Annals of Mathematics, **108** (1978), 1-39

# Closed geodesics and the $\eta$-invariant

By JOHN J. MILLSON

Dedicated to the memory of V.K. Patodi

## Introduction

In [3], Atiyah, Patodi and Singer introduced an invariant of a Riemannian manifold of dimension $4n-1$. This invariant, which they called the $\eta$-invariant, is determined by the spectrum of a certain self-adjoint square root of the Laplacian on differential forms. It is non-local; that is, it is not obtained by integrating a universal polynomial in curvature over the manifold. Thus, unlike earlier invariants determined by the spectrum such as the Euler characteristic or the signature, it cannot be computed from the asymptotic expansion of Trace$^{-t\Delta}$, the trace of the heat operator, as $t$ goes to zero. However, recently, Colin de Verdiere [26], Chazarain [7] and Duistermaat-Guillemin [9] discovered a connection between the spectrum of the Laplacian and non-local information about a Riemannian manifold by studying the distribution trace of the fundamental solution of the wave equation. For generic manifolds the singularities of this distribution on the real line are at the set of lengths of closed geodesics and there is an asymptotic expansion at each singularity with coefficients giving information about the closed geodesic. It is an important question to decide if the $\eta$-invariant can be determined from these data. The following formula shows this is indeed the case for manifolds of constant negative curvature and suggests a general formula.

Let $M$ be a compact oriented $4n-1$ dimensional Riemannian manifold of constant negative curvature. Let $\mathcal{P}$ be the set of primitive closed geodesics on $M$. Then each $\gamma \in \mathcal{P}$ determines the holonomy element $R(\gamma) \in$ SO$(4n-2)$ by parallel translation around $\gamma$, the (linearized) Poincaré map $P(\gamma) \in \mathrm{Sp}(8n-4, \mathbf{R})$ and the length $L(\gamma)$ of $\gamma$. We stop to give a definition of $P(\gamma)$. Let $\varphi_t$ denote the geodesic flow on $S(M)$, the unit tangent bundle of $M$. Then a closed geodesic of length $L$ corresponds to a fixed-point of $\varphi_L$. Then $d\varphi_L$ maps the tangent space of that fixed-point to itself and preserves the geodesic flow direction and hence induces a transformation $P(\gamma)$ normal

0003-486X/78/0108-0001 \$ 01.95

to that direction. Now we return to the statement of our results. Let $\phi$ be the standard representation of $SO(4n-2)$ regarded as a complex representation. Then the middle exterior power $\Lambda^{2n-1}\phi$ decomposes into the sum of two irreducible representations of $SO(4n-2)$ corresponding to the $+i$ and $-i$ eigenspaces of * acting on the space of $\Lambda^{2n-1}\phi$. We let $ch_+$ be the character of the $+i$ eigenspace and $ch_-$ the character of the $-i$ eigenspace and define a class function $\chi$ on $SO(4n-2)$ by

$$\chi(g) = ch_+(g) - ch_-(g) .$$

We set $N(\gamma) = e^{L(\gamma)}$ and using the idea of Selberg [21] we define a zeta function by the following series which is absolutely convergent for $\operatorname{Re} s > 2n-1$:

$$\log \tilde{Z}(2n-1+s) = \sum_{\mathcal{P}} \sum_{k=1}^{\infty} \frac{\chi(R(\gamma)^k)}{|\det(I - P(\gamma)^k)|^{1/2}} \frac{N(\gamma)^{-ks}}{k} .$$

Then $\tilde{Z}(s)$ admits a meromorphic continuation to the entire complex plane and

$$\eta(0) = \frac{1}{\pi i} \log \tilde{Z}(2n-1) .$$

In fact $\tilde{Z}$ satisfies the remarkable functional equation

$$\tilde{Z}(s)\tilde{Z}(4n-2-s) = e^{2\pi i \eta(0)}$$

and satisfies the Riemann hypothesis (it has *all* its zeroes on the line $\operatorname{Re} s = 2n-1$).

The properties of $\tilde{Z}$ developed in Chapter III should be considered as joint work with Takuro Shintani. He first proposed the group theoretic version of the formula for $\tilde{Z}$ in the three dimensional case and proved the functional equation (with an unknown constant on the right hand side). Theorem 3.1 (and consequently the calculation of the constant) is due to the author. It is a great pleasure to thank Takuro Shintani for all these contributions and for many helpful and agreeable conversations. We would also like to thank Jim Arthur for much help with the Selberg trace formula and the unitary representation theory of the noncompact simple groups. His thesis [1] was a great help to us in learning this difficult subject. Lastly we would like to thank M.F. Atiyah who encouraged us to begin these calculations four years ago and I.M. Singer who showed us how to rewrite our original group-theoretic formula to obtain the present one and also helped us with some other problems that arose while we were working on this paper.

## Chapter I

### 1. The $\eta$-invariant of Atiyah, Patodi and Singer

Let $M^{4n-1}$ be a compact oriented Riemannian manifold. Consider the operator $A$ on odd forms on $M$, $\bigoplus_{p=0}^{2n}\Omega^{2p-1}$ defined on $\Omega^{2p-1}$ by

$$A = (-1)^{n+p}(*d + d*) .$$

It is easily seen that

(i) $A$ is self-adjoint;

(ii) $A^2 = \Delta$, the Hodge Laplacian.

From (i) and (ii) it follows that $A$ is diagonalizable with real eigenvalues. The point of the definition of the $\eta$-invariant is that the eigenvalues $\lambda_n$ of $A$ can be either positive or negative—they are square roots of the eigenvalues of $\Delta$. We define

$$\eta(s) = \sum_{n=0}^{\infty} \frac{\operatorname{sign}\lambda_n}{|\lambda_n|^s} = \sum_{\lambda_n>0} \frac{1}{\lambda_s^n} - \sum_{\lambda_n<0} \frac{1}{|\lambda_n|^s} .$$

In [3] the following result is proved. $\eta(s)$ has a meromorphic continuation to the entire complex plane and does not have a pole at zero. From this it follows that $\eta(0)$ is well-defined. This value is the $\eta$-invariant. The above result is proved with the help of the formula

$$\eta(s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{s+1/2} \operatorname{tr} A e^{-t\Delta} \frac{dt}{t} .$$

This formula will be the basis of our calculations. We will not however use the operator $A$ but replace $A$ by its restriction to the space of (coclosed) $2n-1$ forms. If we call this new operator $\bar{A}$ then $\bar{A} = *d$ and we have the following result from [3]:

$$\eta(s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{s+1/2} \operatorname{tr} \bar{A} e^{-t\Delta} \frac{dt}{t} .$$

For the rest of this paper $A$ will denote this operator.

### 2. Review of homogeneous vector bundles

Given a homogeneous space $X$ of dimension $m$ with $X = G/K$ where $K$ is compact and a unitary representation $\sigma: K \to \mathrm{GL}(V_\sigma)$, we can form a vector bundle $E = Gx_\sigma V$. The points of $E$ are equivalence classes of pairs $(g, v)$ under $(gk, v) \sim (g, \sigma(k)v)$. We denote such equivalence classes by $[g, v]$. Note that $E$ admits a left $G$ action $g_0[g, v] = [g_0 g, v]$. We call $\sigma$ the isotropy representation of $E$.

Given such a bundle $E$ we can form $\Gamma(E)$, its $L^2$ sections ($G/K$ has an invariant measure induced from Haar measure on $G$). $G$ acts on a section $s$ by $g_0 s(x) = g_0 s(g_0^{-1}x)$. By this procedure we associate to every unitary representation $\sigma$ of $K$, a unitary representation of $G$ whose spaces are $\Gamma(E)$. We denote this representation $\mathrm{Ind}_K^G \sigma$ and call it the representation of $G$ induced from $\sigma$. $\mathrm{Ind}_K^G \sigma$ is not necessarily irreducible but in case $G$ is compact the Frobenius Reciprocity Theorem, Wallach [27, p. 118], tells us how to decompose it into irreducibles. We make the important remark that an element $f$ of $\Gamma(E)$ may be identified with a $K$-equivariant mapping $\tilde{f}: G \to V_\sigma$.

We assume henceforth that $G$ is simple and non-compact, that $K$ is a maximal compact subgroup of $G$ (hence $X = G/K$ is symmetric) and that $C$ is the Killing form of $G$. Let $E_1, E_2, \cdots, E_m$ be a basis for the right invariant fields on $G$ which are orthonormal with respect to the metric $(x, y) = C(x, \theta y)$. The operator $\Omega' = \sum_{i=1}^m E_i^2$ operating on functions on $G$ is called the Casimir operator of $G$. We recall that $\Omega'$ is in the center of the universal enveloping algebra of $G$. (For the case we are interested in, $G = \mathrm{SO}(n, 1)$, we define a new operator $\Omega$ by the formula $\Omega = -(2n-2)\Omega'$). If $\rho$ denotes the action of $G$ on the bundle of forms on $G/K$ then we have

KUGA'S LEMMA.

$$\rho(\Omega') = -\Delta$$

*where $\Delta$ is the Hodge Laplacian for the metric induced on $G/K$ by $C$.*

*Proof.* See Matsushima-Murakami [20].

*Remark.* For each element $e_i$ of the Lie algebra of $G$ we associate a Killing field $\bar{E}_i$ on $G/K$ using the left action of $G$ on $G/K$ as follows:

$$\bar{E}_i(m) = \frac{d}{dt}\left(\exp t(-te_i)\cdot m\right)\Big|_{t=0}.$$

Clearly the action of the Lie algebra of $G$ on forms associated to the action of $G$ is $e_i \to \mathfrak{L}_{\bar{E}_i}$ where $\mathfrak{L}_X$ denotes the Lie derivative operation by $X$. Thus,

$$\rho(\Omega') = \sum_{i=1}^m \mathfrak{L}_{\bar{E}_i} \circ \mathfrak{L}_{\bar{E}_i}.$$

With $\Omega$ as above for $G = \mathrm{SO}(n, 1)$ we have $\rho(\Omega) = \Delta$ where $\Delta$ is the Laplacian of the metric with constant curvature $-1$.

In this paper we shall be solely concerned with homogeneous vector bundles $E$ which are exterior powers of the cotangent bundle of $G/K$. Let us first recall what the tangent bundle of $G/K$ looks like. The Lie algebra of $G$ which we will denote by $\mathfrak{g}$ splits as a vector space

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

Here $\mathfrak{k}$ is the Lie algebra of $K$. This splitting is just the splitting of $\mathfrak{g}$ into a subspace $\mathfrak{k}$ on which the Cartan-Killing form $C$ is negative definite and a subspace $\mathfrak{p}$ on which $C$ is positive definite. Of course we may identify $\mathfrak{p}$ with the tangent space to $G/K$ at the identity coset which we will henceforth refer to as the origin of the homogeneous space $G/K$. $C$ provides us with an invariant metric on $G/K$.* This metric, in the case $G = \mathrm{SO}(n, 1)$, $K = \mathrm{SO}(n)$, has constant sectional curvature $-(1/2n - 2)$. We will use the metric $\langle x, y\rangle = (1/2n - 2)C(x, y)$ which will have constant curvature $-1$. Now the action of $K$ on $\mathfrak{g}$ given by restricting the adjoint action of $G$ to $K$ also splits into the direct sum of an action on $\mathfrak{k}$ (just the adjoint action of $K$) and an action of $\mathfrak{p}$. This later action will be referred to as $\phi$. Then $\phi$ is the isotropy representation of the tangent bundle. It is also the isotropy representation of the cotangent bundle and $\Lambda^p\phi$ will be the isotropy representation of the bundle of $p$-forms. Now the *left*-invariant distribution on $G$ corresponding to $\mathfrak{p}$ is the horizontal distribution of the Riemannian connection on the tangent bundle of $G/K$ associated to $C$. In other words the left-invariant vector fields correspond to the canonical horizontal fields from the theory of principal bundles (which never correspond to vector fields on the base). Thus $\Lambda^p\mathfrak{p}^*$ extended to be left-invariant is the horizontal distribution associated to the connection on $p$-forms. Let $\omega_1, \omega_2, \cdots, \omega_n$ be a basis for the left-invariant 1-forms on $G$ which are horizontal (that is $\omega_{i/e} \in \mathfrak{p}^*$). Now given any $p$ form $\omega$ on $G/K$ we may lift it to $G$ to obtain a horizontal $p$-form $\omega' = \pi^*\omega$ on $G$. We may then write out the components of $\omega'$ relative to the basis for the horizontal $p$-forms obtained by forming all possible products $\{\omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_p} : 1 \leqq i_1 < \cdots < i_p \leqq n\}$. The component functions $(f_{i_1 i_2 \ldots i_p})$ give us a map $f\colon G \to \Lambda^p\mathbf{R}^n$ which satisfies $f(gk) = \sigma(k^{-1})f(g)$ where $\sigma = \Lambda^p\phi^*$.

We now return to a general homogeneous vector bundle $E$ to discuss invariant differential and integral operators on homogeneous bundles. We begin with a study of first order invariant differential operators on $\Gamma(E)$. Let $D$ be such an operator.

We define a differential operator $\tilde{D}$ on $C^\infty(G) \otimes_K V_\sigma$ by

$$\tilde{D}\tilde{f} = \widetilde{Df}\,.$$

Let $\{x_i\}$, $i = 1, 2, \cdots, m$, be an orthonormal basis for $\mathfrak{p}$ and $X_i^L$ the corresponding left-invariant vector fields on $G$. Let $\{\omega_i\}$, $i = 1, 2, \cdots, m$, be a dual basis for the left-invariant 1-forms which are horizontal.

* This metric is $n-1$ times the metric $C'$ given by $C'(x, y) = \text{trace } xy$ where $x, y$ are thought of as symmetric $n+1$ by $n+1$ matrices.

Before describing $\tilde{D}$ we recall for the benefit of the reader the definition of the symbol of a first order differential operator on a vector bundle $E \to M$. Let $\xi$ be a contangent vector to $M$ at $m$ and $v \in E_m$. Let $\varphi$ be a real-valued function on $M$ satisfying $\varphi(m) = 0$, $d\varphi(m) = \xi$ and let $f$ be a section of $E$ satisfying $f(m) = v$. Then

$$S_m(\xi)v = D(\varphi f)(m) .$$

LEMMA 1.1. *There exist constant matrices $A_j$, $j = 1, 2, \cdots, m$, and $B$ so that*

$$\tilde{D} = \sum_{i=1}^{m} X_j^L \otimes A_j + B .$$

*Here $A_j = s_1(\omega_j|_1)$ and $B \in \mathrm{Hom}_K(V_\sigma, V_\sigma)$ may be determined as follows. Let $f \in \Gamma(E)$ have the properties that $f(0) = v$ and $f$ is covariant constant at* 0; *then,*

$$Bv = Df(0) .$$

*Proof.* We observe that there exist functions $B, A_j: G \to \mathrm{End}\, V_\sigma$, $j = 1, 2, \cdots, m$, so that $\tilde{D}$ has the above form whether or not $\tilde{D}$ is invariant. Because $\tilde{D}$ is invariant these functions are constant functions. It follows immediately from the definition of the symbol that

$$A_j(g) = s_g(\omega_j) .$$

Lastly the formula for $B$ follows from the lemma, page 115 of Kobayashi-Nomizu [16] which implies that the horizontal left-invariant fields at $g$ annihilate the lift of a section which is covariant constant at $gK$. This proves Lemma 1.2.

We now investigate the properties the matrices $A_j$ must have in order that $\tilde{D}$ preserve the equivariance of $f$ under $K$.

LEMMA 1.2. *Let $s$ be the symbol of a $G$-invariant first order differential operator on a homogeneous vector bundle $E$ with isotropy representation $\sigma$. Then*

$$s_1 \in \mathrm{Hom}_K(\mathfrak{p}^*, \mathrm{End}\, V_\sigma) ;$$

*that is,*

$$s_1(Adk \cdot \xi) = \sigma(k) s_1(\xi) \sigma(k)^{-1} .$$

*A more useful equivalent formulation is that*

$$\sum_{j=1}^{m} X_j^L \otimes A_j \in (\mathfrak{p} \otimes \mathrm{End}\, V_\sigma)^K ;$$

*that is,*

$$\sum_{j=1}^{m} Adk\, X_j^L \otimes \sigma(k) A_j \sigma(k^{-1}) = \sum_{i=1}^{m} X_j^L \otimes A_j .$$

*Proof.* We prove the last formula. Suppose $f \in \Gamma(E)$ and $\tilde{D}f \in \Gamma(E)$. Then

$$\frac{d}{dt}\left(\sum_{j=1}^{m} A_j f(gk \exp tX_j)\right)\bigg|_{t=0} = \sigma(k^{-1})\frac{d}{dt}\left(\sum_{j=1}^{m} A_j f(g \exp tX_j)\right)\bigg|_{t=0}.$$

But

$$\begin{aligned}\frac{d}{dt}\left(\sum_{j=1}^{m} A_j f(gk \exp tX_j)\right)\bigg|_{t=0} &= \frac{d}{dt}\left(\sum_{j=1}^{m} A_j f(gk \exp tX_j k^{-1}k)\right)\bigg|_{t=0}\\ &= \frac{d}{dt}\left(\sum_{j=1}^{m} A_j \sigma(k^{-1}) f(g \exp tkX_j k^{-1})\right)\bigg|_{t=0}\\ &= \sum_{j=1}^{m} \mathrm{Ad}\, k \cdot X_j^L \otimes A_j \sigma(k^{-1}) f(g).\end{aligned}$$

Hence for all $f \in \Gamma(E)$

$$\sum_{j=1}^{m} X_j^L \otimes \sigma(k^{-1}) A_j f(g) = \sum_{j=1}^{m} \mathrm{Ad}\, k \cdot X_j^L \otimes A_j \sigma(k^{-1}) f(g).$$

This proves the lemma.

*Remark.* For $k^{\text{th}}$ order operators one has $s_1 \in \mathrm{Hom}_K(S^k\mathfrak{p}^*, \mathrm{End}\, V_\sigma)$ where $S^k\mathfrak{p}^*$ is the $k^{\text{th}}$ symmetric power of $\mathfrak{p}^*$, hence $D \in (S^k\mathfrak{p} \otimes \mathrm{End}\, V_\sigma)^K$.

We now examine how the kernels of invariant integral operators appear in our setting. $E$ will denote as usual a homogeneous vector bundle defined by an isotropy representation $\sigma$. The kernel of an integral operator $\xi$ will be a section of $E \boxtimes E^*$, the external tensor product of $E$ with $E^*$ over $G \times G$. Because $E$ is a trivial bundle, an $L^2$ section of this bundle will be merely an element of $L^2(G \times G) \otimes \mathrm{End}\, V_\sigma$, that is an endomorphism valued function on $G \times G$. If $e$ denotes the kernel of $\xi$ then $e$ will satisfy *K-equivariance*

$$\begin{aligned} e(g_1 k, g_2) &= \sigma(k^{-1}) \circ e(g_1, g_2),\\ e(g_1, g_2 k) &= e(g_1, g_2) \circ \sigma(k).\end{aligned}$$

If the operator $\xi$ is symmetric then $e$ will satisfy *symmetry*:

$$e(g_1, g_2) = e^*(g_2, g_1).$$

Here $^*$ denotes the adjoint operation in $\mathrm{End}\, V_\sigma$.

If the operator $\xi$ commutes with $G$ then $e$ will satisfy *G-invariance*:

$$e(gg_1, gg_2) = e(g_1, g_2).$$

We will assume all three of these properties henceforth. We now deduce an elementary (but important) consequence of $G$-invariance.

LEMMA 1.3. *If $\xi$ is G-invariant then there is a function $\bar{e}: G \to \mathrm{End}\, V$ such that:*

$$e(g_1, g_2) = \bar{e}(g_2^{-1} g_1).$$

*Proof.* Set $\bar{e}(g) = e(g, 1)$. We say $\bar{e}$ is a representation function for $\xi$.

*Remark.* $\bar{e}(k_1 g k_2) = \sigma(k_2^{-1})\bar{e}(g)\sigma(k_1^{-1})$. The space of all smooth functions satisfying the above law will be denoted by $L_\sigma(G)$.

We need one more definition concerned with integral operators. Given an integral operator $\xi$ with representation function $\bar{e}$, we define the local trace of $\xi$ written $\tilde{e}$ to be the scalar function on $G$ given by

$$\tilde{e}(g) = \operatorname{trace} \bar{e}(g) .$$

Here the trace is taken in End $V_\sigma$. Then under suitable hypotheses on $\xi$ which will be discussed in Chapter 2 we have that $\xi$ is trace class and the trace of $\xi$ denoted $\operatorname{Tr} \xi$ is given by

$$\operatorname{Tr} \xi = \int_G \tilde{e}(g) dg .$$

We now give the examples that will concern us. We first describe how $d$ looks in this framework. The symbol of $d$ at $\omega_i$ is the map $s(\omega_i)\colon \Lambda^p \mathfrak{p}^* \to \Lambda^{p+1}\mathfrak{p}^*$ given by $s(\omega_i) \circ \omega = \omega_i \wedge \omega$. Noting that $d$ acting at a point annihilates sections which are covariant constant at that point we have

$$d = \sum_{j=1}^m X_j^L \otimes s(\omega_j) .$$

Hence

$$A = *d \circ s(\omega_j) = \sum_{j=1}^m X_j^L \otimes * s(\omega_j) .$$

We shall reserve the symbol $s$ to designate the symbol of $d$ henceforth. We will also denote the representation function of the heat operator $e^{-t\Delta}$ by $\bar{e}_t$ and its local trace by $\tilde{e}_t$ and the representation function of the integral operator $Ae^{-t\Delta}$ by $\bar{a}_t$ and its local trace by $\tilde{a}_t$.

### 3. Unitary representation theory of the Lorentz group

In this section we will establish some notation and define the unitary principal series for certain real rank 1 groups. Of course in this paper we will be interested only in $\mathrm{SO}(4n-1, 1)$ but we will maintain a certain generality for the sake of clarity.

Thus we let $G$ be a rank 1 group and $K$ a maximal compact subgroup which we assume has rank strictly less than the rank of $G$. We can find a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which splits $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} \oplus \mathfrak{a}_\mathfrak{p}$ where $\mathfrak{a}_\mathfrak{p}$ is one dimensional. We exponentiate to obtain a Cartan subgroup which splits $A = A' A_\mathfrak{p}$ where $A' \subseteqq K$. We let $M$ denote the centralizer of $A_\mathfrak{p}$. Then $A_\mathfrak{p}$ is isomorphic to $\mathbf{R}$. We now give an explicit isomorphism. For $\mathrm{SO}(m, 1)$ there is a unique restricted root which we denote by $r_0$. Let $H \in \mathfrak{a}_\mathfrak{p}$ be such that $r_0(H) = 1$. Then we define the element $a_r \in A_\mathfrak{p}$ corresponding to $r \in \mathbf{R}$ by

$$a_r = \exp rH\,.$$

We have $C(H, H) = 2m - 2$ hence $\langle H, H\rangle = 1$ where $\langle\ ,\ \rangle$ is the curvature $-1$ metric and $C$ is the Killing form.

Now let $\mathfrak{g}_C$ denote the complexification of $\mathfrak{g}$ and $\mathfrak{a}_C$ the complexification of $\mathfrak{a}$. We let $P_+$ denote the set of positive roots of $\mathfrak{g}_C$ which do not vanish on $\mathfrak{p}$. Let $\mathfrak{n}_C = \sum_{\alpha \in P_+} CX_\alpha$ where for any $\alpha \in P$, $X_\alpha$ is a fixed root vector. Let $\mathfrak{n} = \mathfrak{n}_C \cap \mathfrak{g}$ and $N$ be the analytic subgroup of $G$ corresponding to $\mathfrak{n}$. The well-known Iwasawa decomposition states that the map

$$(n, a, k) \longrightarrow nak\ , \quad n \in N,\ a \in A_\mathfrak{p},\ k \in K$$

gives a diffeomorphism of $N \times A_{\mathscr{P}} \times K$ with $G$. We denote by $B$ the group $MA_\mathfrak{p}N$.

We now construct a basis $\{X_j\colon j = 1, 2, \cdots, m\}$ for $\mathfrak{p}$. Let $X_1 = H$. Let $N_2, N_3, \cdots, N_m$ be root vectors for the root $r_0$ chosen so that $[N_j, \theta(N_i)] = 0$, $i \neq j$, and so that $[N_j, \theta(N_j)] = -2H$ where $\theta$ is the Cartan involution. We define $X_j$ in the orthogonal complement of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{p}$ (which will be denoted by $\mathfrak{q}$) by

$$X_j = \frac{1}{2}\big(N_j - \theta(N_j)\big)\,.$$

Then the basis $\{X_j\colon j = 1, 2, \cdots, m\}$ is an orthonormal basis for $\big(\mathfrak{p}, \langle\ ,\ \rangle\big)$. We define $Y_j \in k$ for $j = 1, 2, \cdots, m$ by

$$Y_j = \frac{1}{2}\big(N_j + \theta(N_j)\big)\,.$$

Then $N_j = X_j + Y_j$ and $[X_j, Y_j] = -H$. Moreover, all the vectors $X_j$, $Y_j$, $H$ have the same length for $C$ and are unit length for $\langle\ ,\ \rangle$. The above basis for $\mathfrak{p}$ will be used in all subsequent calculations with differential operators.

Now we construct the different principal series of $G$. They will be parametrized by the various irreducible representations of $M$ (which is compact) and the real numbers. To an irreducible representation $\tau$ of $M$ and a unitary character of $A_\mathfrak{p}$ defined by $r \to e^{i\lambda r}$, $\lambda \in \mathbf{R}$, we correspond a representation of $B = MA_\mathfrak{p}N$ by $ma_rn \to \tau(m)e^{i\lambda r}$. Now we construct an irreducible (infinite-dimensional) unitary representation of $G$ which we denote $\pi_{\tau,\lambda}$. Let $E_\tau$ denote the Hilbert space of functions $f\colon G \to V_\tau$ (the space on which $\tau$ acts) satisfying:

$$f(mg) = \tau(m)f(g)\,,$$
$$f(ng) = f(g)\,,$$

whose restrictions to $K$ are square integrable for the Haar measure on $K$. We realize the principal series representation $\pi_{\tau,\lambda}$ on the subspace $E_{\tau,\lambda}$ of $E_\tau$

of functions satisfying the additional condition:

$$f(a_r g) = e^{-i\lambda r} f(g) .$$

We have defined the space on which $\pi_{\tau,\lambda}$ acts. $\pi_{\tau,\lambda}(g)$ acts on an element $f$ of the space $E_{\tau,\lambda}$ as follows:

$$\pi_{\tau,\lambda}(g) f(x) = f(xg) e^{\rho(H(xg)) - \rho(H(x))} .$$

Here $\rho$ is one half the sum of the positive roots of $\mathfrak{g}_C$. The inner product on $E_{\tau,\lambda}$ is given by:

$$(f, g) = \int_K (f(k), g(k))_\tau dk$$

where $(,)_\tau$ denotes the inner product on $V_\tau$. The family of representations $\{\pi_{\tau,\lambda} : \lambda \in \mathbf{R}, \tau \in \hat{M}\}$ constitute the unitary principal series of $G$. If we allow $\lambda \in \mathbf{C}$ we obtain the general principal series $\pi_{\tau,\lambda}$. We observe that $\pi_{\tau,\lambda} | K = \operatorname{ind}_M^K \tau$.

Having concluded our discussion of general representation theoretic facts we now specialize to the Lorentz groups $\mathrm{SO}(4n-1, 1)$. Here we have

$$A_\mathfrak{p} = \left\{ \begin{pmatrix} \cosh r & \sinh r & 0 \\ \sinh r & \cosh r & 0 \\ 0 & 0 & 1 \end{pmatrix} : r \in \mathbf{R} \right\} ,$$

$$M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} : x \in \mathrm{SO}(4n-2) \right\} ,$$

and $A'$ is the subgroup of $M$ for which $X$ consists of $2 \times 2$ diagonal blocks $R(\theta_j)$, $1 \leqq j \leqq 2n-1$, where

$$R(\theta_j) = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix} .$$

We will say $\gamma \in G$ is a hyperbolic element if there exists $g \in G$ so that $g\gamma g^{-1} \in A = A_\mathfrak{p} A'$ and $r_\gamma \neq 0$. (We will write $r_\gamma, \theta_1(\gamma), \cdots, \theta_{2n-1}(\gamma)$ for the coordinates of $g\gamma g^{-1}$; since these parameters will always be the arguments of a class-function there will be no ambiguity.) If $\Gamma \subseteqq G$ is a uniform, torsion-free, discrete subgroup then every $\gamma \in \Gamma$ which is not the identity is hyperbolic.

Now we note that since we are dealing with a symmetric space of odd dimension, the Cartan involution is orientation reversing. For hyperbolic space the Cartan involution $\theta$ which fixes the standard embedding of $\mathrm{SO}(n)$ in $\mathrm{SO}(n, 1)$ is given by conjugation by the diagonal matrix $\iota$ with diagonal entries $-1, 1, \cdots, 1$. $\theta$ acts on the representations of $G$ according to

$$\pi^\theta(g) = \pi(\theta(g)) .$$

To compute the action of $\theta$ on $\pi_{\tau,\lambda}$, we note that the action is the same as that of $\theta \circ \mathrm{Ad}\, g$ for $g \in G$. Now we make the following judicious choice of inner automorphism. Let $M'$ be the normalizer of $\mathfrak{a}_{\mathfrak{p}}$ in $K$. Then $M'/M$ is a group of order 2, say $\{1, \delta\}$. $\delta$ induces an automorphism of $M$ (modulo the group of inner automorphisms of $M$) hence a transformation of $\hat{M}$. We set $\tau' = \delta(\tau)$, for $\tau \in \hat{M}$. Now $\theta\delta$ has the advantage over $\theta$ that it preserves $N$ and its action on $\pi_{\tau,\lambda}$ can be immediately found to be

$$\pi^{\theta\delta}_{\tau,\lambda} = \pi_{\tau',\lambda} \,.$$

Since it is well-known, Wallach [27], page 254, that $\delta$ induces an equivalence between $\pi_{\tau,\lambda}$ and $\pi_{\tau',-\lambda}$ we have also

$$\pi^{\theta}_{\tau,\lambda} = \pi_{\tau,-\lambda} \,.$$

Note that because $\theta$ centralizes $K$, $\theta$ acts on $\mathrm{Ind}_K^G \sigma$ by

$$f^{\theta}(g) = f(\theta(g)) = f(\iota g \iota) \,.$$

Now we want to apply the Frobenius reciprocity theorem to find out which representations $\pi_{\tau,\lambda}$ contain $\sigma = \Lambda^{2n-1}\phi$ when restricted to $K$.

$$\begin{aligned}[\Lambda^{2n-1}\phi\colon \pi_{\tau,\lambda} | K] &= [\Lambda^{2n-1}\phi\colon \mathrm{Ind}_M^K \tau] \\ &= [\Lambda^{2n-1}\phi | M\colon \tau] \,.\end{aligned}$$

But $\Lambda^{2n-1}\phi | M = \Lambda_+^{2n-1}\bar{\phi} \oplus \Lambda_-^{2n-1}\bar{\phi} \oplus \Lambda^{2n-2}\bar{\phi}$.

Here $\phi$ denotes the standard representation of $\mathrm{SO}(4n-1)$ and $\bar{\phi}$ the standard representation of $\mathrm{SO}(4n-2)$. Since $\tau$ is irreducible we must have one of

$$\begin{cases} \tau = \Lambda_+^{2n-1}\bar{\phi} \,, \\ \tau = \Lambda_-^{2n-1}\bar{\phi} \,, \\ \tau = \Lambda^{2n-2}\bar{\phi} \,. \end{cases}$$

We denote these representations by $\tau_+$, $\tau_-$, and $\tau_1$ respectively. By the Plancherel theorem we may now decompose the $2n-1$ forms on hyperbolic space as a sum of direct integrals

$$\mathrm{Ind}_K^G \Lambda^{2n-1}\phi \cong \int_{-\infty}^{\infty} \pi_{\tau_+,\lambda} p(\tau_+, \lambda) d\lambda \oplus \int_{-\infty}^{\infty} \pi_{\tau_-,\lambda} p(\tau_-, \lambda) d\lambda \oplus \int_{-\infty}^{\infty} \pi_{\tau_1,\lambda} p(\tau_1, \lambda) d\lambda$$

where $p(\tau, \lambda)$ is the Plancherel measure.

The results of the next section show that we can identify the first two terms of the right-hand side with the co-closed forms and the third term with the closed forms.

Our last computation of this section will be to compute $\Omega_{\tau,z}$, the value of the normalized Casimir operator on $\pi_{\tau,z}$, in case $\tau = \tau_+$ or $\tau = \tau_-$.

LEMMA 1.4.

$$\Omega_{\tau_+,z} = \Omega_{\tau_-,z} = z^2 .$$

*Proof.* The following well-known formula for the value of the Casimir operator $\Omega'$ on $\pi_{\tau,z}$ can be found in Arthur [1], Section 6. Let $\mu$ be the highest weight of $\tau$ and let $\rho_M$ be one half the sum of the positive roots of $M$. Then

$$\Omega'_{\tau,z} = C(\mu + \rho_M, \mu + \rho_M) - C(\rho, \rho) - z^2 C(r_0, r_0) .$$

If $\mu_+$ denotes the highest weight of $\Lambda_+^{2n-1}\bar{\phi}$ and $\mu_-$ that of $\Lambda_-^{2n-1}\bar{\phi}$ we have

$$\mu_+ = \theta_1 + \theta_2 + \cdots + \theta_{2n-2} + \theta_{2n-1} ,$$
$$\mu_- = \theta_1 + \theta_2 + \cdots + \theta_{2n-2} - \theta_{2n-1} .$$

We have also

$$\rho_M = (2n-2)\theta_1 + (2n-3)\theta_2 + \cdots + \theta_{2n-1} ,$$
$$\rho = (2n-1)r_0 + (2n-2)\theta_1 + \cdots + 2\theta_{2n-2} + \theta_{2n-1} .$$

Let $\mu$ denote either $\mu_+$ or $\mu_-$ and $\tau$ denote either $\tau_+$ or $\tau_-$ in what follows. Noting the relation between $\Omega_{\tau,z}$ and $\Omega'_{\tau,z}$ we have

$$\Omega_{\tau,z} = z^2 + \frac{1}{2n-2} C(\rho, \rho) - \frac{1}{2n-2} C(\mu + \rho_M, \mu + \rho_M) .$$

But $C(\rho, \rho) - C(\mu + \rho_M, \mu + \rho_M) = 0$ and the result follows.

## 4. Harmonic analysis of $\Gamma(E)$

Given a $C^\infty$ function of compact support on $G$ we define its Fourier transform $\hat{f}$ as a function on $\hat{G}$, the unitary dual of $G$, as follows. Let $\pi \in G$. Then we can associate to $\pi$ a representation of the convolution algebra of compactly supported functions on $G$, also denoted by $\pi$, by defining

$$\pi(f) = \int_G f(g)\pi(g)dg .$$

It is not difficult to show, see Harish-Chandra [14], that if $f$ is $C^\infty$ and of compact support on $G$ then $\pi(f)$ is trace class. Then $\hat{f}(\pi)$ is defined by

$$\hat{f}(\pi) = \text{Trace}\, \pi(f) .$$

Harish-Chandra [14] proved that there is a locally $L^1$ function $\theta_\pi$ defined almost everywhere on $G$ and analytic on the regular elements such that

$$\hat{f}(\pi) = \int_G f(g)\theta_\pi(g)dg .$$

We will sometimes denote $\hat{f}(\pi)$ by $\theta_\pi(f)$. We can extend the Fourier transform to a larger space of functions, those in the Schwartz space $\mathcal{S}(G)$

which is defined in Chapter 2. This is of interest to us because $\tilde{\alpha}_t$ is in $\mathcal{S}(G)$. Now we can state the main theorem of this section.

THEOREM 1.1. *The Fourier transform of* $\tilde{\alpha}_t$ *which we denote* $h_t$ *is given by*:

$$h_t(\tau_+, \lambda) = \lambda e^{-\lambda^2 t} ,$$
$$h_t(\tau_-, \lambda) = -\lambda e^{-\lambda^2 t} ,$$
$$h_t(\tau, \lambda) = 0 \qquad \text{for all } \tau \neq \tau_+, \tau_- .$$

In order to prove this theorem we need some more machinery. We begin by defining the matrix blocks $\bar{\phi}_\sigma^\pi$ of unitary representations $\pi$. We recall the isomorphism in the Frobenius reciprocity theorem

$$\mathrm{Hom}_K(\pi \,|\, K, \sigma) \xrightarrow{\mathcal{F}} \mathrm{Hom}_G(\pi, \mathrm{ind}_K^G \sigma) .$$

$\mathcal{F}$ is defined as follows. Let $P_\sigma \in \mathrm{Hom}_K(\pi \,|\, K, \sigma)$. Then

$$\mathcal{F} P_\sigma = P_\sigma \pi(g^{-1}) .$$

We define the matrix block of $\pi$ corresponding to $P_\sigma$, to be denoted by $\bar{\phi}_\sigma^\pi$, by

$$\bar{\phi}_\sigma^\pi = P_\sigma \pi(g^{-1}) P_\sigma .$$

We note that $\bar{\phi}_\sigma^\pi \in L_\sigma(G)$.

If $D\colon \Gamma(E) \to \Gamma(E)$ is a $G$-map then $D$ acts on $\mathrm{Hom}_G(\pi, \mathrm{ind}_K^G \sigma)$ for each $\pi$ by composition; that is, if $\Phi \in \mathrm{Hom}_G(\pi, \mathrm{ind}_K^G \sigma)$ then $D\Phi = D \circ \Phi \in \mathrm{Hom}_G(\pi, \mathrm{ind}_K^G \sigma)$. We have then the following proposition suggested to the author by G. Zuckerman (the proof is now obvious):

PROPOSITION 1.1. *If* $\pi \,|\, K$ *contains* $\sigma$ *exactly once then there is a scalar* $\mu$ *so that*

$$D\bar{\phi}_\sigma^\pi = \mu \bar{\phi}_\sigma^\pi .$$

*Remark.* Let $v$ be a vector orthogonal to the kernel of $P_\sigma$ in the space of $\pi$. Then $P_\sigma \pi(g^{-1}) v \in \Gamma(E)$. Then Proposition 1.1 implies that there exists a scalar $\mu$ independent of $v$ so that

$$D P_\sigma \pi(g^{-1}) v = \mu P_\sigma \pi(g^{-1}) v .$$

Hence (if we choose a basis for the orthogonal complement of $\ker P_\sigma$ in the space of $\pi$) we may consider $\bar{\phi}_\sigma^\pi$ as a matrix with columns made up from eigensections of $D$ with all columns corresponding to the same eigenvalue $\mu$. With this remark we state two corollaries.

COROLLARY 1. *Suppose* $\xi$ *is an invariant integral operator on* $\Gamma(E)$ *with representation function* $\bar{e}$. *Then the columns of* $\bar{\phi}_\sigma^\pi$ *are simultaneous eigensections for* $\xi$ *corresponding to the eigenvalue* $\mu$. *Hence*

$$\int_G \bar{e}(g_2^{-1}g_1)\bar{\phi}_\sigma^\pi(g_2)dg_2 = \mu\bar{\phi}_\sigma^\pi(g_1) .$$

COROLLARY 2. *Suppose $D$ is an invariant differential operator on $\Gamma(E)$, then the columns of $\bar{\phi}_\sigma^\pi$ are simultaneous eigensections of $D$.*

Let $\tilde{\phi}_\sigma^\pi = \text{trace } \bar{\phi}_\sigma^\pi$ where the trace is taken in End $V_\sigma$.

LEMMA 1.5.

$$(\dim V_\sigma)\int_G \tilde{e}(g^{-1})\tilde{\phi}_\sigma^\pi(g)dg = \mu\tilde{\phi}_\sigma^\pi(1) .$$

*Proof.* This result follows from Corollary 1 by substituting $g_1 = 1$, taking the trace and using the orthogonality relations for the matrix elements of $\sigma$.

Noting that $\tilde{\phi}_\sigma^\pi(1) = \dim V_\sigma$, we have

COROLLARY.

$$\int_G \tilde{e}(g^{-1})\tilde{\phi}_\sigma^\pi(g)dg = \mu .$$

We will see shortly (Lemma 1.7) that $\mu = \theta_\pi(\tilde{e})$.

LEMMA 1.6. *Suppose $\pi | K = \sum_{\alpha=1}^\infty \sigma_\alpha$ and $f \in \mathcal{S}(G)$; then*

$$\theta_\pi(f) = \sum_{\alpha=1}^\infty \int_G f(g^{-1})\tilde{\phi}_{\sigma_\alpha}^\pi(g)dg .$$

*Proof.* Compute trace $\pi(f)$ relative a $K$-finite basis.

LEMMA 1.7. *Suppose $f \in \mathcal{S}(G)$ and $f = \text{trace } \bar{f}$ with $\bar{f} \in L_\sigma(G)$; then*

$$\theta_\pi(f) = \int_G f(g^{-1})\tilde{\phi}_\sigma^\pi(g)dg .$$

*Proof.* The lemma follows immediately from the orthogonality of matrix elements of two distinct irreducible representations of $K$.

COROLLARY 1. *If $f$ is as above and $\pi | K$ does not contain $\sigma$ then* $\hat{f}(\pi) = 0$.

COROLLARY 2. *If $\pi | K$ contains $\sigma$ exactly once then*

$$\bar{e} * \bar{\phi}_\sigma^\pi = \theta_\pi(\tilde{e})\bar{\phi}_\sigma^\pi .$$

These results greatly simplify the problem of computing the Fourier transform of $\tilde{\alpha}_t$. We first note that Corollary 1 combined with the statements near the end of Section 3 imply

$$h_t(\tau, \lambda) = 0 \text{ unless } \tau = \tau_+, \tau_-, \tau_1 .$$

Since $\pi_{\tau_+,\lambda}|K$, $\pi_{\tau_-,\lambda}|K$ and $\pi_{\tau_1,\lambda}|K$ contain $\sigma = \Lambda^{2n-1}\phi$ exactly once, we may apply Corollary 2 to conclude

$$(a)\quad \bar{\alpha}_t * \bar{\phi}_\sigma^{\pi_{\tau_+},\lambda} = h_t(\tau_+, \lambda)\bar{\phi}_\sigma^{\pi_{\tau_+},\lambda} ,$$
$$(b)\quad \bar{\alpha}_t * \bar{\phi}^{\pi_{\tau_-},\lambda} = h_t(\tau_-, \lambda)\bar{\phi}_\sigma^{\pi_{\tau_-},\lambda} ,$$
$$(c)\quad \bar{\alpha}_t * \bar{\phi}^{\pi_{\tau_1},\lambda} = h_t(\tau_1, \lambda)\bar{\phi}_\sigma^{\pi_{\tau_1},\lambda} .$$

Finally since $\bar{\alpha}_t$ is a representation function for the integral operator $Ae^{-t\Delta} = Ae^{-tA^2}$ all we have to do is calculate the eigenvalue of $A$ acting on the matrix block $\bar{\phi}_\sigma^{\pi_{\tau_+},\lambda}$. In fact to prove Theorem 1.1 it is necessary and sufficient that we show

$$A\bar{\phi}_\sigma^{\pi_{\tau_+},\lambda} = \lambda\bar{\phi}_\sigma^{\pi_{\tau_+},\lambda} ,$$
$$A\bar{\phi}^{\pi_{\tau_1},\lambda} = 0 .$$

We first prove that $h_t(\tau_1, \lambda) = 0$.

LEMMA 1.8.

$$A\bar{\phi}_\sigma^{\pi_{\tau_1},\lambda} = 0 .$$

*Proof.* In what follows we abbreviate $\pi_{\tau_1,\lambda}$ to $\pi$. Consider the transform $\bar{\psi}_\sigma^\pi$ of $\bar{\phi}_\sigma^\pi$ by $\theta$. Since $\bar{\phi}_\sigma^\pi \in \mathrm{Hom}_G(\pi, \mathrm{ind}_K^G \sigma)$ we have $\bar{\psi}_\sigma^\pi \in \mathrm{Hom}_G(\pi^\theta, \mathrm{ind}_K^G \sigma)$. But $\pi^\theta = \pi$ and $\mathrm{Hom}_G(\pi, \mathrm{ind}_K^G \sigma)$ is one dimensional; hence, there exists $\mu \in \mathbf{C}$ so that

$$\bar{\phi}_\sigma^\pi(\iota x \iota) = \mu\bar{\phi}_\sigma^\pi(x) .$$

Evaluating at $x = 1$ we find $\mu = 1$. But since $A$ and $\theta$ anti-commute we have the result.

We now use a formula of Harish-Chandra (see Warner [29], p. 42), the Eisenstein integral formula. For the principal series representation $\pi = \pi_{\tau,\lambda}$, $\tau = \tau_\pm$ we have $\bar{\phi}_\sigma^\pi(x) = \check{E}_\lambda(\psi_\tau\colon x) = E_\lambda(\psi_\tau\colon x^{-1})$,

$$E_\lambda(\psi_\tau\colon x) = \int_K \sigma(k)\psi_\tau(k^{-1}x)e^{(i\lambda r_0+\rho)(H(k^{-1}x))}dk .$$

Here $\psi_\tau$ is obtained as follows. Since $\sigma$ appears once in the restriction of $\pi$ to $K$, $\tau$ must appear once in the restriction of $\sigma$ to $M$.

Choose a copy of $V_\tau$ in $V_\sigma$ and let $\psi = \psi_\tau(1) \in \mathrm{Hom}_M(V_\sigma, V_\sigma)$ be the projection of $V_\sigma$ onto $V_\tau$. We extend $\psi_\tau$ to a map $G \to \mathrm{End}\, V_\sigma$ by requiring

$$\psi_\tau(nak) = \psi_\tau(1)\sigma(k) .$$

We would like to calculate the eigenvalue $\mu$ so that

$$A \circ \check{E}_\lambda(\psi_\tau\colon x) = \mu\check{E}_\lambda(\psi_\tau\colon x) .$$

Since we are interested only in calculating the eigenvalue $\mu$ it is enough to calculate $A \circ E_\lambda(\psi_\tau\colon x^{-1})|_{x=1}$.

Let us denote the integrand $\psi_\tau(x)e^{(i\lambda r_0+\rho)(H(x))}$ by $F(x)$. Then

$$\begin{aligned}A \circ E_\lambda(\psi_\tau : x^{-1}) &= \textstyle\sum_{j=1}^m X_j^L \otimes A_j \int_K \sigma(k) F(k^{-1}x^{-1}) dk \\ &= \frac{d}{dt} \int_K \textstyle\sum_{j=1}^m A_j \sigma(k) F\big(k^{-1} \exp(-tX_j) \circ x^{-1}\big) dk \Big|_{t=0} \\ &= \frac{d}{dt} \int_K \textstyle\sum_{j=1}^m A_j \sigma(k) F\big(k^{-1} \exp(-tX_j) k k^{-1} x^{-1}\big) dk \Big|_{t=0} \\ &= \frac{d}{dt} \int_K \sigma(k) \textstyle\sum_{j=1}^m A_j F\big(\exp(-tX_j) k^{-1} x^{-1}\big) dk \Big|_{t=0} .\end{aligned}$$

Putting $x = 1$ we obtain

$$A \circ E(\psi_\tau : 1) = \frac{d}{dt} \int_K \sigma(k) \textstyle\sum_{j=1}^m A_j F\big(\exp(-tX_j) \circ 1\big) \sigma(k^{-1}) dk \Big|_{t=0} .$$

Now we choose for $X_j$ the elements of the basis for $\mathfrak{p}$ constructed in Section 3. We note

$$F\big(\exp(-tN_j) \circ 1\big) = F(1) \qquad \text{for all} \quad t;$$

hence

$$\frac{d}{dt} \int_K \sigma(k) A_j F\big(\exp(-tN_j) \circ 1\big) \sigma(k^{-1}) dk \Big|_{t=0} = 0 .$$

Noting $X_j = N_j - Y_j$, we obtain

$$\frac{d}{dt} \textstyle\sum_{j=2}^m A_j F\big(\exp(-tX_j) \circ 1\big) \Big|_{t=0} = \frac{d}{dt} \textstyle\sum_{j=2}^m A_j F(\exp tY_j \circ 1) \Big|_{t=0} .$$

Hence

$$\begin{aligned}A \circ E_\lambda(\psi_\tau : 1) = &-\int_K \sigma(k) A_1 H F(1) \sigma(k^{-1}) dk \\ &+ \frac{d}{dt} \Big( \int_K \sigma(k) \textstyle\sum_{j=2}^m A_j F(1 \circ \exp tY_j) \sigma(k^{-1}) dk \Big)_{t=0} .\end{aligned}$$

Now

$$\begin{aligned}\int_K \sigma(k) A_j F(1 \circ \exp tY_j) \sigma(k^{-1}) dk &= \int_K \sigma(k) A_j F(1) \sigma(\exp tY_j k^{-1}) dk \\ &= \int_K \sigma(k \exp tY_j) A_j F(1) \sigma(k^{-1}) dk .\end{aligned}$$

Taking the derivative at $t = 0$ we obtain

$$\begin{aligned}A \circ E_\lambda(\psi_\tau : 1) = &-\int_K \sigma(k) A_1 H F(1) \sigma(k^{-1}) dk \\ &+ \int_K \sigma(k) \textstyle\sum_{j=2}^m \sigma(Y_j) A_j F(1) \sigma(k^{-1}) dk .\end{aligned}$$

Now the evaluation of $\mu$ is easy. We begin by computing the second integral (the zero order part of $A$ acting on $E_\lambda$). We have from general Lie theory, $Y_j = [H, X_j]$, $j = 2, 3, \cdots, m$. Hence $\sigma(Y_j) = \mathrm{Ad}[H, X_j]$. Now we define for each pair of vector $u, v \in \mathfrak{p}$ a derivation of the full tensor algebra $\mathcal{T}_{\mathfrak{p}}$ as follows. If $w \in \mathfrak{p}$ then

$$u \wedge v \cdot w = (u, w)v - (v, w)u .$$

Then $u \wedge v$ is the unique extension to $\mathcal{T}_{\mathfrak{p}}$.

LEMMA 1.9. *As derivations of $\mathcal{T}_{\mathfrak{p}}$ we have*

$$\mathrm{Ad}[H, X_j] = -H \wedge X_j , \qquad j = 2, 3, \cdots, m .$$

*Proof.* It is enough to check that we have equality on $\mathfrak{p}$. This follows easily from the bracket relations

$$[X_j, X_k] = 0 ,$$
$$[X_j, Y_k] = -\delta_{jk} H .$$

Let us denote by $\omega_j$ the dual of $X_j$. We have

LEMMA 1.10. *For the curvature $-1$ metric*

$$* \sum_{j=2}^{m} \sigma(Y_j) A_j \psi = i(2n-1)\psi .$$

*Proof.*

$$* \sum_{j=2}^{m} \sigma(Y_j) A_j \psi = - * \sum_{j=2}^{4n-2} (H \wedge X_j) s(\omega_j)\psi .$$

To compute this last composition we construct $\psi$ more explicitly. Let $\{\nu_k : k = 1, 2, \cdots, N\}$ be a basis for the $+i$ eigenspace of the induced Hodge star $\hat{*}$ on $\Lambda^{2n-1}\mathfrak{q}^*$. Then

$$\psi = \sum_{k=1}^{N} \nu_k \otimes \nu_k^* .$$

Putting $S = \sum_{j=2}^{4n-2} H \wedge X_j \circ s(\omega_j)$, we have

$$S\psi = \sum_{k=1}^{N} S\nu_k \otimes \nu_k^* .$$

We claim that for any form $\nu \in \Lambda^p \mathfrak{q}^*$ we have $S\nu = (4n-2-p) r_0 \wedge \nu$. This follows immediately from the observation that if $\nu = \omega_{j_1} \wedge \omega_{j_2} \wedge \cdots \wedge \omega_{j_p}$ then

$$(H \wedge X_j)\nu = \begin{cases} r_0 \wedge \nu & \text{if } j_1 \neq j \text{ any } j \\ 0 & \text{if some } j_1 = j . \end{cases}$$

To complete the proof of the lemma we have only to note that since $r_0$ has unit length for the curvature $-1$ metric we have

$$*(r_0 \wedge \nu) = \hat{*}\nu .$$

Hence

$$- * S\psi_{\tau_+} = i(2n-1)\psi_{\tau_+} .$$

Now we calculate the first-order piece of $A$, that is, $*(-H \otimes A_1)$ where $A_1 = s(r_0)$. Once again we work with the curvature $-1$ metric. Then

$$\begin{aligned} -HF(1) &= -\frac{d}{dt}F(\exp tH)\Big|_{t=0} \\ &= -\psi\frac{d}{dt}\left(e^{(i\lambda+\rho(H))t}\right)\Big|_{t=0} \\ &= -(i\lambda + \rho(H))\psi\,. \end{aligned}$$

Applying $*\,s(r_0)$ we obtain

$$\begin{aligned} -*(H\otimes A_1)F(1) &= -(i\lambda + \rho(H))*r_0\wedge\psi \\ &= -(i\lambda + \rho(H))\hat{*}\psi \\ &= -i(i\lambda + \rho(H))\psi \\ &= (\lambda - i\rho(H))\psi\,. \end{aligned}$$

Finally adding the two contributions and noting $\rho(H) = 2n - 1$ we find

$$\mu = \lambda\,.$$

Thus

$$A\bar{\phi}_\sigma^{\pi_{\tau_+},\lambda} = \lambda\bar{\phi}_\sigma^{\pi_{\tau_+},\lambda}$$

and consequently

$$A\bar{\phi}_\sigma^{\pi_{\tau_-},\lambda} = -\lambda\bar{\phi}_\sigma^{\pi_{\tau_-},\lambda}$$

and the theorem is proved.

We are now ready to prove a theorem which implies the Riemann hypothesis for $\tilde{Z}$ (see Theorem 3.2). We thank the referee for correcting our original argument; the main part of the following proof was suggested by the referee.

THEOREM 1.2. *Let $\psi$ be an irreducible unitary representation of $G = \mathrm{SO}(4n-1, 1)$ occurring in the decomposition of $L^2(\Gamma\backslash G)$ which contributes to the coclosed $2n-1$ forms on $\Gamma\backslash G/K$ but which is not closed. Then there exists $\lambda\in\mathbf{R}$ so that $\psi = \pi_{\tau_+,\lambda}$ or $\psi = \pi_{\tau_-,\lambda}$.*

*Proof.* Let $\omega$ be a non-zero coclosed $2n-1$ form satisfying $A\omega = \mu\omega$. $\omega$ corresponds to a mapping $f\colon G\to V_\sigma$ which is $K$ equivariant; that is, $f(gk) = \sigma(k^{-1})f(g)$. Set $h(g, x) = \int_K f(gk^{-1}xk)dk$. Choose $g\in G$ so that $f(g)\neq 0$. Then for that choice of $g$ we define $\varphi(x) = h(g, x)$. $\varphi$ has the properties that $\varphi(1)\neq 0$, $*d\varphi = \mu\varphi$, $\varphi(k^{-1}x) = \sigma(k^{-1})\varphi(x)$ and $\varphi(xk) = \sigma(k^{-1})\varphi(x)$. Consequently $\varphi$ can be expressed in terms of the Eisenstein integrals $E_u(\psi_{\tau_+}\colon x)$, $E_v(\psi_{\tau_-}\colon x)$ or $E_w(\psi_{\tau_1}\colon x)$ for some appropriate $u, v, w\in\mathbf{C}$. But we have seen (Lemma 1.8) that $AE_w$ is zero when $w$ is real. Since $AE_w$ is a holomorphic function of $w$ it must be identically zero. Hence if $\mu\neq 0$ then $\varphi$ can be expressed in terms of $E_u(\psi_{\tau_+}\colon x)$ and $E_v(\psi_{\tau_-}\colon x)$ (if $\varphi$ is closed then it does not contribute to $\mathrm{Tr}\,Ae^{-t\Delta}$). But we have seen that the value of the normalized Casimir operator on each of these two representations is $u^2$. But by Kuga's lemma we have $\Delta\omega = u^2\omega$, consequently $u^2$ is positive and $u$ is real. We have shown then that if $\psi$ contributes to the coclosed $2n-1$ forms then $\omega$ gives rise to an element of $\mathrm{Hom}_G(\psi, \pi_{\tau_\pm,\lambda})$. Hence either $\psi = \pi_{\tau_+,\lambda}$ or $\psi = \pi_{\tau_-,\lambda}$.

## Chapter II

### 1. The Selberg trace formula and the calculation of $\mathrm{Tr}\, Ae^{-t\Delta}$

Suppose that $E$ is a homogeneous vector bundle over $G/K$ defined by the isotropy representation $\sigma: K \to V_\sigma$. Let $\xi$ be an invariant integral operator on $\Gamma(E)$. Then the kernel of $\xi$ will be an element of $L^2(G \times G) \otimes \mathrm{End}\, V_\sigma$, that is, an endomorphism valued function on $G \times G$. As explained in Chapter I, $\xi$ has a representation function $\bar{e}: G \to \mathrm{End}\, V_\sigma$ and a local trace $\tilde{e} = \mathrm{trace}\, \bar{e}$.

We shall require one more condition on our integral operator $\xi$: that it anti-commute with orientation reversing isometrics of $G/K$. In terms of $e$, the kernel, this says for $\nu$ orientation reversing

$$e(g_1, \nu g_2) = -e(\nu^{-1} g_1, g_2)\ .$$

In this case we say $e$ is an odd kernel. (We must assume that orientation reversing isometrics may also act on $\Gamma(E)$, a hypothesis satisfied in practice.) In this case we have $\bar{e}(1) = 0$ and $\tilde{e}(1) = 0$; consequently there will be no "identity contribution" to the trace formula.

Now given an invariant integral operator $\xi$ on sections of a homogeneous bundle over $G/K$ satisfying certain growth conditions, we get an induced integral operator $\xi^\#$ on the induced locally homogeneous bundle over $\Gamma\backslash G/K$. If $e(g_1, g_2)$ is the kernel of $\xi$ then the kernel of $\xi^\#$ will be given by

$$\sum_\Gamma e(\gamma g_1, g_2) = \sum_\Gamma \bar{e}(g_2^{-1}\gamma g_1)\ .$$

We shall restrict ourselves to integral operators $\xi$ so that $\xi^\#$ is of trace class and

$$\mathrm{Tr}\, \xi^\# = \sum_\Gamma \int_{\mathcal{D}} \tilde{e}(g^{-1}\gamma g) dg\ ,$$

where $\mathcal{D}$ is a fundamental domain for $\Gamma$ in $G$. (Here we assume Haar measure on $G$ is normalized by the condition that $\mathrm{vol}\, K = 1$ and the induced measure on $G/K$ agrees with that inherited from the metric of curvature $-1$.) If $\xi$ satisfies the above conditions we will say that the function $\bar{e}$ is admissible.

Following the now well-worn path of Selberg on pages 63 to 66 of his famous paper [22], denoting by $\{\gamma\} - 1$ the non-identity conjugacy classes, we obtain (for an odd admissible kernel $\bar{e}$)

$$\operatorname{Tr}\xi^{\#} = \sum_{\{\gamma\}-1} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} \tilde{e}(g^{-1}\gamma g)dg\ .$$

Now it is easy to see that though each of the two factors on the right-hand side depend on the normalization of the Haar measure of $G_\gamma$ their product does not; hence we may normalize the Haar measure of $G_\gamma$ in any convenient way. Also we may replace $\gamma$ in the above formula by any conjugate of $\gamma$ in $G$, in particular by a conjugate $a_\gamma \in A$. We will assume this has been done throughout this section. Thus we obtain the parameters of Chapter I, Section 3; $r_\gamma$, $\theta_1(\gamma)$, $\theta_2(\gamma)$, $\cdots$, $\theta_{2n-1}(\gamma)$. We may assume that $r_\gamma$ is positive.

As explained we can assume $A \subseteq G_\gamma$ and obtain

$$\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} \tilde{e}(g^{-1}\gamma g)dg = \operatorname{vol}(\Gamma_\gamma\backslash A)\int_{A\backslash G} \tilde{e}(g^{-1}a_\gamma g)dg\ .$$

Once again we can give $A$ any convenient measure. We give it the product measure of Lebesgue measure on $A_\mathfrak{p}$ and the measure on $A'$ so that $\operatorname{vol}(A') = 1$. We find then that $\operatorname{vol}(\Gamma_\gamma\backslash A) = r_{\gamma^*}$ where $\gamma^*$ generates $\Gamma_\gamma$.

But there is a simple formula relating $\int_{A\backslash G} \tilde{e}(g^{-1}a_\gamma g)dg$ and the Fourier transform $h$ of $\tilde{e}$. We write $a_\gamma = a'_\gamma \exp r_\gamma H \in A'A$. We also put $\Delta_p = \prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})$ where $\alpha$ runs over the positive non-compact roots of $\mathfrak{g}_C$ with respect to $\mathfrak{a}_C$. Then we have, see Wallach [28], pages 177, 178,

$$\int_{A\backslash G} \tilde{e}(g^{-1}a_\gamma g)dg = \frac{1}{2\pi}\sum_{\tau\in\hat{M}} \frac{r_\gamma^*}{\Delta_p(a_\gamma)}\int_{-\infty}^{\infty} h(\tau, \lambda)e^{ir_\gamma\lambda} \operatorname{tr}\tau(a'_\gamma)d\lambda$$

and we obtain the trace formula

$$\operatorname{Tr}\xi^{\#} = \frac{1}{2\pi}\sum_{\{\gamma\}-1}\sum_{\tau\in\hat{M}} \frac{r_\gamma^*}{\Delta_p(a_\gamma)}\int_{-\infty}^{\infty} h(\tau, \lambda)e^{ir_\gamma\lambda} \operatorname{tr}\tau(a'_\tau)d\lambda\ .$$

Now we apply the trace formula to $\operatorname{Tr} Ae^{-t\Delta}$. We calculated the Fourier transform of $\tilde{a}_t$ in the previous chapter and found

$$h_t(\tau_+, \lambda) = \lambda e^{-\lambda^2 t}\ ,$$
$$h_t(\tau_-, \lambda) = -\lambda e^{-\lambda^2 t}\ ,$$
$$h_t(\tau, \lambda) = 0 \qquad \text{for} \quad \tau \neq \tau_+, \tau_-\ .$$

We use the following formulas (for formula (3) see Atiyah-Singer [4], page 577):

(1)
$$\mu_j = e^{r_0 + i\theta_j/2}\ ,$$
$$\Delta_p(a_\gamma) = \prod_{j=1}^{2n-1}|\mu_j - \mu_j^{-1}|^2\ ;$$

(2)
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda e^{-\lambda^2 t}e^{i\lambda r}d\lambda = \frac{2\pi i r}{(4\pi t)^{3/2}}e^{-r^2/4t}\ ;$$

(3)
$$\operatorname{ch}\tau_+ - \operatorname{ch}\tau_- = i^{2n-1}2^{2n-1}\sin\theta_1\sin\theta_2\cdots\sin\theta_{2n-1}\ .$$

*In what follows* we abbreviate $\theta_j(\gamma)$ by $\theta_j$ and $\mu_j(\gamma)$ by $\mu_j$. Also we put $|\mu| = |\mu_j|$ and $|\mu^*| = |\mu(\gamma^*)|$.

THEOREM 2.1.

$$\operatorname{Tr} Ae^{-t\Delta} = 2^{2n-1} i^{2n-1} 2\pi i \sum_{\{\gamma\}_\Gamma} \frac{\log|\mu^*|^2 (\sin\theta_1 \cdots \sin\theta_{2n-1}) \log|\mu|^2}{|\mu_1 - \mu_1^{-1}|^2 \cdots |\mu_{2n-1} - \mu_{2n-1}^{-1}|^2} \frac{e^{-(\log|\mu|^2)^2/4t}}{(4\pi t)^{3/2}} .$$

From Theorem 1.2 we also have

$$\operatorname{Tr} Ae^{-t\Delta} = \sum_{\lambda_j > 0} (N_{\tau_+, \lambda_j} - N_{\tau_-, \lambda_j}) \lambda_j e^{-\lambda_j^2 t}$$

where $N_{\tau_+,\lambda_j}$ is the multiplicity with which $\pi_{\tau_+,\lambda_j}$ occurs in $L^2(\Gamma\backslash G)$. The equality of these two expressions is a relation between eigenvalues of the Laplacian, the $\lambda_j$'s and the eigenvalues of the element $\gamma \in \Gamma$, the $\mu_j$'s. We caution the reader not to miss the point—there is a vast difference between the $\mu_j$'s and the $\lambda_j$'s.

It follows from Theorem 2.1 that there exist positive constants $a$, $b$, $c$ so that $|\operatorname{Tr} Ae^{-t\Delta}| \leqq ce^{-at-(b/t)}$.

Finally we record the general trace formula which we have obtained for kernels on $\mathrm{SO}(4n-1, 1)$ whose Fourier transforms satisfy:*

(1) $\qquad h(\tau_+, \lambda) = -h(\tau_-, \lambda) = h(\tau_-, -\lambda)$ ,

(2) $\qquad h(\tau, \lambda) = 0$ , $\quad \tau \neq \tau_+, \tau_-$ ,

(3) $h(\tau, \lambda)$ has a holomorphic extension to the strip $\mathcal{F}_\rho$ where

$$\mathcal{F}_\rho = \{\lambda = \eta + i\nu \in \mathbf{C} : |\nu| < 2n - 1\} ,$$

(4) $h(\tau, \lambda)$ has a continuous extension $\overline{\mathcal{F}}_\rho$ and satisfies, for all $l$ and $m$,

$$\sup_{\lambda \in \mathcal{F}_\rho} (1 + \|\lambda\|^2)^l \left| \frac{d^m}{d\eta^m} h(\tau, \eta + i\nu) \right| < \infty .$$

Then abbreviating $h(\tau_+, \lambda)$ to $h(\lambda)$ we have

$$\sum_{\lambda_j > 0} m_j h(\lambda_j) = \frac{2^{2n-1} i^{2n-1}}{2\pi} \sum_{\{\gamma\}_\Gamma} \frac{\log|\mu^*|^2 (\sin\theta_1 \cdots \sin\theta_{2n-1})}{|\mu_1 - \mu_1^{-1}|^2 \cdots |\mu_{2n-1} - \mu_{2n-1}^{-1}|^2} \int_\infty^\infty h(\lambda) e^{i\lambda r} d\lambda$$

where $m_j = N_{\tau_+,\lambda_j} - N_{\tau_-,\lambda_j}$.

## 2. Some estimates

Up to this point we have not justified the application of the Selberg trace formula to $Ae^{-t\Delta}$ and in fact we shall need to apply the Selberg trace formula to a much less rapidly decreasing kernel in Chapter III. Also we have not shown that $\tilde{a}_t$ and $\bar{a}_t$ are Schwartz functions on $G$. We now deal with these points.

* The admissibility of such kernels will be proved in Section 2.

We begin by defining the Schwartz spaces $\mathcal{S}(G)$ and $\mathcal{S}(\hat{G})$. For $x$ in $G$ define

$$\Xi(x) = \int_K e^{-\rho(H(kx))}dk .$$

The geodesic distance $d$ on $G/K$ lifts to a $K$ bi-invariant function on $G$

$$\delta\colon G \longrightarrow R_+ \begin{cases} \delta(g) = d(1, \bar{g}) \\ \delta(g_1^{-1}g_2) = d(\bar{g}_1, \bar{g}_2) . \end{cases}$$

Now we define a collection of semi-norms on $C^\infty(G)$ as follows. Let $D_1$ be a left-invariant differential operator on $G$, $D_2$ a right-invariant differential operator on $G$ and $s \in \mathbf{R}$. Then we define a semi-norm $|| \ ||_{D_1, D_2, s}$ by

$$||f||_{D_1, D_2, s} = \sup_{x \in G} |D_1 D_2 f(x)| \frac{(1 + \delta(x))^s}{\Xi(x)} .$$

Then $\mathcal{S}(G) = \{f \in C^\infty(G)\colon ||f||_{D_1, D_2, s} < \infty\}$ for all $D_1$, $D_2$, $s$. For any vector space $V$ we define $\mathcal{S}(G, V)$, the Schwartz space of $V$ valued functions on $G$, by

$$\mathcal{S}(G, V) = \mathcal{S}(G) \otimes V .$$

Now in our case we can regard $\hat{G}$ as a collection of disjoint copies of the real line. $\mathcal{S}(\hat{G})$ will be just the direct sum of the usual Schwartz spaces of each line.

In his thesis [1], J. Arthur proves that the Fourier transform gives an isomorphism from $\mathcal{S}(G)$ to $\mathcal{S}(\hat{G})$; hence, we get an induced transformation of the topological duals, $\mathcal{S}'(\hat{G})$ to $\mathcal{S}'(G)$. The space $\mathcal{S}'(G)$ will be called the space of tempered distribution on $G$. We have inclusions

$$\mathcal{S}(G) \subset L^2(G) \subset \mathcal{S}'(G) ;$$

hence

$$\mathcal{S}(G, V) \subset L^2(G, V) \subset \mathcal{S}'(G, V) .$$

Now Gaffney [11] has shown that for any complete Riemannian manifold, $e^{-t\Delta}$ is a bounded operator on the square integrable $p$-forms. Hence it defines a tempered distribution and we may take its Fourier transform as a tempered distribution. But since this operator is just $e^{-tC}$ where $C$ is the Casimir operator, the value of the Fourier transform of $\bar{e}_t$ at a representation $\pi$ is just $e^{-tC_\pi}$ where $C_\pi$ is the value of $C$ on $\pi$. Since we are assuming $\pi$ contributes to the forms, we have $C_\pi \geqq 0$. Hence the Fourier transform of $\bar{e}_t$ is a Schwartz function; hence $\bar{e}_t \in \mathcal{S}(G, V)$. But by definition $\mathcal{S}(G, V)$ is stable for operators of the form $X_i \otimes L_i$ where $X_i$ is a left-invariant vector field on $G$ and $L_i \in \operatorname{End} V$. Hence $\bar{a}_t = A \circ \bar{e}_t$ is also in $\mathcal{S}(G, V)$. From this it follows immediately that $\tilde{a}_t \in \mathcal{S}(G)$.

We now attack the problem of showing $\bar{a}_t$ is an admissible function (see Section 1). To do this we give a whole class of admissible functions, of which $\bar{a}_t$ is a member, namely the $L^1$ Schwartz space of $\sigma$ bi-invariant functions $\mathcal{C}^1(G, \sigma)$ which we now define.

For any left-invariant differential operator $D$ and any integer $r \geqq 0$ we define the semi-norm $\nu_{D,r}$ on $L_\sigma(G)$ by

$$\nu_{D,r}(f) = \sup_{x \in G} \| Df(x) \| \frac{(1 + \delta(x))^r}{(\Xi(x))^2} .$$

We now define

$$\mathcal{C}^1(G, \sigma) = \{f \in L_\sigma(G): \nu_{D,r}(f) < \infty \quad \text{for all} \quad D, r\} .$$

PROPOSITION 2.1. *All functions in $\mathcal{C}^1(G, \sigma)$ are admissible.*

*Proof.* This result is standard. A proof may be found in the Rutgers doctoral thesis of R. Miatello written under the direction of N. Wallach.

The rest of this chapter is concerned with giving sufficient conditions for a function $h(\tau, \lambda)$ to be the Fourier transform of an element of $\mathcal{C}^1(G, \sigma)$. To do this we need a new space. Let $\mathcal{F}_\rho$ be the strip in the complex plane given by

$$\mathcal{F}_\rho = \{\lambda = \eta + i\nu \in \mathbf{C}: |\nu| < 2n - 1\}$$

and let $\overline{\mathcal{F}}_\rho$ be the closure of $\mathcal{F}_\rho$ in $\mathbf{C}$. Let $Z^{\text{odd}}(\mathcal{F}_\rho)$ be the space of continuous functions

$$h: \hat{M} \times \overline{\mathcal{F}}\rho \longrightarrow \mathbf{C}$$

such that:

(a) $h(\tau, \lambda)$ is holomorphic on $\mathcal{F}_\rho$ for all $\tau \in \hat{M}$ and satisfies the estimate (for all $l, m \in \mathbf{N}$),

$$\sup_{\lambda \in \mathcal{F}_\rho} (1 + |\lambda|^2)^l \left| \frac{d^m}{d\eta^m} h(\tau; \eta + i\nu) \right| < \infty ,$$

(b) $h(\tau, \lambda) = 0$ unless $\tau = \tau_+$ or $\tau_-$ and

(i) $h(\tau_+, \lambda) = h(\tau_-, -\lambda)$ ,

(ii) $h(\tau_+, \lambda) = -h(\tau_-, \lambda)$ .

Let $p(\tau_+, \lambda)d\lambda = p(\tau_-, \lambda)d\lambda$ be the Plancherel measure. $p(\tau_+, \lambda) = p(\tau_-, \lambda)$ is a polynomial of degree $4n - 2$ in $\lambda$: see Wallach [27], page 294.

We then have the following theorem of Paley-Wiener type.

THEOREM 2.2. *If $h(\lambda) \in Z^{\text{odd}}(\mathcal{F}_\rho)$ then the "wave-packet"*

$$\bar{e}(x) = \int_{-\infty}^{\infty} E_\lambda(\psi_{\tau_+}: x) h(\tau_+, \lambda) p(\tau_+, \lambda) d\lambda + \int_{-\infty}^{\infty} E_\lambda(\psi_{\tau_-}: x) h(\tau_-, \lambda) p(\tau_-, \lambda) d\lambda$$

*is in $\mathcal{C}^1(G)$.*

*Proof.* This theorem follows from the results of the Rutgers doctoral thesis of 0. Campolli written under the direction of N. Wallach. The point is that Theorem 2.3.3 of this thesis applies because $\lambda$ is the value $\pi_{\tau,\lambda}(z')$ of an operator $z'$ from the center of the universal enveloping algebra of the Lie algebra of $SO(4n, C)$. In fact $z' = (-1)^n z$ where $z$ is the operator corresponding to the Weyl group invariant $\gamma(z) = x_1 x_2 \cdots x_{2n}$ under the Harish-Chandra isomorphism $\gamma$. Here $\{x_1, x_2, \cdots, x_n\}$ is the basis for the Lie algebra of the maximal torus $SO(4n, C)$ which is dual to $\{r_0, i\theta_1, \cdots, i\theta_{2n-1}\}$. Letting $\mu_+$ be the highest weight of $\tau_+$ we have, according to a well-known formula (see for example Arthur [1], Section 6, Lemma 7),

$$\begin{aligned}\pi_{\tau_+,\lambda}(z) &= \langle\gamma(z), -\mu_+ - i\lambda r_0\rangle \\ &= \langle x_1 x_2 \cdots x_{2n}, -\mu_+ - i\lambda r_0\rangle \\ &= \langle x_1, -i\lambda r_0\rangle\langle x_2 \cdots x_{2n}, -\mu_+\rangle \\ &= -i\lambda \textstyle\prod_{j=1}^{n} \langle x, \mu_+\rangle \\ &= i\lambda(i)^{2n-1} \\ &= (-1)^n\lambda .\end{aligned}$$

Hence $\pi_{\tau_+,\lambda}(z) = (-1)^n\lambda$. Similarly $\pi_{\tau_-,\lambda}(z) = (-1)^{n+1}\lambda$ and $\pi_{\tau_1,\lambda}(z) = 0$.

In our original proof of Theorem 2.2 we explicitly computed the Eisenstein integrals $E_\lambda(\psi_{\tau_+}: x)$ and $E_\lambda(\psi_{\tau_-}: x)$ and obtained the following result. Let $\psi_{\tau_+}, \psi_{\tau_-}, \psi_{\tau_1} \in \mathrm{Hom}_M(V_\sigma, V_\sigma)$ be the projections on the spaces of $\tau_+$, $\tau_-$ and $\tau_1$ respectively. Let $\pi: \mathfrak{a}_\mathfrak{p} \to A_\mathfrak{p}$ be the exponential map and define

$$\begin{aligned}T_\lambda(r) &= E_\lambda(\psi_{\tau_+}: \pi(r)) , \\ S_\lambda(r) &= E_\lambda(\psi_{\tau_-}: \pi(r)) .\end{aligned}$$

Then $T_\lambda$ and $S_\lambda$ determine the respective Eisenstein integrals. Define a polynomial $q_{2n}(\lambda)$ by

$$q_{2n}(\lambda) = \prod_{k=0}^{2n-1} \frac{(k^2 + \lambda^2)}{(2k + 1)(k + 1)} .$$

Then

$$\begin{aligned}T_\lambda(r) &= a_\lambda(r)\psi_{\tau_+} + b_\lambda(r)\psi_{\tau_-} + c_\lambda(r)\psi_{\tau_1} , \\ S_\lambda(r) &= b_\lambda(r)\psi_{\tau_+} + a_\lambda(r)\psi_{\tau_-} + c_\lambda(r)\psi_{\tau_1} ,\end{aligned}$$

with

$$\begin{aligned}a_\lambda(r) = \frac{(-1)^n}{2n\, q_{2n}(\lambda)} \Big\{&(i\lambda \sinh r + 2n \cosh r)\Big(\frac{1}{\sinh r}\frac{d}{dr}\Big)^{2n} \cos \lambda r \\ &+ \sinh^2 r\Big(\frac{1}{\sinh r}\frac{d}{dr}\Big)^{2n+1} \cos \lambda r\Big\} ,\end{aligned}$$

$$\begin{aligned}b_\lambda(r) = \frac{(-1)^n}{2n\, q_{2n}(\lambda)} \Big\{&(i\lambda \sinh r + 2n \cosh r)\Big(\frac{1}{\sinh r}\frac{d}{dr}\Big)^{2n} \cos \lambda r \\ &+ \sinh^2 r\Big(\frac{1}{\sinh r}\frac{d}{dr}\Big)^{2n+1} \cos \lambda r\Big\} ,\end{aligned}$$

$$c_\lambda(r) = \frac{(-1)^n}{q_{2n}(\lambda)}\left(\frac{1}{\sinh r}\frac{d}{dr}\right)^{2n}\cos \lambda r\ .$$

With these explicit formulas the Paley-Wiener theorem is immediate. We thank G. Zuckerman who pointed out to us the fact that $\pi_{\tau_+,\lambda}((-1)^n z) = \lambda$.

## Chapter III

### The Selberg zeta function of the second kind

We define $\tilde{Z}(s)$, the Selberg zeta function of the second kind, by an Euler product convergent for Re $s$ sufficiently large. Let $\beta_+$ be the character on the maximal torus of $SO(4n-2)$ which is the maximal weight of $\Lambda_+^{2n-1}$ and $\beta_-$ that of $\Lambda_-^{2n-1}$. Thus

$$\beta_+(\theta_1, \theta_2, \cdots, \theta_{2n-1}) = e^{i(\theta_1+\theta_2+\cdots+\theta_{2n-2}+\theta_{2n-1})}\ ,$$
$$\beta_-(\theta_1, \theta_2, \cdots, \theta_{2n-1}) = e^{i(\theta_1+\theta_2+\cdots+\theta_{2n-2}-\theta_{2n-1})}\ .$$

Then define:

$$\tilde{Z}(s) = \prod\nolimits_{w\in\mathfrak{S}}\prod\nolimits_{\mathcal{P}}\prod\nolimits_{n_i=1}^{\infty}\frac{1-\mu_1^{-2n_1}\bar{\mu}_1^{-2n_2}\cdots\mu_{2n-1}^{-2n_{4n-3}}\bar{\mu}_{2n-1}^{-2n_{4n-2}}w\circ\beta_+|\mu|^{-2s}}{1-\mu_1^{-2n_1}\bar{\mu}_1^{-2n_2}\cdots\mu_{2n-1}^{-2n_{4n-3}}\bar{\mu}_{2n-1}^{-2n_{4n-2}}w\circ\beta_-|\mu|^{-2s}}\ .$$

Here $\mathfrak{S}$ is the subgroup of the Weyl group of $SO(4n-2)$ consisting of those elements that do not permute the $\theta_j$'s but make an even number of sign changes, $\mathcal{P}$ is the set of primitive $\Gamma$ conjugacy classes, and the $\mu_j$'s and $\beta_+$, $\beta_-$ are evaluated at $a_\gamma$. In Proposition 3.1 we will see that $\mu_j^2$, $\mu_j^{-2}$, $\bar{\mu}_j^2$, $\bar{\mu}_j^{-2}$, $1 \leqq j \leqq 2n-1$, are the eigenvalues of the Poincaré map $P(\gamma)$; the $\theta_j$'s are the negatives of the rotation angles of $R(\gamma)$, the holonomy element associated to $\gamma$; and $\mu^2 = N(\gamma) = e^{L(\gamma)}$, in case we think of $\gamma$ as a closed geodesic. Thus the above definition makes sense formally in great generality. Note that $\gamma$ may be diagonalized over the complex numbers as the diagonal matrix with diagonal entries $\mu_1^{\pm 2}, \mu_2^{\pm 2}, \cdots, \mu_{2n-1}^{\pm 2}$. (Note that $|\mu_j| > 1$ for $1 \leqq j \leqq 2n-1$.) In case $\Gamma \subseteq SO(3, 1) = SL(2, C)/\pm I$ we have

$$\tilde{Z}(s) = \prod\nolimits_{\mathcal{P}}\prod\nolimits_{n_1,n_2\geqq 1}\frac{1-\mu^{-2n_1}\bar{\mu}^{-2n_2}e^{2i\theta}}{1-\mu^{-2n_1}\bar{\mu}^{-2n_2}e^{-2i\theta}}|\mu|^{-2s}\ .$$

We now calculate the logarithmic derivative of $\tilde{Z}$.

$$\frac{\tilde{Z}'(s)}{\tilde{Z}(s)} = \Bigg[\sum\nolimits_{\sigma\in\mathfrak{S}}\sum\nolimits_{\mathcal{P}}\sum\nolimits_{n_i=1}^{\infty}\frac{\log|\mu|^2\mu_1^{-2n_1}\bar{\mu}_1^{-2n_2}\cdots\mu_{2n-1}^{4n-3}\bar{\mu}_{2n-1}^{-2n_{4n-2}}\sigma\circ\beta_+|\mu|^{-2s}}{1-\mu_1^{-2n_1}\bar{\mu}_1^{-2n_2}\cdots\mu_{2n-1}^{-2n_{4n-3}}\bar{\mu}_{2n-1}^{-2n_{4n-2}}\sigma\circ\beta_+|\mu|^{-2s}}$$
$$-\sum\nolimits_{\sigma\in\mathfrak{S}}\sum\nolimits_{\mathcal{P}}\sum\nolimits_{n_i=1}^{\infty}\frac{\log|\mu|^2\mu_1^{-2n_1}\bar{\mu}_1^{-2n_1}\cdots\mu_{2n-1}^{-2n_{4n-3}}\bar{\mu}_{2n-1}^{-2n_{4n-2}}\sigma\circ\beta_-|\mu|^{-2}}{1-\mu_1^{-2n_1}\bar{\mu}_1^{-2n_2}\cdots\mu_{2n-1}^{-2n_{4n-3}}\bar{\mu}_2^{-2n_{4n-3}}\sigma\circ\beta_-|\mu|^{-2s}}\Bigg]\ .$$

Expanding the denominator in an infinite geometric series and observing that $\{\gamma^k: k\in N, \gamma\in\mathcal{P}\}$ gives all the elements of $\Gamma$, we obtain:

$$\frac{\widetilde{Z}'(s)}{\widetilde{Z}(s)} = \Big[\sum_{w\in\mathfrak{S}}\sum_{\Gamma}\sum_{n_i=1}^{\infty}\log|\mu^*|^2\mu_1^{-2n_1}\bar{\mu}_1^{-2n_2}\cdots\mu_{2n-1}^{-2n_{4n-3}}\bar{\mu}_{2n-1}^{-2n_{4n-2}}w\circ\beta|\mu|^{-2s}$$

$$-\sum_{w\in\mathfrak{S}}\sum_{\Gamma}\sum_{n_i=1}^{\infty}\log|\mu^*|^2\mu_1^{-2n_1}\bar{\mu}_1^{-2n_2}\cdots\mu_{2n-1}^{-2n_{4n-3}}\bar{\mu}_{2n-1}^{-2n_{4n-2}}w\circ\beta_-|\mu|^{-2s}\Big]$$

$$= 2^{2n-1}i^{2n-1}\sum_{\Gamma}\sum_{n_i=1}^{\infty}\log|\mu^*|^2\sin\theta_1\cdots\sin\theta_{2n-1}\mu_1^{-2n_1}\bar{\mu}_1^{2n_2}$$

$$\cdots\mu_{2n-1}^{-2n_{4n-3}}\bar{\mu}_{2n-1}^{-2n_{4n-2}(1/2)}|\mu|^{-2s}.$$

THEOREM 3.1.

$$\int_0^\infty e^{-s^2t}\operatorname{Tr}Ae^{-t\Delta}\,dt = \frac{i}{2}\frac{\widetilde{Z}'(2n-1+s)}{\widetilde{Z}(2n-1+s)}, \quad \operatorname{Re}s^2 > -a.$$

*Proof.* Recall the expression we had derived for $\operatorname{Tr}Ae^{-t\Delta}$ from the Selberg trace formula:

$$\operatorname{Tr}Ae^{-t\Delta} = 2^{2n-1}i^{2n-1}i\sum_{\{\gamma\}_\Gamma}\frac{\log|\mu^*|^2\sin\theta_1\cdots\sin\theta_{2n-1}2\pi\log|\mu|^2}{|\mu_1-\mu_1^{-1}|^2|\mu_2-\mu_2^{-1}|^2\cdots|\mu_{2n-1}-\mu_{2n-1}^{-1}|^2}\frac{e^{-(\log|\mu|^2)^2/4t}}{(4\pi t)^{3/2}}.$$

We will expand the denominator of our expression for $\operatorname{Tr}Ae^{-t\Delta}$ according to the infinite geometric series

$$\operatorname{Tr}Ae^{-t\Delta} = 2^{2n-1}i^{2n-1}i\sum_{\{\gamma\}_\Gamma}\frac{\log|\mu^*|^2\sin\theta_1\sin\theta_2\cdots\sin\theta_{2n-1}2\pi\log|\mu|^2}{|\mu|^{2(2n-1)}|1-\mu_1^{-2}|^2|1-\mu_2^{-2}|^2\cdots|1-\mu_{2n-1}^{-2}|^2}$$
$$\times\frac{e^{-(\log|\mu|^2)^2/4t}}{(4\pi t)^{3/2}}.$$

Before we expand the denominator let us observe the formula

$$\int_0^\infty e^{-s^2t}\frac{e^{-r^2/4t}}{(4\pi t)^{3/2}}dt = \frac{e^{-sr}}{4\pi r}.$$

Applying this to our expression for $\operatorname{Tr}Ae^{-t\Delta}$ we obtain

$$\int_0^\infty e^{-s^2t}\operatorname{Tr}Ae^{-t\Delta}dt = 2^{2n-1}i^{2n-1}\frac{i}{2}\sum_{\{\gamma\}_\Gamma}\frac{\log|\mu^*|^2\sin\theta_1\sin\theta_2\cdots\sin\theta_{2n-1}}{|\mu|^{2(2n-1)}|1-\mu_1^{-2}|^2\cdots|1-\mu_{2n-1}^{-2}|^2}$$
$$\times e^{-s\log|\mu|^2}.$$

But

$$e^{-s\log|\mu|^2} = |\mu|^{-2s},$$

$$\int_0^\infty e^{-s^2t}\operatorname{Tr}Ae^{-t\Delta}dt = 2^{2n-1}i^{2n-1}\frac{i}{2}\sum_{\{\gamma\}_\Gamma}\frac{\log|\mu^*|^2\sin\theta_1\sin\theta_2\cdots\sin\theta_{2n-1}}{(1-\mu_1^{-2})(1-\bar{\mu}_1^{-2})\cdots(1-\mu_{2n-1}^{-2})(1-\bar{\mu}_{2n-1}^{-2})}$$
$$\times|\mu|^{-2s-2(2n-1)}.$$

Now we expand the denominator (note $|\mu|>1$) to obtain

$$= 2^{2n-1}i^{2n-1}\frac{i}{2}\sum_{\{\gamma\}_\Gamma}\sum_{n_i=1}^{\infty}\log|\mu^*|^2\sin\theta_1\cdots\sin\theta_{2n-1}\mu_1^{-2n_1}\bar{\mu}_1^{-2n_2}$$
$$\cdots\mu_{2n-1}^{-2n_{n-2}}|\mu|^{-2s-2(2n-1)}.$$

If we let $\Psi(s) = \int_0^\infty e^{-s^2t}\operatorname{Tr}Ae^{-t\Delta}dt$, then from our last line we see

$$\Psi(s-2n+1) = \frac{i}{2}\frac{\widetilde{Z}'(s)}{\widetilde{Z}(s)} .$$

Thus

$$\Psi(s) = \frac{i}{2}\frac{\widetilde{Z}'(2n-1+s)}{\widetilde{Z}(2n-1+s)} .$$

Now $\eta(s)$ is an integral transform of $\operatorname{Tr} Ae^{-t\Delta}$ and we have just seen that $\widetilde{Z}'(2n-1+s)/\widetilde{Z}(2n-1+s)$ is also. We just have to relate the two transforms: to do this all we have to do is observe:

$$\int_0^\infty \sigma^{-s} e^{-\sigma^2 t} d\sigma = \frac{1}{2} t^{s-1/2} \Gamma\left(\frac{1-s}{2}\right) , \qquad \operatorname{Re} s < 1 .$$

Now we are almost done.

$$\eta(s) = \frac{1}{\Gamma\left(\frac{1+s}{2}\right)} \int_0^\infty t^{s-1/2} \operatorname{Tr} Ae^{-t\Delta} dt ,$$

$$= \frac{1}{\Gamma\left(\frac{1+s}{2}\right)} \int_0^\infty \frac{2}{\Gamma\left(\frac{1-s}{2}\right)} \left(\int_0^\infty \sigma^{-s} e^{-\sigma^2 t} d\sigma\right) \operatorname{Tr} Ae^{-t\Delta} dt , \quad \operatorname{Re} s < 1 .$$

interchanging

$$= \frac{1}{\Gamma\left(\frac{1+s}{2}\right)} \frac{2}{\Gamma\left(\frac{1-s}{2}\right)} \int_0^\infty \sigma^{-s} \left(\int_0^\infty e^{-\sigma^2 t} \operatorname{Tr} Ae^{-t\Delta} dt\right) d\sigma$$

$$= \frac{-\frac{1}{i}\frac{1}{2}2}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)} \int_0^\infty \sigma^{-s} \frac{d}{d\sigma} \log \widetilde{Z}(2n-1+\sigma) d\sigma ,$$

$$\operatorname{Re} s^2 > -a , \qquad \operatorname{Re} s < 1 .$$

At $s = 0$ since $\lim_{x\to+\infty} \widetilde{Z}(x) = 1$, $\Gamma(1/2) = \sqrt{\pi}$ we obtain:

$$\eta(0) = \frac{1}{\pi i} \log \widetilde{Z}(2n-1) \qquad \text{(where} \quad \log 1 = 0) .$$

*Note.* In this proof we were guided by the example of Ray-Singer [21]. We now present a most beautiful theorem which we owe to Takuro Shintani.

THEOREM. *$\widetilde{Z}$ satisfies the functional equation*

$$\widetilde{Z}(s)\widetilde{Z}(4n-2-s) = e^{2\pi i \eta(0)} .$$

*Proof.* Now $\widetilde{Z}(s)$ is defined to the right of some half line. We begin by constructing an analytic continuation of $\widetilde{Z}'(s)/\widetilde{Z}(s)$. First note that since $\Gamma$ is discrete there exists a positive constant $c_0$ such that $\log|\mu_\gamma| > c_0$ for all $\gamma \in \Gamma$, $\gamma \neq 1$ where $\mu_\gamma$ is an eigenvalue of $\gamma$ such that $|\mu_\gamma| > 1$. Take an *odd* $C^\infty$ function $\varphi(t)$ on $\mathbf{R}$ which satisfies

$$\varphi(t) = \begin{cases} 1 & |t| > c_0 \\ 0 & |t| \leqq c_0 . \end{cases}$$

Set

$$\begin{cases} H_s(\tau_+, \lambda) = \int_{-\infty}^{\infty} \varphi(t) e^{-s|t|} e^{-i\lambda t} dt = -H_s(\tau_-, \lambda) \\ H_s(\tau, \lambda) = 0 \qquad \tau \neq \tau_+, \tau_- . \end{cases}$$

Here $s$ is a complex parameter.

*Note.* We are applying the trace formula to the one parameter family of functions $f_s$ on $G$ with Fourier transform $H_s(\tau, \lambda)$, justified by Theorem 2.2 for $\operatorname{Re} s$ sufficiently large.

Now we apply the general trace formula to obtain (note the $1/2\pi$ disappears in the Fourier inversion formula):

$$\begin{aligned} \textstyle\sum_j H_s(\tau_+, \lambda_j) &= 2^{2n-1} i^{2n-1} \textstyle\sum_{\{\gamma\}_\Gamma} \frac{\log|\mu^*|^2 \sin\theta_1 \cdots \sin\theta_{2n-1}}{|\mu_1 - \mu_1^{-1}|^2 \cdots |\mu_{2n-1} - \mu_{2n-1}^{-1}|^2} e^{-s} \log|\mu|^2 \\ &= \frac{\tilde{Z}'(2n-1+s)}{\tilde{Z}(2n-1+s)} . \end{aligned}$$

Thus:

$$\begin{aligned} \frac{\tilde{Z}'(2n-1+s)}{\tilde{Z}(2n-1+s)} &= \textstyle\sum_j H_s(\tau_+, \lambda_j) \\ &= \begin{cases} \sum_j \dfrac{1}{s - i\lambda_j} \displaystyle\int_0^{\infty} \varphi'(t) e^{(-2s+i\lambda_j)t} dt \\ -\sum_j \dfrac{1}{s + i\lambda_j} \displaystyle\int_{-\infty}^{0} \varphi'(t) e^{(2s+i\lambda_j)t} dt . \end{cases} \end{aligned}$$

This last expression is obtained by integrating the expression for $H_s(\tau_+, \lambda_j)$ by parts. All this is valid for $\operatorname{Re} s$ sufficiently large. By integrating by parts $k$ times we obtain an expression of the form

$$* \qquad \frac{Z'(2n-1+s)}{Z(2n-1+s)} = \begin{cases} \sum_j \dfrac{1}{(s - i\lambda_j)^k} \displaystyle\int_0^{\infty} \varphi^{(k)}(t) e^{(-2s+i\lambda_j)t} dt \\ -\sum_j \dfrac{i}{(s + i\lambda_j)^k} \displaystyle\int_{-\infty}^{0} \varphi^{(k)}(t) e^{(2s+i\lambda_j)t} dt . \end{cases}$$

Now the sequence of numbers $\{\lambda_j^2 : j = 1, 2, \cdots\}$ is just the eigenvalues of the Laplacian on coclosed $2n-1$ forms as we have seen. But there is a general formula of Gaffney [10] (valid for any compact Riemannian manifold) giving the asymptotic behavior of the $j^{\text{th}}$ eigenvalue of the Laplacian. From this we deduce

$$\lambda_j^2 \sim c j^{2/4n-1} \qquad \text{where } c \text{ is a constant.}$$

Thus when $k$ is sufficiently large the above series converge for all $s$. We have thus obtained an analytic continuation of

$$\frac{\tilde{Z}'(2n-1+s)}{\tilde{Z}(2n-1+s)} .$$

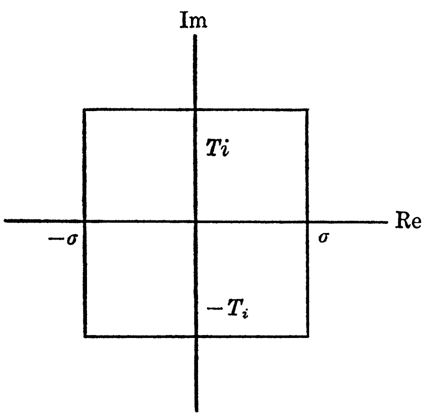
Next we note from * (by multiplying the right-hand side by $s - i\lambda_j$ and taking the limit as $s \to i\lambda_j$) that $\tilde{Z}'(2n-1+s)/\tilde{Z}(2n-1+s)$ has a simple pole at $s = i\lambda_j$ with residue $m_j$ and (by multiplying by $(s + i\lambda_j)$) a simple pole at $s = -i\lambda_j$ with residue $-m_j$ where $m_j$ is the multiplicity with which $\pi_{\tau_+,\lambda} - \pi_{\tau_-,\lambda_j}$ enters into the decomposition of $L^2(\Gamma\backslash \mathrm{SO}(4n-1,1))$. Thus $\tilde{Z}$ *satisfies the Riemann hypothesis* and in fact has no zeroes off its critical line $\operatorname{Re} s = 2n-1$. We see then that

$$\frac{\tilde{Z}'(2n-1+s)}{\tilde{Z}(2n-1+s)} - \frac{\tilde{Z}'(2n-1-s)}{\tilde{Z}(2n-1-s)} = R(s)$$

is an odd entire function of $s$. Now let $h(s)$ be an odd function which in the strip $\{s : |\operatorname{Re} s < 2n-1+\varepsilon, \varepsilon > 0\}$ decreases sufficiently rapidly as $\operatorname{Im} s \to \infty$. Consider the contour integral

$$\frac{1}{2\pi i}\int_L h(s)\frac{\tilde{Z}'(2n-1+s)}{\tilde{Z}(2n-1+s)}ds$$

where $L$ is the following contour




$$4n-2 < \sigma < 4n-2+\varepsilon .$$

As $T$ goes to infinity the above integral approaches:

$$\frac{1}{2\pi i}\int_{i\infty-\sigma}^{i\infty+\sigma} h(s)R(s)ds + 2\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} h(s)\frac{\tilde{Z}'(2n-1+s)}{\tilde{Z}(2n-1+s)}ds .$$

Now we substitute for $Z'(2n-1+s)/Z(2n-1+s)$ to find that the contour integral is given by:

$$2\frac{2^{2n-1}i^{2n-1}}{2\pi i}\sum_{\{\gamma\}_\Gamma}\frac{\log|\mu^*|^2\sin\theta_1\cdots\sin\theta_{2n-1}}{|\mu_1-\mu_1^{-1}|^2\cdots|\mu_{2n-1}-\mu_{2n-1}^{-1}|^2}-\int_{\sigma-i\infty}^{\sigma+i\infty}|\mu|^{-2s}h(s)ds$$
$$+\frac{1}{2\pi i}\int_{i\infty-\sigma}^{-i\infty-\sigma}h(s)R(s)ds\ .$$

On the other hand the residue theorem implies that the integral # is equal to (as $T\to\infty$) $2\sum_j m_j h(i\lambda_j)$. But now we observe that

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}|\mu|^{-2s}h(s)ds=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{-\lambda\log|\mu|^2}h(\lambda)\,d\lambda\ ,$$

by moving the contour.

Now we make the change of variable $\lambda = i\lambda$ to obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda\log|\mu|^2}h(i\lambda)\,d\lambda\ .$$

But now applying the Selberg trace formula for $\lambda\to h(i\lambda)$ we see

$$\int_{i\infty-\sigma}^{-i\infty-\sigma}h(s)R(s)ds=0\ .$$

Since this equality holds for odd holomorphic $h(s)$ satisfying suitable growth conditions we conclude $R(s)=0$.

We obtain then

$$\frac{\tilde Z'(2n-1+s)}{\tilde Z(n-1+s)}=\frac{\tilde Z'(2n-1-s)}{\tilde Z(2n-1-s)}\ ,$$

$\tilde Z(s)\tilde Z(4n-2-s)=$ constant. The constant is evaluated by noting it is just $(\tilde Z(2n-1))^2$. This proves the theorem.

*Remark.* If one assumes the functional equation for $\tilde Z$ then applying the residue theorem to the contour integral # reproduces the trace formula for *odd functions* $h(\gamma)$. It is for this reason that we call $\tilde Z$ the Selberg zeta function of the *second kind.* The original Selberg zeta function has the same property for even functions. From Theorem 3.1 we have

$$\frac{\tilde Z'(2n-1+s)}{\tilde Z(2n-1+s)}=2^{2n-1}i^{2n-1}\sum_{\{\gamma\}_\Gamma}\frac{\log(\mu^*)^2\sin\theta_1\cdots\sin\theta_{2n-1}1}{|\mu_1-\mu_1^{-1}|^2\cdots|\mu_{2n-1}-\mu_{2n-1}|^2(\mu)^{2s}}\ .$$

To identify this formula with the formula stated in the introduction, we give some notation and prove a proposition relating the group theory to geometry.

To each $\gamma\in\Gamma$ we make correspond a smoothly closed geodesic in $\Gamma\backslash G/K$ as follows. $\gamma$ leaves fixed a unique geodesic in $G/K$ denoted $\alpha\colon(-\infty,\infty)\to G/K$ which we call the axis of $\Gamma$. We assume $\alpha$ is parametrized by arc length $s$ and let $T$ denote the tangent vector field to $\alpha$. Let $x_0=\alpha(0)$ and $x_1=\gamma x_0$.

We have $x_1 = \alpha(L)$. The closed segment of $\alpha$ from $x_0$ to $x_1$ projects to the smoothly closed geodesic also denoted $\gamma$ on $\Gamma\backslash G/K$ corresponding to $\gamma$. We denote the quotient map $G/K \to \Gamma\backslash G/K$ by $\pi$ and parallel translation along $\alpha$ from $x$ to $x'$ by $\pi_{x,x'}$. We denote $\pi(x_0)$ by $y_0$ and let $\gamma = p(\gamma)k(\gamma)$ be the polar decomposition of $\gamma$.

*Proposition* 3.1. *Under the above correspondence between elements in the discrete group and smoothly closed geodesics,*

(1) $L(\gamma) = \log \lambda(\gamma)^2$.

(2) *The eigenvalues of* $R(\gamma)$ *are*

$$\{e^{-i\theta_1}, e^{i\theta_1}, \cdots, e^{-i\theta_{2n-1}}, e^{i\theta_{2n-1}}\} .$$

(3) *The eigenvalues of* $P(\gamma)$ *are* $\{\mu_j^2, \bar{\mu}_j^2, \mu_j^{-2}, \bar{\mu}_j^{-2}: 1 \leqq j \leqq 2n-1\}$ *and*

$$\det|1 - P(\gamma)|^{1/2} = |\mu_1 - \mu_1^{-1}|^2 \cdots |\mu_{2n-1} - \mu_{2n-1}^{-1}|^2 .$$

*Proof.* We take for our model of hyperbolic space the manifold $H_{4n-1}$ defined by

$$H_{4n-1} = \{(t, x_1, \cdots, x_{4n-1}) \in \mathbf{R}^{4n}: t^2 - x_1^2 - \cdots - x_{4n-1}^2 = 1, t > 0\}$$

with the Riemannian metric induced from the Minkowski metric on $\mathbf{R}^{4n}$. Choose $g_0 \in G$ so that $g_0$ maps the axis of $\gamma$ to the great hyperbola $H_1$ defined by

$$H_1 = \{(t, x_1, 0, \cdots, 0) \in \mathbf{R}^{4n}: t^2 - x_1^2 = 1, t > 0\}$$

with $g_0(0) = x_0$. We further refine our choice of $g_0$ to obtain $g_0\gamma g_0^{-1} \in A$.

To calculate $R(\gamma)$ we note that $R(\gamma)$ is given by

$$R(\gamma) = d\pi_{x_0} \circ d\gamma^{-1} \circ \tau_{x_0,x_1} \circ d\pi_{x_0}^{-1} .$$

Now consider the polar decomposition $\gamma = p'k'$ relative the maximal compact subgroup of $G$ leaving $x_0$ fixed. $p'$ is a transvection; it leaves invariant a unique geodesic $\beta$ through $x_0$ and $dp'$ induces parallel translation along $\beta$. But $\beta$ must contain the unique geodesic segment joining $x_0$ and $p'x_0$ because $p'x_0 \in \beta$. However $\gamma x_0 = p'x_0$ and consequently $\alpha = \beta$. Since $dp'$ induces parallel translation along $\beta$ it preserves the tangent direction to $\beta$; consequently $\mathrm{Ad}\,k'$ leaves fixed the tangent direction to $\beta$ at $x_0$. Substituting in the above expression for $R(\gamma)$ and noting $dp' | T(H_{4n-1}, x_0) = \tau_{x_0,x_1}$ we obtain

$$R(\gamma) = d\pi_{x_0} \circ \mathrm{Ad}\,k'^{-1} \circ d\pi_{x_0}^{-1} .$$

But the angles $\theta_1(\gamma), \theta_2(\gamma), \cdots, \theta_{2n-1}(\gamma)$ are obtained by conjugating $\gamma$ by $g \in G$ so that $g\gamma g^{-1} \in A$; see Chapter II, Section 1. Since $g_0\gamma g_0^{-1} \in A$ we can calculate the rotation angles of $\gamma$ from the rotation angles of $g_0k'g_0^{-1}$, but these latter angles are the negatives of the rotation angles of $R(\gamma)$ by the above formula.

This proves (2).

To prove (1) and (3) we may again replace $\gamma$ by $g_0\gamma g_0^{-1}$; that is, we may assume $\gamma \in A$. But then $\mu(\gamma) = e^{r/2}$. $L(\gamma)$ is just the distance $d(e_0, \gamma e_0)$ between $e_0 = (1, 0, \cdots, 0)$ and $\gamma \circ (1, 0, \cdots, 0) = (\cosh r, \sinh r, 0, \cdots, 0)$. But the Riemannian distance between two vectors $x, y \in H_{4n-1}$ is just the hyperbolic angle between them,

$$d(x, y) = \cosh^{-1}\langle x, y\rangle$$

where $\langle x, y\rangle$ is the inner product of $x$ and $y$ in the Minkowski space. Thus

$$\begin{aligned} L(\gamma) &= \cosh^{-1}(\cosh r) \\ &= r \end{aligned}$$

and (1) is proved.

We are left then with the problem of computing $P(\gamma)$. We use the following well-known formula, see Appendix 1. Suppose $(x, u)$ is fixed under $\varphi_L$. Let $y(s)$ be the closed geodesic leaving $y(0) = x$ with tangent vector $y'(0) = u$ corresponding to this fixed-point. Let $e_1, e_2, \cdots, e_m$ be the frame chosen previously in $T_xM$ and let $Y_2, Y_3, \cdots, Y_m$ be Jacobi fields along $y(s)$ satisfying for $2 \leqq j \leqq 4n - 1$,

$$\begin{aligned} Y_j(0) &= e_j , \\ \nabla_u Y_j(0) &= 0 . \end{aligned}$$

Let $\bar{Y}_j$ be Jacobi fields along $y(s)$ satisfying

$$\begin{aligned} \bar{Y}_j(0) &= 0 , \\ \nabla_u \bar{Y}_j(0) &= e_j . \end{aligned}$$

Let us define four $m \times m$ matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, and $D = (d_{ij})$:

$$\begin{aligned} Y_i(L) &= \textstyle\sum_{j=2}^m a_{ij} e_j , & Y_i'(L) &= \textstyle\sum_{j=2}^m c_{ij} e_j , \\ \bar{Y}_i(L) &= \textstyle\sum_{j=2}^m b_{ij} e_j , & \bar{Y}_i'(L) &= \textstyle\sum_{j=2}^m d_{ij} e_j . \end{aligned}$$

Then relative to a suitable frame for $T_{(x,u)}(TM)$, $P(\gamma)$ has the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

Thus to complete the proof of Proposition 3.1 we must solve the Jacobi equation on hyperbolic space $H_{4n-1}$ along $H_1$. We begin by extending the frame $e_1, e_2, \cdots, e_{4n-1}$ along $H_1$ by parallel translation and let $E_j$ denote the parallel translate of $e_j$ for $2 \leqq j \leqq 4n - 1$. Now the Jacobi equation for a vector field $Y$ along a geodesic with tangent vector $T$ is

$$\nabla_T^2 Y = R_{T,Y}T .$$

Since we are dealing with a space of constant curvature $-1$ we have

$R_{T,Y}T = Y$ and the Jacobi equation for hyperbolic space becomes

$$\nabla_T^2 Y = Y .$$

Writing out $Y(s) = \sum_{j=2}^{4n-1} c_j(s)E_j$, we find that the general solution of the Jacobi equation is

$$c_j(s) = \alpha \cosh s + \beta \sinh s$$

where $\alpha$ and $\beta$ are arbitrary constants. We find that

$$Y_j(s) = \cosh sE_j ,$$
$$\bar{Y}_j(s) = \sinh sE_j .$$

Hence relative to a suitable basis for $T_{(x,u)}(TM)$ (here $R = R(\gamma)$ is the holonomy element associated to $\gamma$),

$$P(\gamma) = \begin{pmatrix} \cosh LR & \sinh LR \\ \sinh LR & \cosh LR \end{pmatrix} .$$

We adopt some notation for the rest of the proof. Given $A, B \in M(m, \mathbf{C})$ we write $A \sim B$ if $A$ and $B$ are similar; that is, there exists $Q \in M(m, \mathbf{C})$ so that

$$QAQ^{-1} = B .$$

Then $P \sim P'$ where $P'$ is the matrix consisting of $4 \times 4$ diagonal blocks $P_j$ for $1 \leqq j \leqq 2n - 1$ where $P_j: \mathbf{R}^2 \otimes \mathbf{R}^2 \to \mathbf{R}^2 \otimes \mathbf{R}^2$ is given by

$$P_j = \begin{pmatrix} \cosh L & \sinh L \\ \sinh L & \cosh L \end{pmatrix} \otimes \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix} .$$

Clearly $P_j$ is similar to the $4 \times 4$ diagonal matrix with diagonal entries $e^{L+i\theta_j}$, $e^{L-i\theta_j}$, $e^{-L+i\theta_j}$, $e^{-L-i\theta_j}$ and hence

$$\begin{aligned}\det(1 - P(\gamma)) &= \prod_{j=1}^{2n-1}(1 - e^{L+i\theta_j})(1 - e^{L-i\theta_j})(1 - e^{-L+i\theta_j})(1 - e^{-L-i\theta_j}) \\ &= \prod_{j=1}^{2n-1}(\mu_j - \mu_j^{-1})(\bar{\mu}_j - \bar{\mu}_j^{-1})(\bar{\mu}_j - \bar{\mu}_j^{-1})(\mu_j - \mu_j^{-1}) \\ &= \Delta_P(\{\gamma\})^2 ,\end{aligned}$$

and we have completed the proof of Proposition 3.1.

*Remark.* Since $P(\gamma)$ is a symplectic matrix, if $\lambda$ is an eigenvalue of $P(\gamma)$, $\bar{\lambda}$, $\bar{\lambda}^{-1}$ and $\lambda^{-1}$ are eigenvalues also.

COROLLARY.

$$\frac{\tilde{Z}'(2n - 1 + s)}{\tilde{Z}(2n - 1 + s)} = -\sum_{\gamma \in \mathfrak{p}} \sum_{k=1}^{\infty} \frac{\chi(R(\gamma)^k)}{|\det(1 - P(\gamma))|^{1/2}} \frac{\log N(\gamma)}{N(\gamma)^{ks}} .$$

To obtain the formula in the introduction integrate each side from $s$ to $\infty$.

It seems natural to ask in what generality our formula holds. That there might be some generalization to manifolds of strictly negative curvature is suggested by the results of Margulis [19] from which one can deduce that the series for $\widetilde{Z}(s)$ converges absolutely for $\mathrm{Re}\, s > 4n - 2$ (where the dimension of $M$ is $4n - 1$).

## Appendix

### The Poincaré map for the geodesic flow

In this appendix we determine a formula for the Poincaré map associated to a closed orbit of length $L$ under the geodesic flow $\varphi_t$ in the tangent sphere bundle $SM$ of a Riemannian manifold $M$ in terms of Jacobi fields on $M$. For the definition and elementary properties of Jacobi fields, see Cheeger-Ebin [8], Chapter 1, Section 4.

We begin by recalling that the Riemannian connection on $TM$ gives us a splitting

$$T(TM) = T^{\mathrm{hor}}(TM) \oplus T^{\mathrm{vert}}(TM)$$

where $T^{\mathrm{vert}}(TM)$ is the space of vectors tangent to the fibers and $T^{\mathrm{hor}}(TM)$ is the complement to $T^{\mathrm{vert}}(TM)$ provided by the connection. We define $T^{\mathrm{hor}}_{(x,u)}(TM)$ as the space of tangent vectors to horizontal curves in $TM$ originating at $(x, u)$. We define a curve $\big(x(t), U(t)\big)$ in $TM$ to be horizontal if $U(t)$ is parallel along $x(t)$; that is, $\nabla_T U = 0$ where $\nabla$ is the (Koszul) connection associated to the Riemannian metric. Clearly $T^{\mathrm{hor}}_{(x,u)}(TM)$ is a complement to $T^{\mathrm{vert}}_{(x,u)}(TM)$ at $T_{(x,u)}(TM)$. We write the decomposition for $w \in T(TM)$ as $w = w^{\mathrm{hor}} + w^{\mathrm{vert}}$ with $w^{\mathrm{hor}} \in T^{\mathrm{hor}}(TM)$ and $w^{\mathrm{vert}} \in T^{\mathrm{vert}}(TM)$. We have a canonical isomorphism

$$K\colon T_x M \longrightarrow T^{\mathrm{vert}}_{(x,u)}(TM) = T_{(x,u)}(T_x M)$$

given by

$$K\big((x, u), v\big) = \frac{d}{dt}(x, u + tv)\Big|_{t=0}$$

where $(x, v) \in T_x M$. We note that this is just the usual isomorphism between a vector space $V$ and the various tangent spaces $T_u V$ where $u \in V$.

We also have an isomorphism induced by the connection

$$H\colon T_x M \longrightarrow T^{\mathrm{hor}}_{(x,u)}(TM)\ .$$

$H\big((x, u), v\big)$ is the unique horizontal vector in $T_{(x,u)}(TM)$ so that

$$d\textstyle\prod_{(x,u)} H\big((x, u), v\big) = (x, v)$$

where of course $d\prod_{(x,u)}\colon T_{(x,u)}(TM) \to T_x M$. $H\big((x, u), v\big)$ may be described as follows. Let $x(t)$ be a curve on $M$ starting at $x$ with tangent vector $v$. Let $U(t)$ be the vector field along $x(t)$ obtained by parallel translating $u$ along $x(t)$. Then $\alpha(t) = \big(x(t), U(t)\big)$ is a curve in $TM$ and $H\big((x, u), v\big)$ is the tangent vector to this curve at $t = 0$. We generalize this slightly in the following lemma.

LEMMA A. *Let $x(t)$, $0 \leqq t \leqq 1$, be an arc in $M$ with tangent vector $T$ and $x(0) = x$ and let $U(t)$ be a vector field along $x(t)$; that is, a section of the pull-back bundle of $TM$ along $[0, 1]$. We define $\alpha(t) = (x(t), U(t))$ a curve in $TM$. Then $d\alpha(d/dt) \in T_{(x(t),U(t))}(TM)$ and we have*

$$d\alpha\left(\frac{d}{dt}\right)^{\text{hor}} = H\big((x(t), U(t)), T(t)\big),$$

$$d\alpha\left(\frac{d}{dt}\right)^{\text{vert}} = K\big((x(t), U(t)), \nabla_T U\big).$$

*Proof.* $\alpha^*TM$ is a bundle over $[0, 1]$ and consequently is trivial. Parallel translation gives us an explicit trivialization compatible with the connection on $\alpha^*TM$,

$$[0, 1] \times T_x M \xrightarrow{\varphi} \alpha^*TM.$$

Under the trivialization $\alpha(t) = (t, B(t))$ where $B(t)$ is a curve in $T_x M$. $d\varphi$ gives an isomorphism compatible with the splittings into horizontal and vertical parts (because $\varphi$ is connection preserving),

$$T([0, 1] \times T_x M) \xrightarrow{d\varphi} T(\alpha^*TM).$$

Since $dB/dt = \nabla_T U$ is the vertical component for $T([0, x] \times T_x M)$ the lemma is clear.

Now given a frame $e_1, e_2, \cdots, e_m$ for $T_x M$ we receive a basis for $T_{(x,u)}(TM)$ given by

$$\{H((x, u), e_1), \cdots, H((x, u), e_m); K((x, u), e_1), \cdots, K((x, u), e_m)\}.$$

Now assume that $u$ is of unit length and that $e_1 = u$; then clearly a basis for $T_{(x,u)}(SM)$ is given by

$$\{H((x, u), e_1), \cdots, H((x, u), e_m); K((x, u), e_2), \cdots, K((x, u), e_m)\}.$$

A more convenient way to represent $K((x, u), e_j)$ in this case is $(2 \leqq j \leqq m)$

$$K((x, u), e_j) = \frac{d}{dt}(\cos tu + \sin te_j)\Big|_{t=0}.$$

In order to calculate $P(\gamma)$ we must determine the vector field on $SM$ that generates the geodesic flow which we denote $Z$. But $Z$ is just the canonical horizontal field; that is, $Z(x, u)$ satisfies

(1) $Z(x, u)$ is horizontal,

(2) $d\prod_{(x,u)} Z(x, u) = u$.

From the definition of $H$ it follows that

$$Z(x, u) = H\big((x, u), u\big) .$$

Finally then we find that if $e_1 = u$ and $e_2, \cdots, e_m$ is a frame for $T_x(M)$ we wish to compute the matrix of $d\varphi_L$ relative the basis $\beta$ for the complement of the line spanned by $Z$ in $T_{(x,u)}(SM)$ given by

$$\{H\big((x, u), e_2\big), \cdots, H\big((x, u), e_m\big); K\big((x, u), e_2\big), \cdots, K\big((x, u), e_m\big)\} = \beta .$$

We assume $(x, u)$ is fixed under $\varphi_L$ and put $y(s) = \pi \circ \varphi_s(x, u)$, $0 \leqq s \leqq L$. We shall assume some elementary properties of Jacobi fields, see Cheeger-Ebin [8], Chapter 1, Section 4. We first calculate $d\varphi_L \circ H\big((x, u), e_j\big)$; $2 \leqq j \leqq m$. Let $x(t)$ be a geodesic on $M$ starting at $x$ with tangent vector $e_j$ at $t = 0$ and let $U(t)$ be the parallel translate of $u$ along $x(t)$. Let $\alpha(t) = \big(x(t), U(t)\big)$,

$$d\varphi_L \circ H\big((x, u), e_j\big) = \frac{d}{dt}\varphi_L\big(\alpha(t)\big)\Big|_{t=0} .$$

The curve $\varphi_L\big(\alpha(t)\big)$ fits into a variation $\tilde{\chi}\colon \mathbf{R} \times \mathbf{R} \to SM$ given by $\tilde{\chi}(s, t) = \varphi_s\big(x(t), U(t)\big)$. For fixed $t$, the projection of $\varphi_s\big(x(t), U(t)\big)$ is a geodesic $y_t(s)$ leaving $x(t)$ in the direction $U(t)$. The family $y_t(s)$ is a variation $\chi(s, t)$ of $y(s)$ in $M$ through geodesics. We lift $\chi$ to a variation $\gamma_t(s)$ in $SM$ given by $\gamma_t(s) = \big(y_t(s), T_t(s)\big)$ where $T_t(s)$ is the tangent vector to the geodesic $y_t(s)$ and consequently is just the parallel translate of $U(t)$ along $y_t(s)$. Hence $\gamma_t(s) = \tilde{\chi}(s, t)$ and

$$\frac{d}{dt}\big(\varphi_L\big(\alpha(t)\big)\big)\Big|_{t=0} = \frac{d}{dt}\big(\gamma_t(L)\big)\Big|_{t=0} = \Big(\frac{d}{dt}\gamma_t(s)\Big)\Big|_{\substack{t=0\\ s=L}} .$$

Now we have $\big(d\gamma_t(s)/dt\big)\big|_{t=0} \in T_{(y(s),T(s))}(SM)$. We apply Lemma A to deduce that for each $s$, $0 \leqq s \leqq L$,

$$\frac{d_{\gamma t}}{dt}(s)\Big|_{t=0}^{\text{hor}} = H\big(\big(y(s)T(s)\big), Y_j(s)\big) ,$$

where $Y_j(s) = (d/dt)y_t(s)\big|_{t=0}$,

$$\frac{d\gamma_{t(s)}}{dt}\Big|_{t=0}^{\text{vert}} = K\big(\big(y(s), T(s)\big), \nabla_{Y_j(s)}T(s)\big) .$$

But we have $\nabla_{Y_j(s)}T(s) - \nabla_{T(s)}Y_j(s) = [Y_j(s), T(s)] = 0$. Hence

$$\frac{d\gamma_{t(s)}}{dt}\Big|_{t=0}^{\text{vert}} = K\big(\big(y(s), T(s)\big), \nabla_{T(s)}Y_j(s)\big) .$$

The point is that $Y_j(s)$ is the unique Jacobi field along $y(s)$ satisfying

$$Y_j(0) = e_j ,$$
$$\nabla_T Y_j(0) = \nabla_u Y_j(0) = \nabla_{e_j} U(0) = 0 .$$

We now calculate $d\varphi_L \circ K\big((x, u), e_j\big)$, $2 \leqq j \leqq m$. We define $\alpha(t)\colon \mathbf{R} \to T_xM$

by $\alpha(t) = \cos tu + \sin te_j$. Then

$$d\varphi_L \circ K((x, u), e_j) = \frac{d}{dt}\varphi_L(\alpha(t))\Big|_{t=0} .$$

Once again the curve $\varphi_L(\alpha(t))$ fits into a variation $\tilde{\chi}: \mathbf{R} \times \mathbf{R} \to SM$ given by $\tilde{\chi}(s, t) = \varphi_s(\alpha(t))$. For each $t$ the curve $y_t(s) = \varphi_s(\sigma(t))$ is a geodesic emanating from $x$ in the direction $\alpha(t)$. Thus

$$y_t(s) = \exp_x s\alpha(t) .$$

Once again we define $\gamma_t(s) = (y_t(s), T_t(s))$ where $T_t(s)$ is the tangent vector to the geodesic $y_t(s)$ and consequently is just the parallel translate of $\alpha(t)$ along $y_t(s)$. We have

$$\frac{d}{dt}(\varphi_L(\alpha(t)))\Big|_{t=0} = \frac{d}{dt}(\gamma_t(L))\Big|_{t=0} = \left(\frac{d}{dt}\gamma_t(s)\right)\Big|_{\substack{t=0\\ s=L}} .$$

We apply Lemma A to deduce that for each $s$, $0 \leqq s \leqq L$,

$$\frac{d\gamma_t(s)}{dt}\Big|_{t=0}^{\text{hor}} = H((y(s), T(s)), \bar{Y}_j(s))$$

where $\bar{Y}_j(s) = (d/dt)y_t(s)|_{t=0}$,

$$\frac{d\gamma_t(s)}{dt}\Big|_{t=0}^{\text{vert}} = K((y(s), T(s)), \nabla_{\bar{Y}_j(s)} T(s))$$
$$= K((y(s), T(s)), \nabla_{T(s)} \bar{Y}_j(s)) .$$

It is easily seen then that $\bar{Y}_j(s)$ is the unique Jacobi field along $y(s)$ satisfying

$$\bar{Y}_j(0) = 0 ,$$
$$\nabla_u \bar{Y}_j(0) = e_j .$$

Since the maps $v \to H((x, u), v)$ and $v \to K((x, u), v)$ are linear maps from $T_xM$ to $T_{(x,u)}(TM)$ we find the following formula for the matrix of $P(\gamma)$ relative to $\beta$. Let $Y_j$ and $\bar{Y}_j$, $2 \leqq j \leqq m$, be the Jacobi fields described previously. Then $Y_j(L)$, $\bar{Y}_j(L) \in T_xM$. Determine $m - 1 \times m - 1$ matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$ by

$$Y_i(L) = \sum_{j=2}^m a_{ij} e_j , \qquad Y_i'(L) = \sum_{j=2}^m c_{ij} e_j ,$$
$$\bar{Y}_i(L) = \sum_{j=2}^m b_{ij} e_j , \qquad \bar{Y}_i'(L) = \sum_{j=2}^m d_{ij} e_j .$$

Then $P(\gamma)$ has the following $2m - 2 \times 2m - 2$ matrix relative to $\beta$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

This concludes the appendix.

THE INSTITUTE FOR ADVANCED STUDY, PRINCETON, N.J.
YALE UNIVERSITY, NEW HAVEN, CONNECTICUT

REFERENCES

[1] J. ARTHUR, Harmonic analysis of tempered distributions on semisimple Lie groups of real rank one, thesis, Yale, 1970.
[2] M.F. ATIYAH, R. BOTT and V.K. PATODI, On the heat equation and the index theorem, Invent. Math. **19** (1973), 279-330.
[3] M.F. ATIYAH, V.K. PATODI and I.M. SINGER, Spectral asymmetry and Riemannian geometry I, Proc. Camb. Phil. Soc. **77** (1975), 43-69.
[4] M.F. ATIYAH and I.M. SINGER, The index of elliptic operators III, Ann. of Math. **87** (1968), 546-604.
[5] L. BÉRARD-BERGERY, Sur les longeurs des géodésiques périodiques et le spectre des formes d'espace hyperbolique compactes, *Séminaire de Géométrie Riemannienne*, 1970-1972, 84-126.
[6] A. BOREL and J-P. SERRE, Corners and arithmetic groups, Comm. Math. Helv. **48**, fasc. 4 (1973), 436-491.
[7] J. CHAZARAIN, Formule de Poisson pour les variétés Riemanniennes, Inven. Math. **24**, fasc. 1 (1974), 65-82.
[8] J. CHEEGER and D. EBIN, *Comparison Theorems in Riemannian Geometry*, North Holland Publishing Co., 1975.
[9] J.J. DUISTERMAAT and V. GUILLEMIN, The spectrum of positive elliptic operators and periodic geodesics, *Proc. Symp. Pure Math.*, A M.S. **27**, part II (1975), 205-210.
[10] M. GAFFNEY, Asymptotic distributions associated with the Laplacian for forms, Comm. Pure and Appl. Math. vol. XI (1958), 535-545.
[11] ———, The heat equation method of Milgram and Rosenbloom for open Riemannian manifolds, Ann. of Math. **60** (1954), 458-466.
[12] R. GODEMENT, A theory of spherical functions, I, Trans. A.M.S. **73** (1952), 496-556.
[13] HARISH-CHANDRA, Representations of semi-simple Lie groups, II, Trans. A.M.S. **76** (1954), 26-65.
[14] ———, Invariant eigendistributions on a semi-simple Lie group, Trans. A.M.S. **119** (1965), 457-508.
[15] S. HELGASON, *Differential Geometry and Symmetric Space*, Academic Press, New York, 1962.
[16] F. HIRZEBRUCH, Hilbert modular surfaces, L' Enseignement Math. **29** (1973), 183-281.
[17] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry*. vol. 1, Interscience, 1963.
[18] S. LANG, $SL_2(\mathbf{R})$, Addison-Wesley, 1975.
[19] G.A. MARGULIS, Applications of ergodic theory to the investigation of manifolds of negative curvature, J. Funct. Anal. and Appl. **3** (1969), 335-336.
[20] Y. MATSUSHIMA and S. MURAKAMI, On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds, Ann. of Math. **78** (1963), 417-449.
[21] D.B. RAY and I.M. SINGER, Analytic torsion for complex manifolds, Ann. of Math. (2) **98** (1973), 154-177.
[22] A. SELBERG, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. **20** (1956), 47-87.
[23] H. SAMELSON, *Notes on Lie Algebras*, Van Nostrand-Reinhold, 1969.
[24] I.M. SINGER, Eigenvalues of the Laplacian and invariants of manifolds, Proc. Internat. Cong. Math. Vancouver, B.C. 1974, 187-200.
[25] G. VAN DIJK, Spherical functions on the $\mathscr{P}$-adic group PGL(2), Proc. Koninklijke Nederlandse Akademie van Wetenschappen, Series A, **72** (1969), 213-225.
[26] Y. COLIN DE VERDIERE, Spectre du Laplacien et longeurs des géodésignes périodiques, C.R. Acad. Sci. Paris, Sér. A-B, **275** (1972), A805-A808.
[27] N. WALLACH, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, Inc., N.Y. 1973.

[28] N. WALLACH, On the Selberg trace formula in the case of compact quotient, Bull. A.M.S. **82**, No. 2 (1976), 171–195.
[29] G. WARNER, *Harmonic Analysis on Semi-Simple Lie Groups*, II, Springer-Verlag, New York, 1972.

(Received October 4, 1976)
(Revised August 10, 1977)