^n$ be the compact torus consisting of the diagonal matrices with unit length elements on the diagonal and $\underline{T}_0$ be the complexification of T_0. We let $\mathfrak{t}_0^* \cong \mathbb{R}^n$ be the dual of the Lie algebra of T_0. We define $\mu_0 \colon W \to \mathfrak{t}_0^*$ by

$$\mu_0((z_1, \dots, z_n)) = \Big(-\frac{|z_1|^2}{2}, \dots, -\frac{|z_n|^2}{2}\Big).$$

Then μ_0 is a momentum map for the Hamiltonian action of T_0 on W. We note that all possible momentum maps are obtained from μ_0 by adding a vector $\mathbf{c} = (c_1, \dots, c_n) \in \mathfrak{t}_0^*$. Thus μ_0 is the unique momentum map vanishing at the origin of W.

Let $E = W \times \mathbb{C}$ be the total space of the trivial line bundle L over W. We first describe the Hermitian holomorphic structure on L. We give L a holomorphic structure by requiring that the nowhere vanishing section s_0 of E defined by $s_0(w) = (w, 1)$ is holomorphic. Thus if U is an open subset of W and s is a local section over U, then s is holomorphic if and only if the function f on U defined by $s = fs_0 \mid U$ is holomorphic.

We define a Hermitian structure on L by defining

$$\|s_0\|(w) = \exp\left(-\frac{\pi}{2}\|w\|^2\right).$$

Hence the section σ_0 given by

$$\sigma_0(w) = \exp\left(\frac{\pi}{2}\|w\|^2\right) s_0(w)$$

has unit length at every point. We note that

$$-\frac{1}{2\pi\sqrt{-1}}\,\overline{\partial}\partial\ \log\|s_0\|^2 = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

in agreement with [Sj, p. 115]. Let E_0 denote the principal circle bundle of unit length vectors in E.

Our goal is to describe the lift of the action of T_0 to E so that the normalized momentum map corresponding to this lifted action is μ_0. We note there is a distinguished lift given by

$$t \circ (w, z) = (tw, z), t \in T_0, w \in W, z \in \mathbb{C}.$$

By definition this lift leaves invariant the holomorphic section s_0, but it also leaves invariant the unit length section σ_0 because the function $\exp(-\frac{\pi}{2}\|w\|^2)$ is invariant under $\mathrm{U}(n)$ and hence under T_0. Hence T_0 leaves fixed the Hermitian structure on E, and $\underline{T}_0$ leaves the holomorphic section s_0 fixed. We will call such lifts *untwisted*, and we will say the linearization consisting of the trivial bundle together with the previous action is untwisted. We will now prove the following.

Proposition A.4 *The normalized momentum map of T_0 corresponding to the untwisted linearization is μ_0.*

The proposition will be a consequence of the next lemma and corollary.

We will compute μ_0 in the trivialization of E_0 given by σ_0. We note that since σ_0 is invariant under T_0, the untwisted lift of the compact torus above (relative to the trivialization by s_0) remains untwisted in the trivialization by σ_0. We let ψ denote the coordinate in the fiber circle of E_0 so $\nu_{E_0} = \partial/\partial\psi$. Hence if $z_i = x_i + \sqrt{-1}y_i, 1 \leq i \leq n$, then $x_1, y_1, \ldots, x_n, y_n, \psi$ are coordinates in E_0.

Lemma A.5 *The canonical connection on E is given by*

$$\nabla\sigma_0 = -\pi\sqrt{-1}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i)\otimes\sigma_0.$$

Equivalently in the above coordinates the connection form θ of the canonical connection is given by

$$\theta = \sqrt{-1}d\psi - \pi\sqrt{-1}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i).$$

Proof The reader will verify that ∇ satisfies $\nabla_{\partial/\partial\bar{z}_j}s_0 = 0, 1 \leq j \leq n$ and has curvature $-2\pi\sqrt{-1}\omega$. Hence ∇ is the unique Hermitian connection with curvature $d\theta = -2\pi\sqrt{-1}\omega$. ■

We then have the following corollary.

Corollary A.6 *The horizontal lift of the vector field $x_i\partial/\partial y_i - y_i\partial/\partial x_i$ is*

$$x_i\partial/\partial y_i - y_i\partial/\partial x_i + \pi(x_i^2 + y_i^2)\partial/\partial\psi.$$

Proof By the formula for θ we see that the horizontal lift of $\partial/\partial x_i$ is $\partial/\partial x_i - \pi y_i\partial/\partial\psi$, and the horizontal lift of $\partial/\partial y_i$ is $\partial/\partial y_i + \pi x_i\partial/\partial\psi$. Since the operation of taking horizontal lifts is linear over the functions, the corollary follows. ■

Proposition A.4 follows from the corollary and equation (A.1).

Now suppose T is a compact torus with complexification $\underline{T}$, and T acts on W through a representation $\rho\colon T \to T_0$. Assume further that we linearize the action of $\underline{T}$ on W by the untwisted linearization. It is clear from equation (A.1) that we obtain the *normalized* momentum map μ_T for T by restricting the *normalized* momentum map for T_0. More precisely, let $\rho^*\colon \mathfrak{t}_0^* \to \mathfrak{t}^*$ be the induced map on dual spaces. Then we have $\mu_T = \rho^* \circ \mu_0$. Thus the normalized momentum map corresponding to the untwisted linearization of the linear action of T is homogeneous linear in the squares of the $|z_i|$'s. We now obtain the desired result in this appendix by applying Lemma A.3 and [Sj, Theorem 2.18, p. 122].

Theorem A.7 *Let $\underline{T}$ be a complex torus with maximal compact subtorus T and let χ be a character of $\mathbb{T}$. Then the GIT quotient for a linear action of $\underline{T}$ on a complex vector space W with linearization given by $W \times \mathbb{C}$ and the action*

$$t \circ (w, z) = (tw, \chi(t)z)$$

is homeomorphic to the symplectic quotient by T obtained using the momentum map $(\mu_T)_0 + \dot{\chi}$ if the momentum map for the action of T is proper. Here $(\mu_T)_0$ is the momentum map that vanishes at the origin of W.

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