THE RELATIONS AMONG INVARIANTS OF POINTS ON THE PROJECTIVE LINE

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ABSTRACT. Nous considérons l'anneau des invariants de n points ordonnés sur la droite projective. L'espace $(\mathbb{P}^1)^n//\mathrm{PGL}_2$ est peut-être le premier exemple intéressant d'un quotient GIT. La construction dépend du choix des poids pour les n points. En 1894, Kempe [K] a introduit un ensemble de générateurs (au moins dans le cas où tous les poids sont constants egaux à 1). Ici, nous décrivons les rélations entre les générateurs pour tous les choix possibles de poids. En un sens il n'y a qu'une rélation, qui est quadrique sauf dans le cas classique de la cubique de Segre, i.e. lorsque $n = 6$ et les poids sont 1^6. Pour n plus petit ou egal a 6, la géométrie est classique. Le cas $n = 8$ est plus riche encore et est developpé dans cet article.

1. VERSION FRANÇAISE ABRÉGÉE

Il est bien connu que la géométrie des invariants de n points ordonnés, pour $n \leq 6$, est très riche. Nous nous demandons s'il existe une structure similaire pour plus de points. Le cas de 8 points, qui s'avère être plus riche encore, peut être décrit en utilisant des constructions classique. Dans le cas général de n points , avec des poids arbitraires, nous décrivons les générateurs de l'idéal des relations en termes d'une algèbre graphique (? je ne sais pas si c'est la bonne terminologie). En un certain sens, il y a une seule relation si l'on tient compte des symétries. Quand tous les poids sont egaux à 1 et n est pair, il y a exactement une seule relation en plus des symétries. En général, la rupture de symétrie donne d'autre équations. La cubique de Segre est trompeuse: dans tous les autres cas, les relations sont quadriques. La preuve utilise la structure exceptionnelle de $n = 8$ comme point de depart dune recurrence, puis s'appuie sur la dégénérescence torique de Speyer-Sturmfels et la thorie de la représentation de S_n.

2. INTRODUCTION

We consider the ring of invariants of n points on the projective line, and the GIT quotient $(\mathbb{P}^1)^n//\mathrm{PGL}_2$. The quotient depends on a choice of n weights $\vec{w} := (w_1, \ldots, w_n) \in (\mathbb{Z}^+)^n$. The quotient is given by

$$(\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n//_{\vec{w}}\mathrm{PGL}_2 := \mathrm{Proj}\left(\bigoplus_k R_{k\vec{w}}\right)$$

where $R_{\vec{v}} = \Gamma((\mathbb{P}^1)^n, \mathcal{O}(v_1, \ldots, v_n))^{\mathrm{PGL}_2}$. Small cases ($n \leq 6$) yield familiar beautiful geometry, and we refer in particular to [DO] for a masterful discussion, including of the history. In these examples, we take all weights to be 1, and work over a field. With sufficient care, this entire discussion applies over $\mathbb{Z}$.

The case $n = 4$ gives the cross ratio $(\mathbb{P}^1)^4 \dashrightarrow \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. From the perspective of this project, the cross ratio is best understood as a rational map not to $\mathbb{P}^1$, but to the line in $\mathbb{P}^2$ where the three coordinates sum to 0; the S_4-action is more transparent in this way, as we will see shortly.

The case $n = 5$ yields the quintic del Pezzo surface $(\mathbb{P}^1)^5 \dashrightarrow \overline{\mathcal{M}}_{0,5} \hookrightarrow \mathbb{P}^5$. The map is given by the degree 2 invariants; there are no degree 1 invariants. The quotient is cut out by 5 quadrics.

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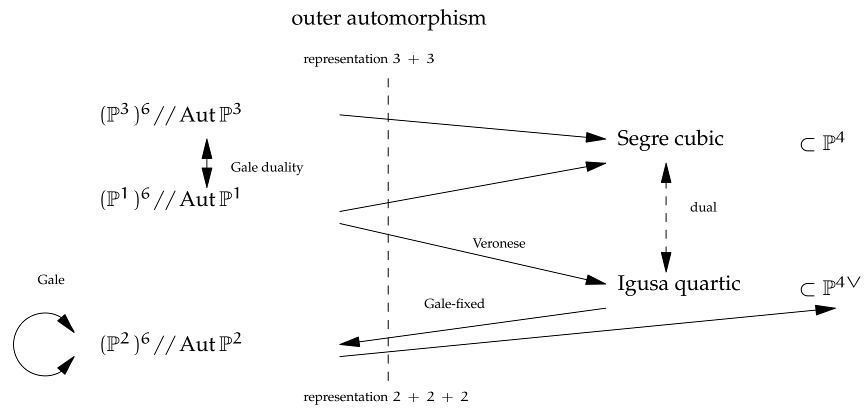

FIGURE 1. The classical geometry of six points in projective space

The case $n = 6$ is particularly beautiful, see Figure 1. The ring is generated in degree 1, and this piece has dimension 5, so the quotient threefold, the *Segre cubic*, is naturally a hypersurface in $\mathbb{P}^4$. The equations (for characteristic not 3) are cleanest written in six variables $x_1, \ldots, x_6$:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0.$$

(This description was given by Joubert in 1867, and in 1911 Coble gave an invariant-theoretical interpretation of the Joubert identities [C1], see also Coble's book [C2, Ch. III].) There are two obvious S_6 actions on the invariants, the first via permutations of the points, and the second by permutations of the variables x_i. One might hope that this would arise from a correspondence between the points and the variables, but the actions are related by the outer automorphism of S_6.

The quotient $(\mathbb{P}^3)^6 // \operatorname{Aut}(\mathbb{P}^3)$ is canonically isomorphic to $(\mathbb{P}^1)^6 // \operatorname{Aut}(\mathbb{P}^1)$, via the Gale transform. The quotient $(\mathbb{P}^2)^6 // \operatorname{Aut}(\mathbb{P}^2)$ is a double cover of $\mathbb{P}^4$, branched over the Igusa quartic hypersurface, given by the equations

$$w_1 + \cdots + w_6 = 4(w_1^4 + \cdots + w_6^4) - (w_1^2 + \cdots + w_6^2)^2 = 0.$$

The points and the variables are again related by the outer automorphism of S_6. Gale duality exchanges the two sheets of the double cover, and thus the Gale-fixed points, where the six points lie on a conic, correspond to the Igusa quartic. The Igusa quartic is dual (in the sense of classical projective geometry) to the Segre cubic, and the map on moduli sends six points on $\mathbb{P}^1$ to six points on the conic.

Our motivating question is the following: *is there similarly rich structure in the case of more points*? In §3, we describe a structure for eight points parallel to, and in some sense generalizing, that of six points. Further discussion (and proofs) will be given in [HMSV2]. Although the results are geometric, the proofs are essentially representation theory. In §4, we describe the generators for the ideal of relations for n points in general. They are "inherited" from the $n = 8$ case. This completes the program initiated in [HMSV1], and details and proofs will be given in [HMSV3].

3. EIGHT POINTS

The quotient $M_8 := (\mathbb{P}^1)^8 // \operatorname{Aut}(\mathbb{P}^1)$ naturally lies in $\mathbb{P}^{13}$. (The graded ring is generated in degree 1, and its degree 1 piece has rank 14.) A number of authors have shown by computer calculation that the ideal of relations in characteristic 0 is generated by 14 quadrics (Koike, Kondo, Freitag, Salvati Manni, Maclagan, ...). We describe the structure of the quadrics more directly, motivated by a suggestion of Dolgachev. See Figure 2.

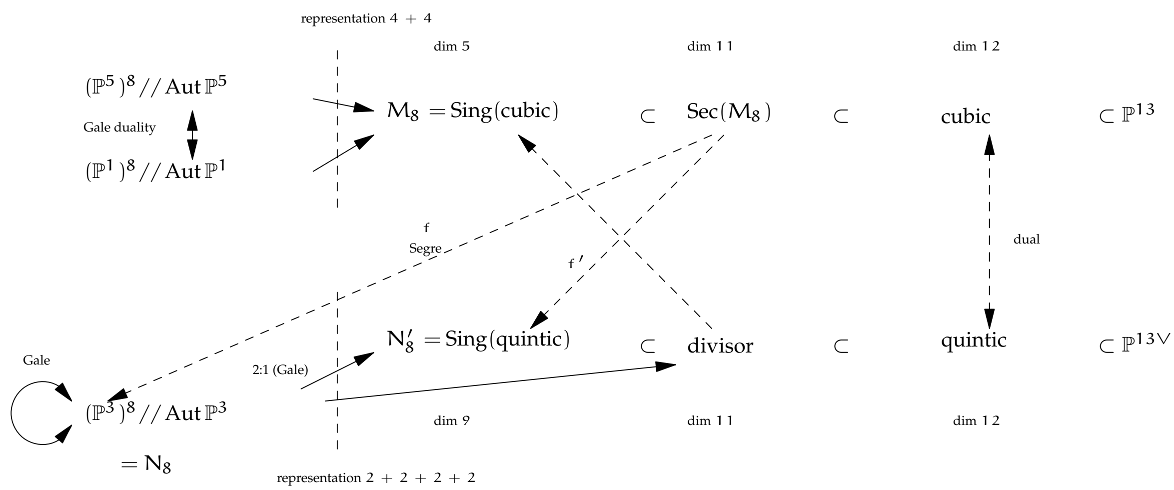

FIGURE 2. Relations among moduli spaces of eight points in projective space

The symmetric group acts on the graded ring. There is a unique skew-invariant cubic, and it lies in the ideal of M_8 because it is divisible by the Vandermonde polynomial upon pullback to $(\mathbb{P}^1)^8$. (Sam Grushevsky and Riccardo Salvati Manni have pointed out to us that this fact that it contains M_8 readily follows in the language of theta functions, see [FSM, §5].) In fact more is true: M_8 is the singular locus of the skew cubic, and furthermore the cone over M_8 is scheme-theoretically the singular locus of affine cone of the cubic. Thus the 14 quadrics are the 14 partial derivatives of the skew cubic (and in characteristic 3, the cubic itself is a necessary generator of the ideal). We emphasize that no computer verification is required.

The dual to the skew cubic has surprisingly low degree — it is a skew quintic in $\mathbb{P}^{13}$, whose singular locus N_8' has dimension 9. The moduli space N_8 of 8 points in $\mathbb{P}^3$ is a double cover of this singular locus. The sheets of the double cover are exchanged by Gale-duality, which sends 8 points in $\mathbb{P}^3$ to 8 points in $\mathbb{P}^3$ (both up to projective equivalence).

By Bezout's theorem, the secant variety to M_8 is contained in the skew cubic; it is a divisor. The duality birational map f from the cubic to the quintic blows down this divisor: given a secant line to M_8 meeting M_8 at points p and q, the 14 quadrics (which give the duality map) vanish at p and q, and thus are scalar multiples of each other. Hence we have identified one of the two dimensions of the $\operatorname{Sec} M_8$ contracted by f. We now describe the other dimension contracted, by lifting $f : \operatorname{Sec}(M_8) \dashrightarrow N_8'$ to $\operatorname{Sec}(M_8) \dashrightarrow N_8$. Fix two distinct points of M_8, and hence a secant line. From this data of a pair of octuples of points on $\mathbb{P}^1$, we obtain an octuple of points on $\mathbb{P}^1 \times \mathbb{P}^1$, which via the Segre embedding yields 8 points in $\mathbb{P}^3$ (up to projective equivalence). Conversely, given 8 general points in $\mathbb{P}^3$, we can find a smooth quadric through them (indeed, a one-parameter of smooth quadrics), yielding an unordered pair of octuples of points on $\mathbb{P}^1$. (In fact this construction is used to show that the skew quintic is dual to the skew cubic.)

Even more structure may be present. The trisecant variety to N_8' is contained in the skew quintic, and the quadrisecant variety to N_8' is contained in the divisor. A dimension estimate shows that this is quite unusual; both should easily fill out the full $\mathbb{P}^{13\vee}$. One might expect that both containments are equality, although we have not yet shown this.

This generalizes the $n = 6$ story in a number of ways. For example, the analogue of the cubic for $n = 8$ is the Segre cubic for $n = 6$. (The analogue for $n = 4$ is the union of the three boundary points.) Also, the similarity between Figures 1 and 2 is not coincidental: one can describe a fibration over the constructions of Figure 1 in the boundary of Figure 2, commuting with all dualities. For example, if we consider octuples of points in $\mathbb{P}^3$ where the two points coincide, projecting from those two points yields six points in $\mathbb{P}^2$, and the Gale dualities in the two figures correspond.

4. The general case

We describe the invariants in terms of a *graphical algebra*. To a directed graph Γ (with no loops) on n ordered vertices (in bijection with the n points), we associate

$$\prod_{\vec{ab}\in\Gamma}(x_a y_b - y_a x_b),$$

an invariant element of $\mathcal{O}(\vec{v})$, where $\vec{v}$ is the n-tuple of valences of the vertices. The degree $\vec{w}$ invariants are generated (as a vector space or module) by these elements.

This description can be used to show that the ring of invariants for any $\vec{w}$ is generated in degree 1. In the unit weight case, this is Kempe's Theorem [K].

Remark. Weyl's theorems on rings of invariants for a group representation in [W] are of the form: *First Main Theorem:* describe generators of the ring of invariants. *Second Main Theorem:* describe relations among the generators. One of his main results is for the symplectic group acting diagonally on the direct sum of n copies of the standard representation ([W, Thm. 6.1.A and 6.1.B] are the first and second main theorems respectively); for the case of SL_2, we obtain the affine cone over the Grassmannian $Gr(2,n)$. The first main theorem gave generators for the invariants, the Plücker coordinates. The second main theorem gave the relations, the Plücker relations. Kempe [K] proved the first main theorem when the direct sum is replaced by the tensor product. Theorem 4.1 (or more correctly the main theorem of [HMSV3]) is the second main theorem.

We make a series of observations about this graphical algebra.

Multiplication. Multiplication of (elements associated to) graphs is by superposition. (See for example Figure 3(a). The vertex labels 1 through 4 are omitted for simplicity. In later figures, even the vertices will be left implicit.)

Sign (linear) relations. Changing the orientation of a single edge changes the sign of the invariant (e.g. Figure 3(b)).

Plücker (linear) relation. Direct calculation shows the relation of Figure 3(c).

Bigger relations from smaller ones. The "four-point" Plücker relation immediately "extends" to relations among more points, e.g. Figure 3(d) for 6 points. Any relation may be extended in this way. For example, the sign relation in general should be seen as an extension of the two-point sign relation.

Remark. The sign and (extended) Plücker relations generate all the linear relations. They can be used to show that the "non-crossing graphs" (with no pairs of edges crossing) with all edges oriented "upwards" (the arrow points toward the higher-numbered label) form a basis — the graphical form of the straightening algorithm. But breaking symmetry obscures the structure of the ring.

The Segre cubic. The relation of Figure 3(e) is patently true: the superposition of the three graphs on the left is the same as that of the three graphs on the right. This is a cubic relation on the six point space. It turns out to be nonzero, and is thus necessarily the Segre cubic relation. Of course, all that matters about the orientations of the edges is that they are the same on the both sides of the equation.

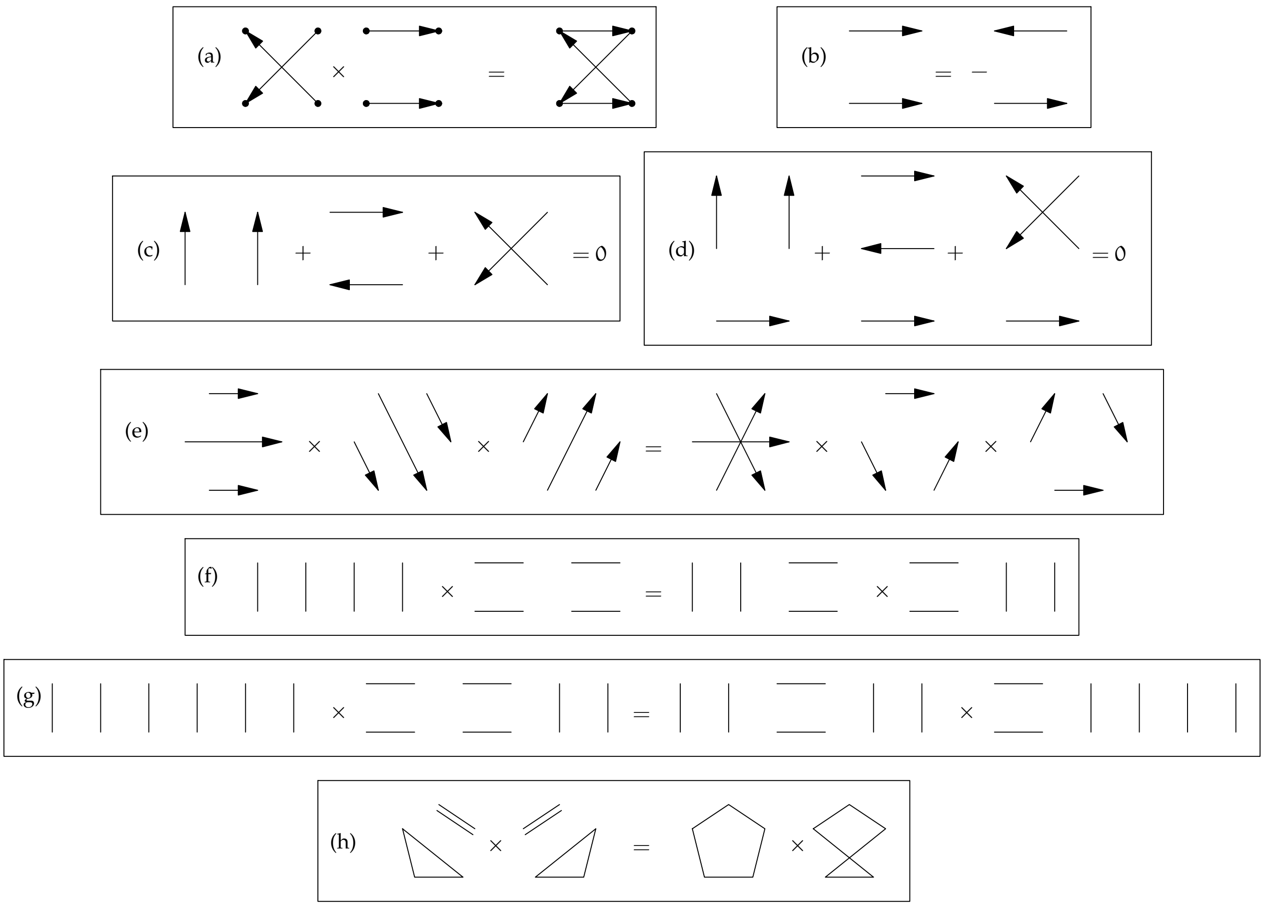

FIGURE 3. Relations in the graphical algebra

The skew cubic on eight points. One can describe the Segre cubic differently: if Γ is any 1-regular graph, consider $\sum_{\sigma \in S_6} \operatorname{sgn}(\sigma)\sigma(\Gamma^3)$. Replacing 6 by 8 yields the skew cubic of §3. Replacing 6 by (even) $n > 8$ yields 0, but there are still nontrivial analogues for arbitrary n.

A simple (binomial) quadric on eight points. Figure 3(f) gives an obvious relation on 8 points. The arrowheads are omitted for simplicity; they should be chosen consistently on both sides, as in Figure 3(e).

Simple quadrics for at least eight points are obtained by "extending" the eight-point relations, e.g. Figure 3(g) is the extension to 12 points, where the same two edges are added to each graph in Figure 3(f).

We may now state the main theorem of [HMSV3], in a special case.

4.1. *Main Theorem for the n even "unit weight" case $\vec{w} = 1^n$. — If $n \neq 6$, the simple quadrics (i.e. the S_n-orbit of the quadric above) generate the ideal of relations.*

By [HMSV1, Thm. 1.2], the arbitrary weight case readily reduces to the "unit weight" case $\vec{w} = 1^n$ (n even), so this solves the problem for arbitrary weight. As an example, an explicit description of the quadrics in the del Pezzo case of five points are as the five rotations of the patently true relation in Figure 3(h).

Themes of proof: (i) Use symmetry when necessary. (ii) Break symmetry when necessary. (iii) The eight point case is the base of the induction.

More precisely, we use a construction of Speyer-Sturmfels [SS] to degenerate the quotient into a toric variety, and we show the toric variety is cut out in degree two and three. We identify the cubics, and lift them to explicit cubic relations for the original (quotient) variety. We show by representation theory (and elementary combinatorics) that they lie in the ideal cut out by the quadrics. Then we use representation theory to see that the quadrics are generated as a module by the simple quadrics defined above.

The degree 1 part R_1 of the ring carries the irreducible S_n representation corresponding to $n/2 + n/2$. A key fact is the following: $\operatorname{Sym}^2 R_1$ is multiplicity free, and consists of irreducibles corresponding to all partitions with at most four parts, all even. R_2 corresponds to those partitions with at most three parts (all even). Thus the degree 2 part of the ideal corresponds to partitions with precisely four parts, all even. From this perspective, the $n = 8$ case is clearly special. Less obviously, even the form of the simple binomial quadrics is suggested by the representation theory.

Conclusion. We have thus answered our motivating question: there is sufficient structure in the general case that we can describe (generators of) the relations completely. The structure is inherited from the case of $n = 8$, where it is a consequence of exceptional geometry.

Future prospects. In our situation, we have a family of graded rings where relations in one case extend to relations in larger cases. One might hope that analogues of our results hold in a more general abstract setting. The third author has a precise algebraic conjecture implying this in a wide variety of circumstances generalizing this instance. In the particular case of n points in $\mathbb{P}^1$, this "coherence conjecture" would imply that the relations "stabilize" after a certain number of points, but would not predict an n for which it would happen.

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