CASIMIR OPERATORS AND MONODROMY REPRESENTATIONS OF GENERALISED BRAID GROUPS

JOHN J. MILLSON AND VALERIO TOLEDANO LAREDO

Abstract. Let $\mathfrak{g}$ be a complex, simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Weyl group W. We construct a one–parameter family of flat connections ∇_κ on $\mathfrak{h}$ with values in any finite–dimensional $\mathfrak{g}$–module V and simple poles on the root hyperplanes. The corresponding monodromy representation of the braid group $B_\mathfrak{g}$ of type $\mathfrak{g}$ is a deformation of the action of (a finite extension of) W on V. The residues of ∇_κ are the Casimirs κ_α of the subalgebras $\mathfrak{sl}_2^\alpha \subset \mathfrak{g}$ corresponding to the roots of $\mathfrak{g}$. The irreducibility of a subspace $U \subseteq V$ under the κ_α implies that, for generic values of the parameter, the braid group $B_\mathfrak{g}$ acts irreducibly on U. Answering a question of Knutson and Procesi, we show that these Casimirs act irreducibly on the weight spaces of all simple $\mathfrak{g}$–modules if $\mathfrak{g} = \mathfrak{sl}_3$ but that this is not the case if $\mathfrak{g} \not\cong \mathfrak{sl}_2, \mathfrak{sl}_3$. We use this to disprove a conjecture of Kwon and Lusztig stating the irreducibility of quantum Weyl group actions of Artin's braid group B_n on the zero weight spaces of all simple $U_\hbar\mathfrak{sl}_n$–modules for $n \geq 4$. Finally, we study the irreducibility of the action of the Casimirs on the zero weight spaces of self–dual $\mathfrak{g}$–modules and obtain complete classification results for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g}_2$ and conjecturally complete results for $\mathfrak{g}$ orthogonal or symplectic.

Contents

Date: May 2003.

The work of J.J.M. was partially supported by NSF grants DMS-98-03520 and 01-04006.

The work of V.T.L. was partially supported by an MSRI postdoctoral fellowship for the academic year 2000–2001.

1. Introduction

It has been known since the seminal work of Knizhnik and Zamolodchikov how to construct representations of Artin's braid groups B_n by using the representation theory of a given complex, semi–simple Lie algebra $\mathfrak{g}$ [KZ]. Realising B_n as the fundamental group of the quotient of the configuration space

$$X_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | \, z_i \neq z_j, \, 1 \leq i < j \leq n\} \tag{1.1}$$

by the natural action of the symmetric group $\mathfrak{S}_n$, one obtains these representations as the monodromy of the Knizhnik–Zamolodchikov connection

$$\nabla_{\text{KZ}} = d - h \sum_{1 \leq i < j \leq n} \frac{d(z_i - z_j)}{z_i - z_j} \cdot \Omega_{ij} \tag{1.2}$$

with values in the n–fold tensor product $V^{\otimes n}$ of a finite–dimensional $\mathfrak{g}$–module V. Here, the one–form ∇_{KZ} is regarded as an $\mathfrak{S}_n$–equivariant flat connection on the topologically trivial vector bundle over X_n with fibre $V^{\otimes n}$ and then pushed down to $X_n/\mathfrak{S}_n$. Its coefficients $\Omega_{ij} \in \text{End}(V^{\otimes n})$ are given by

$$\Omega_{ij} = \sum_{a=1}^{\dim \mathfrak{g}} \pi_i(X_a) \pi_j(X^a) \tag{1.3}$$

where $\{X_a\}, \{X^a\}$ are dual basis of $\mathfrak{g}$ with respect to the Killing form and $\pi_k(\cdot)$ denotes the action on the kth tensor factor of $V^{\otimes n}$. Finally, the complex number h may be regarded as a deformation parameter which, upon being set to 0, gives a monodromy representation of B_n factoring through the natural action of the symmetric group on $V^{\otimes n}$.

Aside from their intrinsic interest, these representations appear naturally in a number of different contexts. They define for example the commutativity and associativity constraints in the tensor category of highest weight representations of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ [KL, Wa, TL1] and, by the Kohno–Drinfeld theorem, on the finite–dimensional representations of the quantum group $U_\hbar \mathfrak{g}$ [Dr3, Dr4, Ko1]. As such, they define invariants of knots and links and, for suitable rational values of h, of three–manifolds [Tu].

The purpose of the present paper is to use the representation theory of $\mathfrak{g}$ in a similar vein to construct monodromy representations of a different braid group, namely the generalised braid group $B_\mathfrak{g}$ of type $\mathfrak{g}$. The latter may be defined as the fundamental group of the quotient $\mathfrak{h}_{\text{reg}}/W$ of the set $\mathfrak{h}_{\text{reg}}$ of regular elements in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by the action of the corresponding Weyl group W. Like Artin's braid groups, $B_\mathfrak{g}$ is presented on generators $T_1, \ldots, T_n$ labelled by a choice $\alpha_1, \ldots, \alpha_n$ of simple roots of $\mathfrak{g}$ with relations

$$T_i T_j \cdots = T_j T_i \cdots \tag{1.4}$$

for any $i \neq j$, where the number of factors on each side is equal to the order of the product $s_i s_j$ of the orthogonal reflections corresponding to α_i and α_j in W [Br].

To state our first main result, let $R = \{\alpha\} \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ so that $\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha\in R}\text{Ker}(\alpha)$. For each $\alpha \in R$, let $\mathfrak{sl}_2^\alpha = \langle e_\alpha, f_\alpha, h_\alpha\rangle \subseteq \mathfrak{g}$ be the corresponding $\mathfrak{sl}_2(\mathbb{C})$–subalgebra of $\mathfrak{g}$ and

$$\kappa_\alpha = \frac{\langle\alpha,\alpha\rangle}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha) \tag{1.5}$$

the truncated Casimir operator of $\mathfrak{sl}_2^\alpha$ where $\langle\cdot,\cdot\rangle$ is a fixed multiple of the Killing form of $\mathfrak{g}$. Let V be a $\mathfrak{g}$–module, then we prove in section 2 the following[1]

Theorem 1.1. *The one–form*

$$\nabla_\kappa = d - h\sum_{\alpha\in R}\frac{d\alpha}{\alpha}\cdot\kappa_\alpha \tag{1.6}$$

defines, for any $h \in \mathbb{C}$, a flat connection on the topologically trivial bundle over $\mathfrak{h}_{\text{reg}}$ with fibre V which is reducible with respect to the weight space decomposition of V.

As a consequence, each weight space of V carries a canonical one–parameter family of monodromy representations of the *pure* braid group $P_\mathfrak{g} = \pi_1(\mathfrak{h}_{\text{reg}})$. This action extends to one of the full braid group $B_\mathfrak{g}$ on the direct sum of weight spaces corresponding to a given Weyl group orbit, and in particular on the zero weight space of V, by pushing ∇_κ down to the quotient space $\mathfrak{h}_{\text{reg}}/W$. Since the Weyl group itself does not act on V, this requires choosing an action of $B_\mathfrak{g}$ on V which permutes the weight spaces compatibly with the projection $B_\mathfrak{g} \to W$. This may for example be achieved by taking the simply–connected complex Lie group G corresponding to $\mathfrak{g}$ and mapping $B_\mathfrak{g}$ to one of the Tits extension $\widetilde{W}$ of W, a class of subgroups of the normaliser in G of the torus T corresponding to $\mathfrak{h}$ which are extensions of W by the sign group $\mathbb{Z}_2^n$, where $n = \dim(\mathfrak{h})$ [Ti]. The choice of a specific Tits extension is somewhat immaterial since any two are conjugate by an element of T and the corresponding representations of $B_\mathfrak{g}$ are therefore equivalent.

The rest of the paper is devoted to the study of the irreducibility of our monodromy representations. Define a subspace $U \subseteq V$ invariant under the monodromy action of $B_\mathfrak{g}$ to be *generically irreducible* if is irreducible for all values of h lying outside the zero set of some holomorphic function. In section 3, we prove the following

[1]Theorem 1.1 was independently discovered by De Concini around 1995 (unpublished). A variant of the connection ∇_κ also appears in the recent paper [FMTV]. Unlike ∇_κ however, the connection introduced in [FMTV] is not W–equivariant and therefore only defines representations of the *pure* braid group $P_\mathfrak{g}$ instead of the full braid group $B_\mathfrak{g}$

Theorem 1.2. *A subspace $U \subseteq V$ is generically irreducible under the braid group $B_{\mathfrak{g}}$ (resp. the pure braid group $P_{\mathfrak{g}}$) if, and only if it is irreducibly acted upon by the Casimirs κ_α and $\widetilde{W}$ (resp. the κ_α and $\widetilde{W} \cap T$).*

This naturally prompts the question, originally asked us by C. Procesi and A. Knutson, of whether the *Casimir algebra i.e.,* the algebra

$$\mathcal{C}_{\mathfrak{g}} = \langle \kappa_\alpha \rangle_{\alpha \in R} \vee \mathfrak{h} \subset U\mathfrak{g} \tag{1.7}$$

generated by the Casimirs κ_α and $\mathfrak{h}$ inside the enveloping algebra of $\mathfrak{g}$ acts irreducibly on the weight spaces of any simple $\mathfrak{g}$–module or, stronger still, whether it is equal to the algebra $U\mathfrak{g}^{\mathfrak{h}}$ of $\mathfrak{h}$–invariants in $U\mathfrak{g}$.

The answer to both questions is clearly positive for $\mathfrak{g} = \mathfrak{sl}_2$ and we show in section 4 that it this almost so for $\mathfrak{g} = \mathfrak{sl}_3$. More precisely,

Theorem 1.3. *If $\mathfrak{g} = \mathfrak{sl}_3$, $\mathcal{C}_{\mathfrak{g}}$ is a proper subalgebra of $U\mathfrak{g}^{\mathfrak{h}}$, but the latter is generated by $\mathcal{C}_{\mathfrak{g}}$ and the centre $Z(U\mathfrak{g})$ of $U\mathfrak{g}$. In particular, $\mathcal{C}_{\mathfrak{g}}$ acts irreducibly on the weight spaces of any simple $\mathfrak{g}$–module.*

As a consequence, all monodromy representations of P_3 on weight spaces of simple $\mathfrak{sl}_3$–modules, and of B_3 on their zero weight spaces are generically irreducible, a fact which refines a result proved by Kwon [Kw] in the context of quantum Weyl groups, and to which we shall return below. For $\mathfrak{g} \not\cong \mathfrak{sl}_2, \mathfrak{sl}_3$, the situation is radically different and we prove

Theorem 1.4. *If $\mathfrak{g} \not\cong \mathfrak{sl}_2, \mathfrak{sl}_3$, there exists a simple $\mathfrak{g}$–module V the zero weight space of which is reducible under the joint action of $\mathcal{C}_{\mathfrak{g}}$ and of W. In particular, $\mathcal{C}_{\mathfrak{g}}$ and $Z(U\mathfrak{g})$ do not generate $U\mathfrak{g}^{\mathfrak{h}}$.*

For $\mathfrak{g} \not\cong \mathfrak{sl}_n$, our V is in fact the kernel of the commutator map $[\cdot,\cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$. For $\mathfrak{g} \cong \mathfrak{sl}_n$, $\mathrm{Ker}([\cdot,\cdot])$ is reducible and the construction of a suitable V relies on the following general reducibility criterion, valid for any $\mathfrak{g}$. Let V be a simple $\mathfrak{g}$–module with zero weight space $V[0] \neq \{0\}$. If V is self–dual, it is acted on by a linear involution Θ_V such that, for any $X \in \mathfrak{g}$, $\Theta_V X \Theta_V^{-1} = \Theta(X)$ where Θ is the Chevalley involution of $\mathfrak{g}$ relative to a given choice of simple root vectors. Since Θ acts as -1 on $\mathfrak{h}$ and fixes the Casimirs κ_α and $\widetilde{W}$, Θ_V leaves $V[0]$ invariant and commutes with $\mathcal{C}_{\mathfrak{g}}$ and $\widetilde{W}$. $V[0]$ is therefore reducible under $\mathcal{C}_{\mathfrak{g}}$ and $\widetilde{W}$ whenever Θ_V does not act as a scalar on it.

To prove that this is the case for some V we note further that if $\mathfrak{r} \subset \mathfrak{g}$ is a reductive subalgebra normalised by Θ and V is such that its restriction to $\mathfrak{r}$ contains a zero–weight vector u lying in a simple $\mathfrak{r}$–summand U which isn't self–dual, then $\Theta_V u$ cannot be proportional to u since $U \cap \Theta_V U = \{0\}$. To summarise, our initial problem reduces to finding simple, self–dual $\mathfrak{g}$–modules V whose restriction to some reductive subalgebra $\mathfrak{r} \subset \mathfrak{g}$ contains non–self dual summands intersecting $V[0]$ non–trivially. For $\mathfrak{g} = \mathfrak{sl}_n$, we construct such V's by using the Gelfand–Zetlin branching rules for the inclusion $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$ [GZ1].

In section 5, we use our results to disprove a conjecture of Kwon and Lusztig on quantum Weyl group actions of the braid group B_n [Kw]. To state it, recall that the Drinfeld–Jimbo quantum group $U_\hbar\mathfrak{g}$ corresponding to $\mathfrak{g}$ defines, on any of its integrable representations $\mathcal{V}$, an action of the braid group $B_\mathfrak{g}$ called the *quantum Weyl group action*, which is a deformation of the action of $\widetilde{W}$ on the $\mathfrak{g}$–module $V = \mathcal{V}/\hbar\mathcal{V}$ [Lu, KR, So]. In [Kw], Kwon considered the case of $\mathfrak{g} = \mathfrak{sl}_n$ and gave a necessary condition for the zero weight space of $\mathcal{V}$ to be irreducible under $B_\mathfrak{g} = B_n$. He showed in particular that the zero weight spaces of all $U_\hbar\mathfrak{sl}_3$–modules are irreducible under B_3. Based on these findings he and Lusztig conjectured that this should hold for all B_n, $n \geq 4$.

Theorem 1.5. *The Kwon–Lusztig conjecture is false for any simple, complex Lie algebra* $\mathfrak{g} \not\cong \mathfrak{sl}_2, \mathfrak{sl}_3$.

Our disproof is based on the simple observation that the *quantum* Chevalley involution $\Theta_\hbar$ of $U_\hbar\mathfrak{g}$ acts on any self–dual $U_\hbar\mathfrak{g}$–module $\mathcal{V}$ and that its restriction to the zero weight space $\mathcal{V}[0]$ centralises the action of $B_\mathfrak{g}$. We then remark that $\Theta_\hbar$ acts as a scalar on $\mathcal{V}[0]$ iff the classical Chevalley involution acts as a scalar on the zero weight space of the $\mathfrak{g}$–module $V = \mathcal{V}/\hbar\mathcal{V}$ and rely on the results of section 4.

In section 6 we show that, despite the reducibility results of §4, the connection ∇_κ yields none–the–less irreducible monodromy representations of $B_\mathfrak{g}$ of arbitrarily large dimensions. For $\mathfrak{g}$ classical, we show in fact that, with V the adjoint representation if $\mathfrak{g} \cong \mathfrak{sl}_n$ and the vector one otherwise, the weight spaces of all Cartan powers of V are irreducible under the the Casimirs κ_α.

Finally, in section 7, we show that, when $\mathfrak{g} \not\cong \mathfrak{sl}_2, \mathfrak{sl}_3$ is classical or $\mathfrak{g}_2$, the zero weight space of most *self–dual*, simple $\mathfrak{g}$–modules is reducible under the Casimir algebra $\mathcal{C}_\mathfrak{g}$ of $\mathfrak{g}$, thus strengthening the results of section 4. More precisely, let V be a simple, self–dual $\mathfrak{g}$–module with zero weight space $V[0] \neq \{0\}$ and highest weight $\lambda \neq 0$. Then, for $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_n$ or $\mathfrak{g}_2$, we obtain the following complete classification results

Theorem 1.6. *If* $\mathfrak{g} = \mathfrak{sl}_n$, $V[0]$ *is irreducible under* $\mathcal{C}_{\mathfrak{sl}_n}$ *if, and only if* λ *is of one of the following forms*

(i) $\lambda = (p, 0, \ldots, 0, -p)$, $p \in \mathbb{N}$.
(ii) $\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{k})$, $0 \leq k \leq n/2$.
(iii) $\lambda = (p, p, -p, -p)$, $p \in \mathbb{N}$.

Theorem 1.7. *If $\mathfrak{g}=\mathfrak{g}_2$, $V[0]$ is irreducible under $\mathcal{C}_{\mathfrak{g}_2}$ if, and only if V is fundamental representation or its second Cartan power.*

Our calculations rely on the use of the Chevalley involution Θ outlined above and branching to the subalgebras $\mathfrak{gl}_k\subset\mathfrak{sl}_n$ and $\mathfrak{sl}_3\subset\mathfrak{g}_2$ respectively. They show in fact that $V[0]$ is irreducible under $\mathcal{C}_{\mathfrak{g}}$ if, and only if Θ acts as a scalar on it. It seems natural to conjecture that this should be so for any $\mathfrak{g}$. For $\mathfrak{g}=\mathfrak{so}_{2n+1},\mathfrak{so}_{2n},\mathfrak{sp}_{2n}$ we proceed in a similar way by branching to the equal rank subalgebra $\mathfrak{gl}_n\subset\mathfrak{g}$. This leads to the following partial classification results.

Theorem 1.8. *If $\mathfrak{g}=\mathfrak{so}_m$, with $m=2n,2n+1$, $V[0]$ is irreducible under $\mathcal{C}_{\mathfrak{g}}$ if λ has one of the following forms,*

(i) $\lambda=(p,0,\ldots,0)$, $p\in\mathbb{N}$.
(ii) $\lambda=(2,2,0,\ldots,0)$.
(iii) $\lambda=(\underbrace{1,\ldots,1}_{k},0,\ldots,0)$, $1\leq k\leq n$.
(iv) $\lambda=(\underbrace{1,\ldots,1}_{n-1},-1)$.

Conversely, if λ is of none of the above forms and satisfies $\lambda_i=0$ for $i>n/2$, then $V[0]$ is reducible under $\mathcal{C}_{\mathfrak{g}}$.

Theorem 1.9. *If $\mathfrak{g}=\mathfrak{sp}_{2n}$, $V[0]$ is irreducible under $\mathcal{C}_{\mathfrak{g}}$ if λ is of one of the following forms,*

(i) $\lambda=(2p,0,\ldots,0)$, $p\in\mathbb{N}$.
(ii) $\lambda=(2,2,0,\ldots,0)$.
(iii) $\lambda=(\underbrace{1,\ldots,1}_{2k},0,\ldots,0)$, $1\leq k\leq n/4$.

Conversely, if λ is of none of the above forms and satisfies $\lambda_i=0$ for $i>n/2$, then $V[0]$ is reducible under $\mathcal{C}_{\mathfrak{g}}$.

We conjecture in fact that the restriction $\lambda_i=0$ can be removed in the statements of theorems 1.8–1.9.

It is interesting to note how the reducibility results of section 7 contrast with the following theorem of Etingof, which is reproduced with his kind permission in section 8. Let $\beta\in\sum_{i=1}^n\mathbb{N}\cdot\alpha_i$ be a positive linear combination of simple roots and, for $\mu\in\mathfrak{h}^*$, let $M_\mu[\mu-\beta]$ be the subspace of weight $\mu-\beta$ of the Verma module of highest weight μ.

Theorem 1.10 (Etingof)**.** *There exists a Zariski open set $\mathcal{O}_\beta\subset\mathfrak{h}^*$ such that, for any $\mu\in\mathcal{O}_\beta$, $M_\mu[\mu-\beta]$ is irreducible under the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$.*

The above theorem, used in conjunction with Knop's calculation of the centre of the subalgebra $U\mathfrak{g}^{\mathfrak{h}}$ of $\mathfrak{h}$–invariants [Kn], yields in fact the following interesting result, which is also given in §8

Theorem 1.11 (Etingof). *The centraliser of the Casimir algebra $\mathcal{C}_\mathfrak{g}$ in $U\mathfrak{g}$ is generated by $\mathfrak{h}$ and the centre of $U\mathfrak{g}$.*

Acknowledgements. We wish to heartily thank Allen Knutson whose observations on reading [KM] initiated this project. We are also grateful to R. Buchweitz, C. De Concini, P. Etingof, M. Kashiwara, B. Kostant, C. Laskowski, A. Okounkov, C. Procesi, R. Rouquier, S. Yuzvinksy and A. Wassermann for a number of useful discussions. The research for this project was partly carried out while the first author visited the Institut de Mathématiques de Jussieu in June 2000 and June 2001 and while the second author was a post–doctoral fellow at MSRI during the academic year 2000–2001. We are grateful to both institutions for their financial support and pleasant working conditions.

2. Flat connections on $\mathfrak{h}_{\text{reg}}$

2.1. **The flat connection ∇_κ.** Let $\mathfrak{g}$ be a complex, semi–simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system $R = \{\alpha\} \subset \mathfrak{h}^*$. Let

$$\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in R} \text{Ker}(\alpha) \tag{2.1}$$

be the set of regular elements in $\mathfrak{h}$ and V a finite–dimensional $\mathfrak{g}$–module. We shall presently define a flat connection on the trivial vector bundle $\mathfrak{h}_{\text{reg}} \times V$ over $\mathfrak{h}_{\text{reg}}$. We need for this purpose the following flatness criterion due to Kohno [Ko2]. Let B be a complex, finite–dimensional vector space and $\mathcal{A} = \{H_i\}_{i \in I}$ a finite collection of hyperplanes in B determined by the linear forms $\phi_i \in B^*$, $i \in I$.

Lemma 2.1. *Let V be a finite–dimensional vector space and $\{r_i\} \subset \text{End}(V)$ a family indexed by I. Then,*

$$\nabla = d - \sum_{i \in I} \frac{d\phi_i}{\phi_i} \cdot r_i \tag{2.2}$$

defines a flat connection on $(B \setminus \mathcal{A}) \times V$ iff, for any subset $J \subseteq I$ maximal for the property that $\bigcap_{j \in J} H_j$ is of codimension 2, the following relations hold for any $j \in J$

$$[r_j, \sum_{j' \in J} r_{j'}] = 0 \tag{2.3}$$

Remark. Since the relations (2.3) are homogeneous, a solution $\{r_i\}_{i \in I}$ of (2.3) defines in fact a *one–parameter* family of representations

$$\rho_h : \pi_1(B \setminus \mathcal{A}) \longrightarrow GL(V) \tag{2.4}$$

parametrised by $h \in \mathbb{C}$ where ρ_h is the monodromy of the connection (2.2) with r_i replaced by $h \cdot r_i$.

For any $\alpha \in R$, choose root vectors $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha = \alpha^\vee$ and let

$$\kappa_\alpha = \frac{\langle \alpha, \alpha \rangle}{2} (e_\alpha f_\alpha + f_\alpha e_\alpha) \in U\mathfrak{g} \tag{2.5}$$

be the truncated Casimir operator of the three–dimensional subalgebra $\mathfrak{sl}_2^\alpha \subset \mathfrak{g}$ spanned by $e_\alpha, h_\alpha, f_\alpha$ relative to the restriction to $\mathfrak{sl}_2^\alpha$ of a fixed multiple $\langle \cdot, \cdot \rangle$ of the Killing form of $\mathfrak{g}$. Note that κ_α does not depend upon the particular choice of e_α and f_α and that $\kappa_{-\alpha} = \kappa_\alpha$. Let $R^+ \subset R$ be the set of positive roots corresponding to a choice of simple roots $\alpha_1, \ldots, \alpha_n$ of $\mathfrak{g}$.

Theorem 2.2. *The one–form*

$$\nabla_\kappa = d - h \sum_{\alpha \in R^+} \frac{d\alpha}{\alpha} \cdot \kappa_\alpha = d - \frac{h}{2} \sum_{\alpha \in R} \frac{d\alpha}{\alpha} \cdot \kappa_\alpha \tag{2.6}$$

defines, for any $h \in \mathbb{C}$, a flat connection on $\mathfrak{h}_{\text{reg}} \times V$ which is reducible with respect to the weight space decomposition of V.

Proof. By lemma 2.1, we must show that for any rank 2 root subsystem $R_0 \subseteq R$ determined by the intersection of R with a 2–dimensional subspace in $\mathfrak{h}^*$, the following holds for any $\alpha \in R_0^+ = R_0 \cap R^+$

$$[\kappa_\alpha, \sum_{\beta \in R_0^+} \kappa_\beta] = 0 \tag{2.7}$$

This may be proved by an explicit computation by considering in turn the cases where R_0 is of type $A_1 \times A_1$, A_2, B_2 or G_2 but is more easily settled by the following elegant observation of A. Knutson [Kn]. Let $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be the semi–simple Lie algebra with root system R_0, $\mathfrak{h}_0 \subset \mathfrak{h}$ its Cartan subalgebra and $C_0 \in Z(U\mathfrak{g}_0)$ its Casimir operator. Then, $\sum_{\beta \in R_0^+} \kappa_\beta - C_0$ lies in $U\mathfrak{h}_0$ so that (2.7) holds since κ_α commutes with $\mathfrak{h}_0$. The reducibility of ∇_κ with respect to the $\mathfrak{h}$–action on V is an immediate consequence of the fact that the operators κ_α are of weight zero ■

Remark. Altough V admits a hermitian inner product with respect to which the Casimirs κ_α are self–adjoint, it is easy to check that the connection ∇_κ is not unitary with respect to the corresponding constant inner product on $\mathfrak{h}_{\text{reg}} \times V$. However, the fact that the connection ∇_κ for $\mathfrak{g} = \mathfrak{sl}_n$ coincides with the (genus 0) Knizhnik–Zamolodchikov connection on n points for $\mathfrak{g}' = \mathfrak{sl}_k$ via Howe duality [TL2, thm. 3.5], and that the latter is conjectured to be unitary on the subbundle of conformal blocks for suitable rational values of h [Ga] [2], suggests that the connection ∇_κ ought to be unitary for any $\mathfrak{g}$. It is an interesting open problem to determine whether this is so.

Let W be the Weyl group of $\mathfrak{g}$ and $P_\mathfrak{g} = \pi_1(\mathfrak{h}_{\text{reg}})$, $B_\mathfrak{g} = \pi_1(\mathfrak{h}_{\text{reg}}/W)$ the corresponding generalised pure and full braid groups of type $\mathfrak{g}$. The fibration

[2] this is now a theorem, at least for $\mathfrak{g} = \mathfrak{sl}_2$, see [Ra]

$\mathfrak{h}_{\text{reg}} \to \mathfrak{h}_{\text{reg}}/W$ gives rise to the exact sequence

$$1 \longrightarrow P_{\mathfrak{g}} \longrightarrow B_{\mathfrak{g}} \longrightarrow W \longrightarrow 1 \tag{2.8}$$

By theorem 2.2, the monodromy of ∇_κ yields a one–parameter family of representations of $P_{\mathfrak{g}}$ on V preserving its weight space decomposition. We wish to extend this action to one of $B_{\mathfrak{g}}$, by pushing ∇_κ down to a flat connection on the quotient $\mathfrak{h}_{\text{reg}}/W$. Since W does not act on V, this requires choosing an action of $B_{\mathfrak{g}}$ on V. Let for this purpose G be the complex, connected and simply–connected Lie group with Lie algebra $\mathfrak{g}$, T its torus with Lie algebra $\mathfrak{h}$ and $N(T) \subset G$ the normaliser of T so that $W \cong N(T)/T$. We regard $B_{\mathfrak{g}}$ as acting on V by choosing a homomorphism $\sigma : B_{\mathfrak{g}} \to N(T)$ compatible with

$$\begin{tikzcd} B_{\mathfrak{g}} \arrow[r, "\sigma"] \arrow[dr] & N(T) \arrow[d] \\ & W \end{tikzcd} \tag{2.9}$$

Such σ's abund and we describe in §2.5 a class of them which we call Tits extensions [Ti]. Let $\widetilde{\mathfrak{h}_{\text{reg}}} \xrightarrow{p} \mathfrak{h}_{\text{reg}}$ be the universal cover of $\mathfrak{h}_{\text{reg}}$ and $\mathfrak{h}_{\text{reg}}/W$.

Proposition 2.3. *The one–form $p^*\nabla_\kappa$ defines a $B_{\mathfrak{g}}$–equivariant flat connection on $\widetilde{\mathfrak{h}_{\text{reg}}} \times V = p^*(\mathfrak{h}_{\text{reg}} \times V)$. It therefore descends to a flat connection on the vector bundle*

$$\begin{tikzcd} V \arrow[r] & \widetilde{\mathfrak{h}_{\text{reg}}} \times_{B_{\mathfrak{g}}} V \arrow[d] \\ & \mathfrak{h}_{\text{reg}}/W \end{tikzcd} \tag{2.10}$$

which is reducible with respect to the weight space decomposition of V.

Proof. The action of $B_{\mathfrak{g}}$ on $\Omega^\bullet(\widetilde{\mathfrak{h}_{\text{reg}}}, V) = \Omega^\bullet(\widetilde{\mathfrak{h}_{\text{reg}}}) \otimes V$ is given by $\gamma \to (\gamma^{-1})^* \otimes \sigma(\gamma)$. Thus, if $\gamma \in B_{\mathfrak{g}}$ projects onto $w \in W$, we get using $p \cdot \gamma^{-1} = w^{-1} \cdot p$,

$$\gamma\, p^*\nabla_\kappa\, \gamma^{-1} = d - \frac{h}{2} \sum_{\alpha \in R} dp^* w\alpha / p^* w\alpha \otimes \sigma(\gamma)\kappa_\alpha \sigma(\gamma)^{-1} \tag{2.11}$$

Since κ_α is independent of the choice of the root vectors e_α, f_α in (2.5), $\text{Ad}(\sigma(\gamma))\kappa_\alpha = \kappa_{w\alpha}$ and (2.11) is equal to $p^*\nabla_\kappa$ as claimed. $p^*\nabla_\kappa$ is flat and commutes with the fibrewise action of $\mathfrak{h}$ by theorem 2.2 ■

Thus, for any homomorphism $\sigma : B_{\mathfrak{g}} \to N(T)$ compatible with (2.9), proposition 2.3 yields a one–parameter family of monodromy representations

$$\rho_h^\sigma : B_{\mathfrak{g}} \longrightarrow GL(V) \tag{2.12}$$

which permutes the weight spaces of V compatibly with the action of W on $\mathfrak{h}^*$. By standard ODE theory, ρ_h^σ depends analytically on the complex parameter h and, when $h=0$, is equal to the action of $B_{\mathfrak{g}}$ on V given by σ. We record for later use the following elementary

Proposition 2.4. *Let $\gamma\in B_{\mathfrak{g}}=\pi_1(\mathfrak{h}_{\text{reg}}/W)$ and $\widetilde{\gamma}:[0,1]\to\mathfrak{h}_{\text{reg}}$ be a lift of γ. Then,*

$$\rho_h^\sigma(\gamma)=\sigma(\gamma)\mathcal{P}(\widetilde{\gamma}) \tag{2.13}$$

where $\mathcal{P}(\widetilde{\gamma})\in GL(V)$ is the parallel transport along $\widetilde{\gamma}$ for the connection ∇_κ on $\mathfrak{h}_{\text{reg}}\times V$.

Proof. Let $\widetilde{\widetilde{\gamma}}:[0,1]\to\widetilde{\mathfrak{h}_{\text{reg}}}$ be a lift of γ and $\widetilde{\gamma}$ so that $\widetilde{\widetilde{\gamma}}(1)=\gamma^{-1}\widetilde{\widetilde{\gamma}}(0)$. Then, since the connection on $p^*(\mathfrak{h}_{\text{reg}}\times V)$ is the pull–back of ∇_κ, and that on $\left(p^*(\mathfrak{h}_{\text{reg}}\times V)\right)/B_{\mathfrak{g}}$ the quotient of $p^*\nabla_\kappa$, we find

$$\rho_h^\sigma(\gamma)=\mathcal{P}(\gamma)=\sigma(\gamma)\mathcal{P}(\widetilde{\widetilde{\gamma}})=\sigma(\gamma)\mathcal{P}(\widetilde{\gamma}) \tag{2.14}$$

■

Remark. By (2.13), the representation ρ_h^σ depends on the choice of the homomorphism $\sigma:B_{\mathfrak{g}}\to N(T)$ satisfying (2.9). We simply note here that since any two Tits extensions σ,σ' are conjugate by an element of T (see §2.5), the corresponding monodromy representations are equivalent. We note also that the restriction of ρ_h^σ to the zero weight space $V[0]$ of V does not depend on the choice of σ since $W\cong N(T)/T$ acts canonically on $V[0]$.

Remark. Note that, by (2.13), the restriction of ρ_h^σ to the pure braid group $P_{\mathfrak{g}}$ does not coincide with the monodromy of the connection ∇_κ. Rather, it differs from it by the T–valued character given by the restriction of σ to $P_{\mathfrak{g}}$.

Remark. By Brieskorn's theorem, $B_{\mathfrak{g}}$ is presented on generators $S_1,\ldots,S_n$ labelled by the simple simple reflections $s_1,\ldots,s_n\in W$ with relations

$$\underbrace{S_iS_j\cdots}_{m_{ij}}=\underbrace{S_jS_i\cdots}_{m_{ij}} \tag{2.15}$$

for any $1\leq i<j\leq n$ where the number m_{ij} of factors on each side is equal to the order of s_is_j in W [Br]. Each S_i may be obtained as a small loop in $\mathfrak{h}_{\text{reg}}/W$ around the reflecting hyperplane $\text{Ker}(\alpha_i)$ of s_i.

2.2. **Variants of ∇_κ.** If $p_\alpha\in U\mathfrak{h}$, $\alpha\in R$, is a collection of polynomials in $\mathfrak{h}$, the connection

$$d-\frac{h}{2}\sum_{\alpha\in R}\frac{d\alpha}{\alpha}\cdot(\kappa_\alpha+p_\alpha) \tag{2.16}$$

is flat by theorem 2.2 since $[\kappa_\alpha,p_\beta]=[p_\alpha,p_\beta]=0$ for any $\alpha,\beta\in R$. It is moreover W–equivariant if, in addition, $wp_\alpha=p_{w\alpha}$ for any $w\in W$. The corresponding monodromy representation of $P_{\mathfrak{g}}$ is equal to that of the connection ∇_κ tensored with the character $\chi:P_{\mathfrak{g}}\to T$ given by the monodromy of the abelian connection

$$d - \frac{h}{2} \sum_{\alpha \in R} \frac{d\alpha}{\alpha} \cdot p_\alpha \tag{2.17}$$

and therefore does not significantly differ from the monodromy of ∇_κ. A possible choice is to set $p_\alpha = \langle\alpha,\alpha\rangle/2 \cdot h_\alpha^2$ which yields the connection

$$\nabla_C = d - h \sum_{\alpha \in R^+} \frac{d\alpha}{\alpha} \cdot C_\alpha \tag{2.18}$$

where $C_\alpha \in U\mathfrak{sl}_2^\alpha$ is the full Casimir operator of $\mathfrak{sl}_2^\alpha$.

2.3. **The holonomy Lie algebra $\mathfrak{a}(\mathcal{A})$.** Kohno's lemma 2.1 gives a description of the holonomy Lie algebra $\mathfrak{a}(\mathcal{A})$ of a general hyperplane arrangement $\mathcal{A} = \{\mathcal{H}_i\}_{i\in I}$ as the quotient of the free Lie algebra on generators $\{r_i\}_{i\in I}$ by the relations (2.3). When $\mathcal{A} = \mathcal{A}_\mathfrak{g} = \{\mathrm{Ker}(\alpha)\}_{\alpha\in R}$ is the arrangement of root hyperplanes of $\mathfrak{g}$, theorem 2.2 is equivalent to the fact that the assignement $r_\alpha \to \kappa_\alpha$ extends to an algebra homomorphism

$$\phi : U\mathfrak{a}(\mathcal{A}_\mathfrak{g}) \longrightarrow U\mathfrak{g} \tag{2.19}$$

of the universal enveloping algebra of $\mathfrak{a}(\mathcal{A}_\mathfrak{g})$ to that of $\mathfrak{g}$ satisfying

$$\phi(U\mathfrak{a}(\mathcal{A}_\mathfrak{g})_m) \subset U\mathfrak{g}_{2m} \tag{2.20}$$

for any $m \in \mathbb{N}$, where the superscript denotes the degree corresponding to the natural filtrations on both algebras. We simply note here the following

Proposition 2.5. *If one of the simple factors of $\mathfrak{g}$ is not isomorphic to $\mathfrak{sl}_2$, the map $\phi : U\mathfrak{a}(\mathcal{A}_\mathfrak{g}) \longrightarrow U\mathfrak{g}$ is not injective.*

Proof. The following argument was pointed out to us by R. Buchweitz. It suffices to show that, if $\mathfrak{g} \not\cong \mathfrak{sl}_2$, $\mathfrak{a}(\mathcal{A}_\mathfrak{g})$ contains a free Lie algebra on at least two generators, for then $U\mathfrak{a}(\mathcal{A}_\mathfrak{g})$ has exponential growth with respect to its filtration, whereas $U\mathfrak{g}$, being isomorphic to $S\mathfrak{g}$, only grows polynomially. Let $A_1 \times A_1 \not\cong R_0 \subseteq R$ be a rank two root subsystem with positive roots $\beta_1, \ldots, \beta_p$, $p \geq 3$. Let $\mathcal{F}_{p-1}$ be the free Lie algebra on generators $x_1, \ldots, x_{p-1}$ and consider the maps

$$\mathcal{F}_{p-1} \xrightarrow{i} \mathfrak{a}(\mathcal{A}_\mathfrak{g}) \xrightarrow{\pi} \mathcal{F}_{p-1} \tag{2.21}$$

given by $i(x_j) = r_{\beta_j}$, $j = 1 \ldots p-1$ and

$$\pi(r_\alpha) = \begin{cases} 0 & \text{if } \alpha \notin R_0 \\ x_j & \text{if } \alpha = \beta_j, \text{ with } 1 \leq j \leq p-1 \\ -\sum_{j=1}^{p-1} x_j & \text{if } \alpha = \beta_p \end{cases} \tag{2.22}$$

It is easy to see that π is well–defined, so that $\pi \circ i = \mathrm{id}$ and i gives an embedding of $\mathcal{F}_{p-1}$ into $\mathfrak{a}(\mathcal{A}_\mathfrak{g})$ ∎

Remark. It seems an interesting problem to find a generating set of relations for the kernel of the map ϕ above. One such relation may be obtained for any rank two root subsystem $R_2 \subset R$ such that the intersection of its $\mathbb{Z}$–span with R is equal to R_2 but the intersection of its $\mathbb{R}$–span with R strictly contains R_2. This is the case for the root system of type A_2 given by the long roots in the root system of type $\mathfrak{g}_2$ or for the root system of type $A_1 \times A_1$ generated by any pair of long roots in the root system of type C_n. One then has $[\kappa_\alpha, \sum_{\beta\in R_2} \kappa_\beta] = 0$ in $U\mathfrak{g}$, but $[r_\alpha, \sum_{\beta\in R_2} r_\beta] \neq 0$ in $\mathfrak{a}(\mathcal{A}_\mathfrak{g})$.

2.4. **Triviality of $\widetilde{\mathfrak{h}_{\text{reg}}} \times_{P_\mathfrak{g}} V$.** The aim of this subsection is to show that the pull–back to $\mathfrak{h}_{\text{reg}}$ of the bundle $\widetilde{\mathfrak{h}_{\text{reg}}} \times_{B_\mathfrak{g}} V$ constructed in proposition 2.3, namely $\mathcal{V} = \widetilde{\mathfrak{h}_{\text{reg}}} \times_{P_\mathfrak{g}} V$, is topologically trivial[3]. Since $\mathcal{V}$ is W–equivariant, this seemingly contrasts with the fact that W doesn't act on V. The solution of this apparent paradox lies in the fact that the action of W on $\mathcal{V}$ is given by a cocycle, *i.e.,* in a trivialisation $\mathcal{V} \cong \mathfrak{h}_{\text{reg}} \times V$, by $w(t, v) = (wt, A(w, t)v)$ where $A(w, t) \in GL(V)$ satisfies

$$A(w_1w_2, t) = A(w_1, w_2t)A(w_2, t) \tag{2.23}$$

We compute this cocycle explicitly below. These results will not be used elsewhere in the paper.

Let $\sigma : B_\mathfrak{g} \to N(T)$ be a homomorphism making (2.9) commute. The restriction of σ to the pure braid group $P_\mathfrak{g}$ maps into T and therefore factors through the abelianisation of $P_\mathfrak{g}$. The following gives an explicit description of the latter as a $W = B_\mathfrak{g}/P_\mathfrak{g}$–module.

Proposition 2.6. *Let Z be the free abelian group with one generator γ_α for each positive root α of $\mathfrak{g}$ and define an action of W on Z by*

$$w\,\gamma_\alpha = \gamma_{|w\alpha|} \tag{2.24}$$

where $|w\alpha|$ is equal to $\pm w\alpha$ according to whether $w\alpha$ is positive or negative. Then,

(i) *the assignement $\gamma_{\alpha_i} \to S_i^2$ extends uniquely to a W–equivariant isomorphism $Z \cong P_\mathfrak{g}/[P_\mathfrak{g}, P_\mathfrak{g}]$.*
(ii) *Under the Hurewicz isomorphism $P_\mathfrak{g}/[P_\mathfrak{g}, P_\mathfrak{g}] \cong H_1(\mathfrak{h}_{\text{reg}}, \mathbb{Z})$, γ_α is mapped onto a positively oriented simple loop around the hyperplane* $\text{Ker}(\alpha)$.

Proof. (i) is proved in [Ti, Thm. 2.5]. (ii) it is readily checked that, under the Hurewicz isomorphism, the action of W on $P_\mathfrak{g}/[P_\mathfrak{g}, P_\mathfrak{g}]$ coincides with its natural geometric action on $H_1(\mathfrak{h}_{\text{reg}}, \mathbb{Z})$. It follows from the isomorphism $B_\mathfrak{g} \cong \pi_1(\mathfrak{h}_{\text{reg}}/W)$ that $\gamma_{\alpha_i} = S_i^2$ is mapped onto a positively oriented

[3]the second author is grateful to R. Rouquier for a long walk in the Berkeley hills, during which we took turns in convincing each other that the bundle was trivial, then non–trivial, then trivial again, until sheer exhaustion and the late hour of the night suspended, but alas did not resolve, the argument.

simple loop around the hyperplane $\mathrm{Ker}(\alpha_i)$ [Br] so that (ii) follows by $W-$ equivariance ■

For any positive root α, pick an element $\lambda_\alpha \in \mathfrak{h}$ such that $\exp(2\pi i\lambda_\alpha) = \sigma(\gamma_\alpha)$ and consider the flat connection on $\mathfrak{h}_{\mathrm{reg}} \times V$ given by

$$\nabla_\sigma = d - \sum_{\alpha\in R^+} \frac{d\alpha}{\alpha} \cdot \lambda_\alpha \tag{2.25}$$

Fix a basepoint $t_0 \in \mathfrak{h}_{\mathrm{reg}}$ and identify $\widetilde{\mathfrak{h}_{\mathrm{reg}}}$ with the space of paths in $\mathfrak{h}_{\mathrm{reg}}$ pinned at t_0, modulo homotopy equivalence. Denote by $\mathcal{P}_\sigma(p) \in T$ parallel transport with respect to ∇_σ along one such path p. Then,

Proposition 2.7.

(i) *The map* $\widetilde{\mathfrak{h}_{\mathrm{reg}}} \times V \to \mathfrak{h}_{\mathrm{reg}} \times V$ *given by*

$$(p, v) \to (p(1), \mathcal{P}_\sigma(p)v) \tag{2.26}$$

descends to an isomorphism $\iota : \widetilde{\mathfrak{h}_{\mathrm{reg}}} \times_{P_{\mathfrak{g}}} V \cong \mathfrak{h}_{\mathrm{reg}} \times V$.

(ii) *The right action of* $B_{\mathfrak{g}}$ *on* $\widetilde{\mathfrak{h}_{\mathrm{reg}}} \times V$ *descends, via* ι, *to one of* W *on* $\mathfrak{h}_{\mathrm{reg}} \times V$ *given by*

$$w\,(t, v) = (w^{-1}t, \mathcal{P}_\sigma(w^{-1}p_t)\mathcal{P}_\sigma(\widetilde{\gamma})\sigma(\gamma)^{-1}\mathcal{P}_\sigma(p_t)^{-1}v) \tag{2.27}$$

where p_t *is any pinned path in* $\mathfrak{h}_{\mathrm{reg}}$ *with* $p_t(1) = t$, $\gamma \in B_{\mathfrak{g}}$ *is any element with image* w *and* $\widetilde{\gamma}$ *is its lift to a path in* $\mathfrak{h}_{\mathrm{reg}}$ *with* $\widetilde{\gamma}(0) = t_0$.

PROOF. One readily checks, by using proposition 2.6, that the monodromy $P_{\mathfrak{g}} \to T$ of ∇_σ coincides with the restriction of σ to $P_{\mathfrak{g}}$ from which (i) follows at once. (ii) is a simple computation ■

2.5. **Tits extensions.** Let $\sigma : B_{\mathfrak{g}} \to N(T)$ be a homomorphism making the diagram (2.9) commute. Tits has given a simple construction of a canonical, but not exhaustive, class of such σ which differ from each other via conjugation by an element of T. We summarise below the properties of this class obtained in [Ti]. For any simple root α_i, $i = 1 \dots n$, let $SL_2(\mathbb{C}) \cong G_i \subseteq G$ be the subgroup with Lie algebra spanned by $e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i}$, $T_i = \exp(\mathbb{C}\cdot h_{\alpha_i}) \subset G_i$ its torus and N_i the normaliser of T_i in G_i. Denote by $s_i \in W$ the orthogonal reflection corresponding to α_i.

Proposition 2.8.

(i) *For any choice of* $\sigma_i \in N_i \setminus T_i$, $i = 1 \dots n$, *the assignment* $S_i \to \sigma_i$ *extends uniquely to a homomorphism* $\sigma : B_{\mathfrak{g}} \to N(T)$ *making* (2.9) *commute.*

(ii) *If* $\sigma, \sigma' : B_{\mathfrak{g}} \to N(T)$ *are the homomorphisms corresponding to the choices* $\{\sigma_i\}_{i=1}^n$ *and* $\{\sigma'_i\}_{i=1}^n$ *respectively, there exists* $t \in T$ *such that, for any* $S \in B_{\mathfrak{g}}$

$$\sigma(S) = t\sigma'(S)t^{-1} \tag{2.28}$$

(ii) *For any such $\sigma : B_{\mathfrak{g}} \rightarrow N(T)$, the subgroup $\sigma(B_{\mathfrak{g}}) \subset N(T)$ is an extension of W by $\mathbb{Z}_2^n$ canonically isomorphic to the group generated by the symbols a_i, $i = 1 \ldots n$ subject to the relations*

$$\underbrace{a_i a_j \cdots}_{m_{ij}} = \underbrace{a_j a_i \cdots}_{m_{ij}} \tag{2.29}$$

$$a_i^2 a_j^2 = a_j^2 a_i^2 \tag{2.30}$$

$$a_i^4 = 1 \tag{2.31}$$

$$a_i a_j^2 a_i^{-1} = a_j^2 a_i^{-2\langle \alpha_i^\vee, \alpha_j \rangle} \tag{2.32}$$

for any $1 \leq i \neq j \leq n$, where the number m_{ij} of factors on each side of (2.29) is equal to the order of $s_i s_j$ in W. The isomorphism is given by sending a_i to σ_i.

Proof. (i) We must show that the σ_i satisfy the braid relations (2.15). For any $1 \leq i \neq j \leq n$, set $s_{ij} = s_i s_j \cdots \in W$ and $\sigma_{ij} = \sigma_i \sigma_j \cdots \in N(T)$ where each product has $m_{ij} - 1$ factors. The braid relations in W may be written as $s_{ij} s_{j'} = s_j s_{ij}$ where $j' = j$ or i according to whether m_{ij} is even or odd. Thus, $s_{ij}^{-1} s_j s_{ij} = s_{j'}$ and therefore,

$$\delta_{ij} = \sigma_{j'}^{-1} \sigma_{ij}^{-1} \sigma_j \sigma_{ij} \in T \cap \left(\sigma_{j'}^{-1} \sigma_{ij}^{-1} N_j \sigma_{ij} \right) = T \cap \sigma_{j'}^{-1} N_{j'} = T_{j'} \tag{2.33}$$

Repeating the argument with i and j permuted, we find that $\delta_{ji} \in T_{i'}$ with $i' = i$ or j according to whether m_{ij} is even or odd. Thus, $\delta_{ij} = \delta_{ji}^{-1} \in T_{i'} \cap T_{j'} = \{1\}$ where the latter assertion follows from the simple connectedness of G, and the σ_i satisfy (2.15).
(ii) Let $t_i \in T_i$ be such that $\sigma_i = \sigma_i' t_i$ and choose $c_i \in \mathbb{C}$ such that $t_i = \exp(c_i h_{\alpha_i})$. Since

$$(s_i - 1) \sum_{j=1}^{n} c_j \lambda_j^\vee = -c_i h_{\alpha_i} \tag{2.34}$$

where the $\lambda_i^\vee \in \mathfrak{h}$ are the fundamental coweights defined by $\alpha_i(\lambda_j^\vee) = \delta_{ij}$, we find

$$\exp(-\sum_j c_j \lambda_j^\vee) \sigma_i' \exp(\sum_j c_j \lambda_j^\vee) = \sigma_i' \exp(c_i h_{\alpha_i}) = \sigma_i \tag{2.35}$$

so that σ and σ' are conjugate.
(ii) The σ_i satisfy (2.30)–(2.32) since $x_j^2 = \exp(i\pi\alpha_j^\vee)$ for any $x_j \in N_j \setminus T_j$. Let $K_\sigma \cong \mathbb{Z}_2^n$ be the group generated by the σ_i^2 and $K_\sigma \subset \overline{K}_\sigma \subset \sigma(B_{\mathfrak{g}})$ the kernel of the projection $\sigma(B_{\mathfrak{g}}) \rightarrow W$. By (2.29)–(2.32), K_σ is a normal subgroup of $\sigma(B_{\mathfrak{g}})$ and $\sigma(B_{\mathfrak{g}})/K_\sigma$ is generated by the images $\overline{\sigma_i}$ of σ_i which, in addition to the braid relations satisfy $\overline{\sigma}_i^2 = 1$. Thus, $\sigma(B_{\mathfrak{g}})/K_\sigma$ is a quotient of W, $K_\sigma = \overline{K}_\sigma$ and $\sigma(B_{\mathfrak{g}})/K_\sigma \cong W$. The same argument shows that if Γ is the abstract group generated by $a_1, \ldots, a_n$ subject to (2.29)–(2.32), and $A \subset \Gamma$ is the subgroup generated by the a_i^2, then $\Gamma/A \cong W \cong \sigma(B_{\mathfrak{g}})/K_\sigma$. But A is a quotient of $\mathbb{Z}_n^2$ so that the canonical surjection of Γ onto $\sigma(B_{\mathfrak{g}})$

is an isomorphism of A onto K_σ and therefore an isomorphism of Γ onto $\sigma(B_{\mathfrak{g}})$ ■

We shall henceforth only use homomorphisms $\sigma : B_{\mathfrak{g}} \to N(T)$ of the form given by proposition 2.8 and refer to them, or their image $\widetilde{W} = \sigma(B_{\mathfrak{g}}) \subset N(T)$ as *Tits extensions* of W. Note that, given a choice of simple root vectors $e_{\alpha_i}, f_{\alpha_i}$, $i = 1 \dots n$, any element of $N_i \setminus T_i$ is necessarily of the form

$$\begin{aligned}\sigma_i(t_i) &= \exp(t_i e_{\alpha_i}) \exp(-t_i^{-1} f_{\alpha_i}) \exp(t_i e_{\alpha_i}) \\ &= \exp(-t_i^{-1} f_{\alpha_i}) \exp(t_i e_{\alpha_i}) \exp(-t_i^{-1} f_{\alpha_i})\end{aligned} \tag{2.36}$$

for a unique $t_i \in \mathbb{C}^*$ so that a Tits extension may be given by choosing elements $t_1, \dots, t_n \in \mathbb{C}^*$.

3. Generic irreducibility of monodromy representations

3.1. In this section, we study in detail the reducibility of the monodromy of a flat connection of the form (2.2), namely

$$\nabla = d - h \sum_{i \in I} \frac{d\phi_i}{\phi_i} \cdot r_i \tag{3.1}$$

where the residue matrices r_i act on the finite–dimensional vector space V and are assumed to satisfy the relations (2.3). Let

$$\rho_h : \pi_1(B \setminus \mathcal{A}) \longrightarrow GL(V) \tag{3.2}$$

be the corresponding one–parameter family of monodromy representations. If V is reducible under the r_i, ρ_h is clearly reducible for all values of h. The aim of this section is to prove a converse statement. To formulate it, we need the following

Definition. An analytic curve ρ_h of representations of a finitely–generated group Γ is *generically irreducible* if, for all h in the parameter space lying in the complement of an analytic set, the representation ρ_h is irreducible.

We now state the main result of this section.

Theorem 3.1. *If V is irreducible under the r_i, the monodromy representation ρ_h is generically irreducible.*

Our proof also yields an analogue of this theorem in the formal case. Let

$$\widetilde{\rho} : \pi_1(B \setminus \mathcal{A}) \longrightarrow GL(V((h))) \tag{3.3}$$

be the representation obtained by regarding ρ_h as formal in h, letting $\pi_1(B \setminus \mathcal{A})$ act on $V[[h]] = V \otimes \mathbb{C}[[h]]$ and extending coefficients to $V((h)) = V \otimes \mathbb{C}((h))$.

Theorem 3.2. *If V is irreducible under the r_i, $\widetilde{\rho}$ is irreducible.*

The proof of theorems 3.1 and 3.2 occupies the rest of this section. In §3.5, we apply these results to the monodromy of the connection ∇_κ.

Our proof proceeds by noting that, because reducibility is a closed condition, an analytic curve ρ_h of representations is either generically irreducible or reducible for all h. In the latter case, we prove the existence of a *multivalued* (*i.e.,* Puiseux) analytic curve *germ* $U(h^{1/m})$ of proper subspaces of V invariant under the germ of ρ_h at $h=0$. A simple enough calculation then shows that the subspace $U(h^{1/m})\big|_{h^{1/m}=0} \subsetneq V$ is invariant under the r_i. Note that a single–valued analytic germ of invariant subspaces may in general not exist. Indeed, the curve $c(h)$ of reducible representations of $\Gamma=\mathbb{Z}$ given by

$$c(h)=\begin{pmatrix}1&1\\h&1\end{pmatrix},\quad h\in\mathbb{C}\setminus\{1\} \tag{3.4}$$

only admits the multivalued family of eigenlines $(1,\pm\sqrt{h})$. However, with the base change $\widetilde{c}(k)=c(k^2)$, the eigenlines become analytic in k. We will see that such branching is the worst behaviour that can occur.

3.2. **Reducible analytic curve germs and formal curves.** Let Γ be a finitely–generated group. The set of representations $\mathrm{Hom}(\Gamma,GL(V))$ can be given the structure of an affine variety. Indeed, if $\{\gamma_1,\dots,\gamma_r\}$ is a system of generators of Γ, then

$$\mathrm{Hom}(\Gamma,GL(V))\subseteq GL(V)^r \tag{3.5}$$

is the subset of r–tuples of elements satisfying the relations which define Γ.

Theorem 3.3. *The set* $\mathrm{Hom}^{\mathrm{red}}(\Gamma,GL(V))$ *of reducible representations is a Zariski closed subset of* $\mathrm{Hom}(\Gamma,GL(V))$.

The following is an immediate

Corollary 3.4. *An analytic curve* $\rho_h:\Gamma\longrightarrow GL(V)$ *of representations is either generically irreducible or reducible for all values of* h.

Proof of theorem 3.3. For any $0\leq p\leq\dim V$, let $\mathrm{Gr}_p(V)$ be the Grassmannian of p–planes in V. Set

$$\mathcal{R}_p(\Gamma)=\{(\rho,U)\in\mathrm{Hom}(\Gamma,GL(V))\times\mathrm{Gr}_p(V)|\rho(\Gamma)U=U\} \tag{3.6}$$

We claim that $\mathcal{R}_p(\Gamma)$ is a Zariski closed subset of $\mathrm{Hom}(\Gamma,GL(V))\times\mathrm{Gr}_p(V)$. Indeed, regarding $U\in\mathrm{Gr}_p(V)$ as all multiples of a decomposable p–tensor $\Lambda=u_1\wedge\cdots\wedge u_p\in\bigwedge^p V$ via the Plücker embedding, we see that the invariance of U under $\rho\in\mathrm{Hom}(\Gamma,GL(V))$ is equivalent to the relations (quadratic in the Plücker coordinates on $\mathrm{Gr}_p(V)$)

$$\Lambda\wedge\rho(\gamma_i)\Lambda=0 \tag{3.7}$$

for all $i=1\dots r$, where $\wedge$ is the exterior multiplication in $\bigwedge^*(\bigwedge^p V)$. Since the projection $p_1:\mathrm{Hom}(\Gamma,GL(V))\times\mathrm{Gr}_p(V)\to\mathrm{Hom}(\Gamma,GL(V))$ is closed

[Mu, thm. 2.23], the set

$$\begin{aligned}\mathrm{Hom}^{\mathrm{red},p}(\Gamma, GL(V)) &= \{\rho \in \mathrm{Hom}(\Gamma, GL(V)) | \,\exists\, U \in \mathrm{Gr}_p(V),\ \rho\, U = U\} \\ &= p_1(\mathcal{R}_p(\Gamma))\end{aligned} \tag{3.8}$$

is a closed subset of $\mathrm{Hom}(\Gamma, GL(V))$ and therefore so is

$$\mathrm{Hom}^{\mathrm{red}}(\Gamma, GL(V)) = \bigcup_{p=1}^{\dim V-1} \mathrm{Hom}^{\mathrm{red},p}(\Gamma, GL(V)) \tag{3.9}$$

■

Definition.

(i) An analytic curve ρ_h of representations is *reducible* if it is contained in the subvariety $\mathrm{Hom}^{\mathrm{red}}(\Gamma, GL(V))$ of reducible representations.
(ii) An analytic curve germ (ρ_h, ρ) is *reducible* if it has a representative contained in $\mathrm{Hom}^{\mathrm{red}}(\Gamma, GL(V))$. Hence all representatives are contained in this variety.
(iii) A formal curve beginning at ρ *i.e.,* an algebra homomorphism

$$\phi : \widehat{\mathcal{O}}_{\mathrm{Hom}(\Gamma,GL(V)),\rho} \to \mathbb{C}[[h]] \tag{3.10}$$

is *reducible* if ϕ factors through $\widehat{\mathcal{O}}_{\mathrm{Hom}^{\mathrm{red}}(\Gamma,GL(V)),\rho}$.

Note that the infinite jet $\hat{\rho_h}$ at ρ of a reducible analytic curve ρ_h is a reducible formal curve. The converse is also clear since an analytic function that is formally zero is zero.

3.3. **The existence of a multivalued section.** Retain the notation of §3.2. The projection on the first factor induces a regular map

$$\pi_p : \mathcal{R}_p(\Gamma) \to \mathrm{Hom}^{\mathrm{red},p}(\Gamma, GL(V)) \tag{3.11}$$

We will be concerned in this subsection with constructing a multivalued section to π_p over a curve germ contained in $\mathrm{Hom}^{\mathrm{red},p}(\Gamma, GL(V))$. Let Γ_r be the free group on r generators. The commutative diagram

$$\begin{array}{ccc} \mathcal{R}^p(\Gamma) & \longrightarrow & \mathcal{R}^p(\Gamma_r) \\ {\scriptstyle \pi_p}\big\downarrow & & {\scriptstyle \pi_p}\big\downarrow \\ \mathrm{Hom}^{\mathrm{red},p}(\Gamma) & \longrightarrow & \mathrm{Hom}^{\mathrm{red},p}(\Gamma_r) \end{array} \tag{3.12}$$

shows that it suffices to find a section for the case of a free group. Indeed, such a section induces by restriction a section for all quotients of that free group. For the remainder of this subsection we therefore assume that $\Gamma = \Gamma_r$.

We shall prove in fact the existence of a *formal* multivalued section (see below for a precise definition). The same argument yields an analytic multivalued one. Equivalently, one may deduce the existence of such a section by using Artin's theorem as follows. First a reducible analytic curve gives rise to a reducible formal curve which has a formal section. Then, by [Ar, Thm 1.5 (ii)], one may find an analytic section approximating the formal one as closely as one wishes.

We need some notation. Let

$$\begin{aligned} B &= \mathbb{C}[\mathrm{Hom}(\Gamma, GL(V))] \\ &= \mathbb{C}[g^1_{ij}, \ldots, g^r_{ij}]_{1\leq i,j\leq \dim V}[\det(g^1_{ij})^{-1}, \ldots, \det(g^r_{ij})^{-1}] \end{aligned} \tag{3.13}$$

be the coordinate ring of $\mathrm{Hom}(\Gamma, GL(V)) \cong GL(V)^r$. Let $\widetilde{\mathrm{Gr}}_p(V)$ be the cone in $\bigwedge^p(V)$ defined by the Plücker equations and let C_p be its coordinate ring. Thus,

$$C_p = \mathbb{C}[\widetilde{\mathrm{Gr}}_p(V)] = \mathbb{C}[x_I]/(f^p_\alpha(x_I)) \tag{3.14}$$

where the x_I are the the Plücker coordinates on $\mathrm{Gr}_p(V)$ and $\{f^p_\alpha(x_I)\}_{\alpha\in\mathcal{I}}$ are the (quadratic) Plücker relations defining the Grassmannian. Let

$$\begin{aligned} A_p &= \mathbb{C}[\mathrm{Hom}(\Gamma, GL(V)) \times \widetilde{\mathrm{Gr}}_p(V)] \\ &= B \otimes C_p = B[x_I]/(f^p_\alpha(x_I)) \end{aligned} \tag{3.15}$$

be the coordinate ring of $\mathrm{Hom}(\Gamma, GL(V)) \times \widetilde{\mathrm{Gr}}_p(V)$. Denote the equations (3.7) by $\{q^p_\beta(g^k_{ij}, x_I)\}_{\beta\in\mathcal{J}}$, let $\mathfrak{a}_p = (q^p_\beta)_{\beta\in\mathcal{J}} \subset A_p$ be the corresponding ideal and set

$$\begin{aligned} R_p &= \mathbb{C}[\widetilde{\mathcal{R}}^p(\Gamma)] = A_p/\mathfrak{a}_p \\ &= \mathbb{C}[g^k_{ij}, x_I][\det(g^k_{ij})^{-1}]/(f^p_\alpha(x_I), q^p_\beta(g^k_{ij}, x_I)) \end{aligned} \tag{3.16}$$

where $\widetilde{\mathcal{R}}^p(\Gamma)$ is the closure in $\mathrm{Hom}(\Gamma, GL(V)) \times \widetilde{\mathrm{Gr}}_p(V)$ of the preimage of the variety $\mathcal{R}_p(\Gamma)$ defined by (3.6) under the projection

$$\mathrm{Hom}(\Gamma, GL(V)) \times \widetilde{\mathrm{Gr}}_p(V) \setminus \{0\} \to \mathrm{Hom}(\Gamma, GL(V)) \times \mathrm{Gr}_p(V) \tag{3.17}$$

For later use, we note that the polynomials q^p_β are homogeneous (of degree 2) in the Plücker coordinates x_I. Finally, let $\mathfrak{b}, \mathfrak{b}_p \subset B$ be the ideals of

$$\mathrm{Hom}^{\mathrm{red}}(\Gamma, GL(V)),\ \mathrm{Hom}^{\mathrm{red},p}(\Gamma, GL(V)) \subset \mathrm{Hom}(\Gamma, GL(V)) \tag{3.18}$$

respectively, and set

$$S = \mathbb{C}[\mathrm{Hom}^{\mathrm{red}}(\Gamma, GL(V))] = B/\mathfrak{b} \tag{3.19}$$

$$S_p = \mathbb{C}[\mathrm{Hom}^{\mathrm{red},p}(\Gamma, GL(V))] = B/\mathfrak{b}_p \tag{3.20}$$

Note that the projection (3.11) induces a ring homomorphism $\psi_p : S_p \to R_p$.

Let now $\widehat{B}, \widehat{S}$ be the completions of B, S at a fixed reducible representation ρ. Let $\widehat{A}_p$, $\widehat{R}_p$ be the completions of A_p, R_p along the fiber of π_p over ρ, see [Ha, pg. 194]. A formal curve of representations starting at ρ is a homomorphism

$$\phi : \widehat{B} \to \mathbb{C}[[h]] \tag{3.21}$$

ϕ is given in coordinates by n^2r formal power series $g_{ij}^k(h) = \phi(g_{ij}^k)$. ϕ is a curve of reducible representations if it descends to the quotient $\widehat{S}$ of $\widehat{B}$. Let $\mathfrak{p}$ be the kernel of ϕ. $\mathfrak{p}$ is a prime ideal since $\mathbb{C}[[h]]$ is an integral domain and

$$\mathfrak{p} \supset \mathfrak{b} = \prod_{p=1}^{n-1} \mathfrak{b}_p \tag{3.22}$$

so that $\mathfrak{p} \supset \mathfrak{b}_p$ for some p. Since the completion $\widehat{S}_p$ of S_p at ρ is the quotient $\widehat{B}/\mathfrak{b}_p$, we obtain a homomorphism $\phi : \widehat{S}_p \to \mathbb{C}[[h]]$.

Definition. A multivalued formal section is a homomorphism χ fitting into the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}[[k]] & \xleftarrow{\chi} & \widehat{R}_p \\ {\scriptstyle f_m}\big\uparrow & & {\scriptstyle \psi_p}\big\uparrow \\ \mathbb{C}[[h]] & \xleftarrow{\phi} & \widehat{S}_p \end{array} \tag{3.23}$$

where $f_m(h) = k^m$.

Let $\mathcal{F} = \mathbb{C}((h))$ be the field of fractions of $\mathbb{C}[[h]]$, $\overline{\mathcal{F}}$ its algebraic closure and $\iota : \mathcal{F} \to \overline{\mathcal{F}}$ the corresponding inclusion. The key step in finding a formal multivalued section is the following

Proposition 3.5. *There exists a homomorphism of $\mathbb{C}$–algebras $\tau : \widehat{R}_p \to \overline{\mathcal{F}}$ such that the following diagram is commutative*

$$\begin{array}{ccc} \overline{\mathcal{F}} & \xleftarrow{\tau} & \widehat{R}_p \\ {\scriptstyle \iota}\big\uparrow & & {\scriptstyle \psi_p}\big\uparrow \\ \mathbb{C}[[h]] & \xleftarrow{\phi} & \widehat{S}_p \end{array} \tag{3.24}$$

and $\tau(x_I) \neq 0$ for some I.

Proof. Write the sentence in the symbols $\exists, \cap, \cup$ and complement $'$ and field operations that states that the projection

$$\begin{array}{c} \widetilde{\mathcal{R}}_p(\Gamma)(\mathcal{E}) \subset \mathrm{Hom}(\Gamma, GL(V \otimes \mathcal{E})) \times \widetilde{\mathrm{Gr}}_p(V \otimes \mathcal{E}) \\ {\scriptstyle p_1}\big\downarrow \\ \mathrm{Hom}^{\mathrm{red},p}(\Gamma, GL(V \otimes \mathcal{E})) \end{array} \tag{3.25}$$

is onto for an extension field $\mathcal{E}$ of $\mathbb{C}$. This sentence contains the equations for the affine cone $\widetilde{\mathrm{Gr}}_p(V)$ defined by the Plücker relations $f_\alpha^p(x_I) = 0, \alpha \in \mathcal{I}$, and must include the condition ($x_I \neq 0$ for some I). The resulting statement is true for the field $\mathbb{C}$ hence, by model completeness of the theory of algebraically closed fields[MMP, pg. 5], for any algebraically closed extension field of $\mathbb{C}$. In particular, it is true for $\overline{\mathcal{F}}$ ■

Remark. From the above commutative diagram we obtain

$$\tau(g_{ij}^k) = \iota\phi(g_{ij}^k) \in \mathbb{C}[[h]] \tag{3.26}$$

Lemma 3.6. *Let $\mathbb{C}((h^{\frac{1}{\infty}}))$ denote the field obtained by adjoining all the roots of h to $\mathbb{C}$. Then $\mathbb{C}((h^{\frac{1}{\infty}}))$ is algebraically closed. Consequently, $\overline{\mathcal{F}} \cong \mathbb{C}((h^{\frac{1}{\infty}}))$.*

Proof. This is proved in [Wal], Theorem 3.1 ■

We now construct the desired multivalued formal section.

Theorem 3.7. *There exists an $m \geq 1$ and a homomorphism of $\mathbb{C}$–algebras $\chi : \widehat{R}_p \to \mathbb{C}[[k]]$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{C}[[k]] & \xleftarrow{\ \chi\ } & \widehat{R}_p \\ {\scriptstyle f_m}\big\uparrow & & {\scriptstyle \psi_p}\big\uparrow \\ \mathbb{C}[[h]] & \xleftarrow{\ \phi\ } & \widehat{S}_p \end{array} \tag{3.27}$$

and $\chi(x_I) \neq 0$ for some I.

Proof. By the two previous results we obtain a homomorphism τ taking values in the quotient field $\mathbb{C}((k))$ of $\mathbb{C}[[k]]$ for some root k of h. Such a homomorphism amounts to assigning an element of $\mathbb{C}((k))$ to each of the variables g_{ij}^k, x_I in such a way that these elements satisfy the defining equations p_α and q_β^p of $\widetilde{\mathcal{R}}_p(\Gamma)$. As noted in (3.26), the $\tau(g_{ij}^k)$ already lie in $\mathbb{C}[[k]]$. Since the equations are homogeneous in the x_I, we can multiple the $\tau(x_I)$ by an appropriate power of k so that the resulting elements of $\mathbb{C}((k))$ are in $\mathbb{C}[[k]]$ ■

The following result explains the meaning of the $\mathbb{C}[[k]]$–point χ constructed above.

Proposition 3.8. *The homomorphism χ gives canonically rise to a rank p summand $\mathcal{U}$ of the module $V[[k]] = V \otimes \mathbb{C}[[k]]$ together with an r–tuple of invertible (over $\mathbb{C}[[k]]$) matrices $g^1(k^m), \dots, g^r(k^m)$ leaving $\mathcal{U}$ invariant.*

Proof. The homomorphism χ fits into the diagram

$$\begin{array}{ccccc} \mathbb{C}[[k]] & \xleftarrow{\ \chi\ } & \widehat{R}_p & \xleftarrow{\ \pi\ } & \widehat{A}_p \\ {\scriptstyle f_m}\big\uparrow & & {\scriptstyle \psi_p}\big\uparrow & & \big\uparrow \\ \mathbb{C}[[h]] & \xleftarrow{\ \phi\ } & \widehat{S}_p & \longleftarrow & \widehat{B} \end{array} \tag{3.28}$$

The composition $\chi \circ \pi$ is a tensor product $\eta \otimes \zeta$ where η is a $\mathbb{C}[[k]]$–point of $\mathrm{Hom}(\Gamma, GL(V))$ and ζ is a $\mathbb{C}[[k]]$–point of $\widetilde{\mathrm{Gr}}_p(V)$. Regarding ζ as a $\mathbb{C}((k))$–point of $\widetilde{\mathrm{Gr}}_p(V)$, its Plücker coordinates yield a p–dimensional subspace $\widetilde{\mathcal{U}}$

of the $\mathbb{C}((k))$–vector space $V((k))$. Let $\mathcal{U}$ be the intersection of $\widetilde{\mathcal{U}}$ with $V[[k]]$.

$\mathcal{U}$ is cotorsion–free in $V[[k]]$ since, if $v \in V[[k]]$ and $r \in \mathbb{C}[[k]]$ are such that $rv = u \in \mathcal{U}$, then $v = \frac{1}{r} \cdot u \in \widetilde{\mathcal{U}}$, whence $v \in \mathcal{U} = \widetilde{\mathcal{U}} \cap V[[k]] = \mathcal{U}$. Since $\mathbb{C}[[k]]$ is a principal ideal domain, the quotient $V[[k]]/\mathcal{U}$ is free (since it is torsion–free) and the sequence $\mathcal{U} \to V[[k]] \to V[[k]]/\mathcal{U}$ splits. Hence $\mathcal{U}$ is a direct summand and consequently is free (since it is projective and $\mathbb{C}[[k]]$ is a local ring). Since $\dim(\widetilde{\mathcal{U}}) = p$ it follows that $\mathcal{U}$ is free of rank p.

The homomorphism ϕ corresponds to an r–tuple of matrices $g^1(h), \ldots, g^r(h)$ with entries in $\mathbb{C}[[h]]$ and determinant a unit in $\mathbb{C}[[h]]$. The fact that $\eta \otimes \zeta$ descends to χ and the fact that χ is a $\mathbb{C}[[k]]$–point of $\widehat{R}_p$ implies that $g^1(k^m), \ldots, g^r(k^m)$ leave $\mathcal{U}$ invariant ■

3.4. **Proof of theorem 3.1.** Let Γ be the fundamental group $\pi_1(B \setminus \mathcal{A})$. It is well–known that Γ is finitely–generated, see *e.g.*, [BMR, prop. A2., pg. 181]. If the curve ρ_h of monodromy representations is not generically irreducible, it lies, by corollary 3.4, in the variety of reducible representations and therefore in some $\mathrm{Hom}^{\mathrm{red},p}(\Gamma, GL(V))$, $1 \leq p \leq \dim V - 1$. Let χ be a multivalued analytic section and let $\mathcal{U}$ be the corresponding rank p summand of the free $\mathbb{C}\{k\}$–module $\mathcal{V} = V \otimes \mathbb{C}\{k\}$ obtained by applying the analytic version of proposition 3.8 which we may state as

Proposition 3.9. *There is a canonical one-to–one correspondence between analytic curve germs (U_k, U) in $\mathrm{Gr}_p(V)$ and (free) summands $\mathcal{U}$ of rank p of the $\mathbb{C}\{k\}$–module $\mathcal{V}$.*

Remark. To pass from the $\mathbb{C}\{k\}$–submodule $\mathcal{U}$ to the curve germ (U_k, U) proceed as follows. Choose a basis $e_1, \ldots, e_n$ for V over $\mathbb{C}$ and regard it as basis for $\mathcal{V}$ over $\mathbb{C}\{k\}$. Choose a basis $u_1, \ldots, u_p$ for $\mathcal{U}$ over $\mathbb{C}\{k\}$. Write $u_1, \ldots, u_p$ in terms of $e_1, \ldots, e_n$ to obtain a curve $b(k)$ of bases for p–dimensional subspaces of V. The span of $b(k)$ is U_k.

Set $\widetilde{\rho}_k = \rho_{k^m}$ and let $\widetilde{\rho}$ be the element of $\mathrm{Hom}(\Gamma, GL(\mathcal{V}))$ corresponding to $\widetilde{\rho}_k$. Since $\widetilde{\rho}_k$ is an analytic function of $h = k^m$, the elements

$$A(\gamma) = \frac{I - \widetilde{\rho}(\gamma)}{k^m} \tag{3.29}$$

leave $\mathcal{V}$ invariant. It follows that $A(\gamma)\mathcal{U} \subset \mathcal{U}$ since $\widetilde{\rho}(\Gamma)\mathcal{U} = \mathcal{U}$. The following standard result shows that the subspace $\mathcal{U}(0) \subsetneq V$ is invariant under the residues r_i, thus concluding the proof of theorem 3.1.

Lemma 3.10. *For each hyperplane H_j of the arrangement $\mathcal{A}$, there exists an element $\gamma_j \in \pi_1(B \setminus \mathcal{A})$ such that*

$$\rho_h(\gamma_j) = 1 + 2\pi i h \cdot r_j \mod h^2 \tag{3.30}$$

Proof. Let $x_0 \in B \setminus \mathcal{A}$ be a base point and $\gamma_j \in \pi_1(B \setminus \mathcal{A}; x_0)$ a generator of monodromy around H_j (see, *e.g.*, [BMR, pg. 180–1]). Recall that such an element is obtained as follows. Choose a path $p : [0,1] \to B$ such that

$$p(0) = x_0, \quad p([0,1[) \subset B \setminus \mathcal{A} \quad \text{and} \quad p(1) \in H_j \setminus \bigcup_{j' \neq j} H_{j'} \tag{3.31}$$

Let D be a small ball centred at $p(1)$ and contained in $B \setminus \bigcup_{j' \neq j} H_{j'}$, let $u \in [0,1[$ be such that $p(s) \in D$ for any $s \geq u$ and let ℓ be a positively oriented generator of $\pi_1(D \setminus H_j; p(u)) \cong \mathbb{Z}$. Then,

$$\gamma_i = p_u^{-1} \cdot \ell \cdot p_u \tag{3.32}$$

where $p_u(t) = p(ut)$ and the concatenation of paths is read from right to left. Picard iteration readily yields that, mod h^2,

$$\rho_h(\gamma_j) = 1 + h \sum_{i \in I} \int_{\gamma_j} \frac{d\phi_i}{\phi_i} \cdot r_i = 1 + h \sum_{i \in I} \int_{\ell} \frac{d\phi_i}{\phi_i} \cdot r_i = 1 + 2\pi i h \cdot r_j \tag{3.33}$$

where the last equality follows from the residue theorem since the forms $d\phi_{j'}/\phi_{j'}$, $j' \neq j$ do not have any poles in D ■

The proof of theorem 3.2 is the same as that of theorem 3.1 except for the use of a formal multivalued section provided by proposition 3.8 instead of an analytic one.

3.5. **Generic irreducibility of the monodromy of ∇_κ.** Assume now that V is a $\mathfrak{g}$–module and let $\widetilde{W} = \sigma(B_\mathfrak{g})$ be a Tits extension with sign group $\Sigma = \sigma(P_\mathfrak{g}) \cong \mathbb{Z}_2^n$. Let

$$\rho_h^\sigma : B_\mathfrak{g} \longrightarrow GL(V) \tag{3.34}$$

be the corresponding one–parameter family of monodromy representations defined by proposition 2.3 and $\rho^\sigma : B_\mathfrak{g} \longrightarrow GL(V((h)))$ the representation obtained by regarding ρ_h^σ as formal in h and extending coefficients to $\mathbb{C}((h))$.

Theorem 3.11. *Let $U \subseteq V$ be a subspace invariant, and irreducible under the Casimirs κ_α and $\widetilde{W}$. Then,*

(i) $\rho_h^\sigma : B_\mathfrak{g} \longrightarrow GL(U)$ *is generically irreducible.*
(ii) $\rho^\sigma : B_\mathfrak{g} \longrightarrow GL(U((h)))$ *is irreducible.*

Proof. (i) Assume ρ_h^σ to be reducible for all h. Proceeding as in the proof of theorem 3.1, we find a $1 \leq p \leq \dim U - 1$ and a rank p–summand $\mathcal{X}$ of $\mathcal{U} = U \otimes \mathbb{C}\{k\}$ invariant under the germ of ρ_h^σ at $h = 0$. In particular, $\mathcal{X}(0)$ is invariant under $\rho_0^\sigma(B_\mathfrak{g}) = \widetilde{W}$. For any positive root α, let $\gamma_\alpha \in P_\mathfrak{g}$ be the generator of monodromy around the hyperplane $\mathrm{Ker}(\alpha)$ given by lemma 3.10. Note that $\sigma(\gamma_\alpha)$ lies in the sign group Σ and therefore has order 1 or 2. By lemma 3.10 and proposition 2.4 we find that, mod h^2,

$$\rho_h^\sigma(\gamma_\alpha^2) = \sigma(\gamma_\alpha)^2 (1 + 2\pi i h \cdot \kappa_\alpha)^2 = 1 + 4\pi i h \cdot \kappa_\alpha \tag{3.35}$$

so that $\mathcal{X}(0)$ is also invariant under the Casimirs κ_α. The proof of (ii) is identical ■

Similarly, we obtain

Theorem 3.12. *Let $U \subseteq V$ be a subspace invariant, and irreducible under the Casimirs κ_α and Σ. Then,*

(i) *$\rho_h^\sigma : P_\mathfrak{g} \longrightarrow GL(U)$ is generically irreducible.*
(ii) *$\rho^\sigma : P_\mathfrak{g} \longrightarrow GL(U((h)))$ is irreducible.*

We specialise our results further to the case where U is the zero weight space $V[0]$ of V. Recall that the latter is canonically acted upon by $W \cong N(T)/T$ so that the restriction of ρ_h^σ to $V[0]$ does not depend upon the choice of σ. We owe the following somewhat surprising observation to B. Kostant

Proposition 3.13. *$V[0]$ is irreducible under the Casimirs κ_α iff it is irreducible under the κ_α and W.*

Proof. The simple reflection $s_i \in W$ acts on the zero weight space $V_n^i[0]$ of the irreducible $\mathfrak{sl}_2^{\alpha_i}$–module of dimension $2n+1$ as multiplication by $(-1)^n$. Thus, if p_ε^i, $\varepsilon = 0, 1$ are the spectral projections for the restriction of C_{α_i} to $V[0]$ corresponding to the Casimir eigenvalues of V_n^i, with $n = \epsilon \mod 2$, s_i acts on $V[0]$ as $p_0^i - p_1^i$ and is therefore a polynomial in C_{α_i}. It follows that a subspace $U \subseteq V[0]$ invariant under the κ_α is also invariant under W ■

Corollary 3.14. *The following statements are equivalent*

(i) *$V[0]$ is irreducible under the Casimirs κ_α.*
(ii) *$V[0]$ is generically irreducible under $P_\mathfrak{g}$.*
(iii) *$V[0]((h))$ is irreducible under $P_\mathfrak{g}$.*
(iv) *$V[0]$ is generically irreducible under $B_\mathfrak{g}$.*
(v) *$V[0]((h))$ is irreducible under $B_\mathfrak{g}$.*

4. The Casimir algebra $\mathcal{C}_\mathfrak{g}$ of $\mathfrak{g}$

4.1. Since the connection ∇_κ is reducible with respect to the weight space decomposition of a simple $\mathfrak{g}$–module V, theorems 3.11 and 3.12 naturally prompt the question, originally asked us by A. Knutson and C. Procesi, of whether the Casimir operators κ_α act irreducibly on the weight spaces of V or, stronger still, whether the κ_α, together with $\mathfrak{h}$, generate the $\mathfrak{h}$–invariant subalgebra $U\mathfrak{g}^\mathfrak{h}$ of $U\mathfrak{g}$.

Let $\mathcal{C}_\mathfrak{g}$ be the *Casimir algebra* of $\mathfrak{g}$, *i.e.,* the algebra

$$\mathcal{C}_\mathfrak{g} = \langle \kappa_\alpha \rangle_{\alpha \in R_+} \vee \mathfrak{h} \subseteq U\mathfrak{g}^\mathfrak{h} \tag{4.1}$$

generated by the κ_α, or equivalently the Casimirs C_α, and $\mathfrak{h}$, inside $U\mathfrak{g}^\mathfrak{h}$. We show in §4.2 that if $\mathfrak{g} = \mathfrak{sl}_3$, $\mathcal{C}_\mathfrak{g}$ is a proper subalgebra of $U\mathfrak{g}^\mathfrak{h}$ but that the latter is generated by $\mathcal{C}_\mathfrak{g}$ and the centre $Z(U\mathfrak{g})$ of $U\mathfrak{g}$. In particular, the κ_α act irreducibly on the weight spaces of any simple $\mathfrak{sl}_3$–module and the monodromy of the connection ∇_κ yields generically irreducible representations of the pure braid group $P_3 = P_{\mathfrak{sl}_3}$ on these weight spaces and of B_3 on their zero weight space.

For $\mathfrak{g} \not\cong \mathfrak{sl}_2, \mathfrak{sl}_3$, the situation is radically different and we show that there always exists a simple $\mathfrak{g}$–module V such that its zero weight space $V[0]$ is reducible under $\mathcal{C}_\mathfrak{g}$, thus answering in the negative Knutson and Procesi's question. In particular $\mathcal{C}_\mathfrak{g}$ and $Z(U\mathfrak{g})$ do not generate $U\mathfrak{g}^\mathfrak{h}$. A suitable V is readily found in §4.3 for $\mathfrak{g} \not\cong \mathfrak{sl}_n$. For $\mathfrak{g} \cong \mathfrak{sl}_n$, $n \geq 4$, its construction requires the general reducibility criterion outlined in the Introduction and the Gelfand–Zetlin branching rules which are given and reviewed in §4.4 and §4.5 respectively.

4.2. **The Casimir algebra of $\mathfrak{sl}_3$.**

Theorem 4.1. *If $\mathfrak{g} = \mathfrak{sl}_3$, then*

(i) *$\mathcal{C}_\mathfrak{g}$ is a proper subalgebra of $U\mathfrak{g}^\mathfrak{h}$.*
(ii) *$U\mathfrak{g}^\mathfrak{h}$ is generated by $\mathcal{C}_\mathfrak{g}$ and the center of $U\mathfrak{g}$. In particular, the Casimirs κ_α act irreducibly on the weight spaces of any simple $\mathfrak{g}$–module.*

We shall need some preliminary results. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$ and $E_{ij} e_k = \delta_{jk} e_i$ the corresponding elementary matrices. The following is immediate

Lemma 4.2. *The element $E_{i_1 j_1} E_{i_2 j_2} \cdots E_{i_k j_k} \in U\mathfrak{gl}_n$ is of weight zero iff the sequence $(j_1, j_2, \ldots, j_k)$ is a permutation of $(i_1, i_2, \ldots, i_k)$.*

Let $I = (i_1, i_2, \ldots, i_k)$ be a sequence of *distinct* elements of $\{1, \ldots, n\}$ and set

$$E_I = E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_{k-1} i_k} E_{i_k i_1} \in U\mathfrak{sl}_n^\mathfrak{h} \tag{4.2}$$

Proposition 4.3. *If $\mathfrak{g} = \mathfrak{sl}_n$, $U\mathfrak{g}^\mathfrak{h}$ is generated as an algebra by the monomials E_I and by $\mathfrak{h}$.*

Proof. It suffices to show that the images of the E_I and $\mathfrak{h}$ generate the $\mathfrak{h}$–invariant subalgebra of the graded algebra $\mathrm{gr}(U\mathfrak{g}) \cong S\mathfrak{g}$. By the previous lemma, $S\mathfrak{g}^\mathfrak{h}$ is spanned by elements of the form $p\, E_{I,\sigma}$ where $p \in S\mathfrak{h}$, $I = (i_1, \ldots, i_k)$ is a sequence of elements in $\{1, \ldots, n\}$, $\sigma \in \mathfrak{S}_k$ is a permutation and

$$E_{I,\sigma} = E_{i_1 i_{\sigma(1)}} \cdots E_{i_k i_{\sigma(k)}} \tag{4.3}$$

Writing σ as a product of disjoint cycles $\tau_1 \circ \cdots \circ \tau_r$ with $\tau_j = (m_j^1 \cdots m_j^{k_j})$ then shows that, in $S\mathfrak{g}$

$$E_{I,\sigma} = E_{I_1} \cdots E_{I_r} \tag{4.4}$$

where $I_j = (i_{m_j^1}, \cdots, i_{m_j^{k_j}})$ and therefore that $S\mathfrak{g}^\mathfrak{h}$ is generated by $\mathfrak{h}$ and the E_I ■

Remark. The previous proof shows in fact that $U\mathfrak{sl}_n^{\mathfrak{h}}$ is generated by $\mathfrak{h}$ and the E_I corresponding to sequences $I=(i_1,\ldots,i_k)$ such that $i_1=\min_l i_l$.

Corollary 4.4. *If $\mathfrak{g}=\mathfrak{sl}_3$, $U\mathfrak{g}^{\mathfrak{h}}$ is generated by $\mathfrak{h}$ together with the three quadratic $\mathfrak{h}$–invariants*

$$F_{12}=E_{12}E_{21} \qquad F_{13}=E_{13}E_{31} \qquad F_{23}=E_{23}E_{32} \tag{4.5}$$

and the two cubic $\mathfrak{h}$–invariants

$$G_{123}=E_{12}E_{23}E_{31} \qquad G_{132}=E_{13}E_{32}E_{21} \tag{4.6}$$

Remark. Note that if we permute any two factors in the expressions for E_{123} or E_{132} then the difference is a quadratic invariant.

Proof of (ii) of theorem 4.1. Let $\theta_i-\theta_j$, $1\le i<j\le 3$ be the positive roots of $\mathfrak{g}=\mathfrak{sl}_3$. The $\mathfrak{sl}_2$–triple corresponding to $\theta_i-\theta_j$ is $\{E_{ij},E_{ji},E_{ii}-E_{jj}\}$, so that

$$\kappa_{\theta_i-\theta_j}=E_{ij}E_{ji}+E_{ji}E_{ij} \tag{4.7}$$

is equal to $2F_{ij} \mod U\mathfrak{h}$. By the previous corollary, $S\mathfrak{g}^{\mathfrak{h}}$ is generated, as a $\mathcal{C}_{\mathfrak{g}}$–algebra, by the single element $G=G_{123}+G_{132}$ since

$$G_{123}-G_{132}=F_{13}+[F_{23},F_{12}]\in\mathcal{C}_{\mathfrak{g}} \tag{4.8}$$

It therefore suffices to show that G lies in the algebra generated by $\mathcal{C}_{\mathfrak{g}}$ and $Z(U\mathfrak{g})$. It will be convenient to replace $\mathfrak{g}=\mathfrak{sl}_3$ by the isomorphic $\mathfrak{pgl}_3$. We claim that, modulo $\mathcal{C}_{\mathfrak{g}}$, G is equal to the element H in the center $Z(U\mathfrak{pgl}_3)$ given by

$$H=\sum_{1\le i,j,k\le 3}E_{ij}E_{jk}E_{ki} \tag{4.9}$$

Indeed, write H as H_1+H_2 where H_2 is the part of the sum for which i,j,k are all distinct. Clearly, H_1 lies in $\mathcal{C}_{\mathfrak{g}}$. Break up the sum defining H_2 according to whether the first index is $1,2$ or 3. By the remark above, each of the resulting sums then gives G modulo $\mathcal{C}_{\mathfrak{g}}$ ■

Proof of (i) of theorem 4.1. Let $Q=U\mathfrak{g}^{\mathfrak{h}}/hU\mathfrak{g}^{\mathfrak{h}}$ be the quotient of $U\mathfrak{g}^{\mathfrak{h}}$ by the (two–sided) ideal generated by $\mathfrak{h}$. Let σ be the Chevalley involution of $U\mathfrak{g}$ given by $\sigma(X)=-X^t$ for $X\in\mathfrak{g}$. σ leaves $\mathfrak{h}$ invariant and descends to an involution of Q fixing the image of $\mathcal{C}_{\mathfrak{g}}$. It therefore suffices to show that σ does not act trivially on Q or on the associated graded $\mathrm{gr}(Q)$. However the image of $G=G_{123}+G_{132}$ in $\mathrm{gr}(Q)$ satisfies $\sigma(G)=-G$ since, in $\mathrm{gr}(Q)$

$$\sigma(E_{123})=-E_{132} \qquad \text{and} \qquad \sigma(E_{132})=-E_{123} \tag{4.10}$$

■

Corollary 4.5. *Let V be a simple $\mathfrak{sl}_3$–module. Then, Artin's pure braid group P_3 acts generically irreducibly on the weight spaces of V. In particular, Artin's braid group B_3 acts generically irreducibly on the zero weight space of V.*

4.3. **The Casimir algebra of $\mathfrak{g} \not\cong \mathfrak{sl}_n$.** Assume that $\mathfrak{g}$ is simple and not isomorphic to $\mathfrak{sl}_n$ and let V be the kernel of the commutator map $[\cdot,\cdot] : \mathfrak{g}\wedge\mathfrak{g} \to \mathfrak{g}$. It is known that V is a simple $\mathfrak{g}$–module [Re].

Theorem 4.6. *The zero weight space $V[0]$ is reducible under $\mathcal{C}_\mathfrak{g}$. In particular, $\mathcal{C}_\mathfrak{g}$ and $Z(U\mathfrak{g})$ do not generate $U\mathfrak{g}^\mathfrak{h}$.*

Proof. Since $\mathfrak{g}\wedge\mathfrak{g} \cong \mathfrak{g}\oplus V$ and the zero weight space of $\mathfrak{g}\wedge\mathfrak{g}$ has a basis given by $h_i\wedge h_j$, $1 \leq i < j \leq n$, and $e_\alpha\wedge f_\alpha$, $\alpha\in R_+$, where $h_1,\dots,h_n$ is a basis of $\mathfrak{h}$, we find that

$$\dim V[0] = \frac{n(n-1)}{2} + |R_+| = \frac{n(n-1)}{2} + \frac{m-3n}{2} \tag{4.11}$$

where $m = \dim(\mathfrak{g}) > 3n$. Thus, $\mathfrak{h}\wedge\mathfrak{h}$ is a proper subspace of the zero weight space of V and it suffices to show that it is invariant under the κ_α. This follows at once from the fact that, for any $t_1, t_2 \in \mathfrak{h}$,

$$\begin{aligned} e_\alpha f_\alpha\, t_1\wedge t_2 &= e_\alpha\left(\alpha(t_1) f_\alpha\wedge t_2 + \alpha(t_2) t_1\wedge f_\alpha\right)\\ &= \alpha(t_1) h_\alpha\wedge t_2 + \alpha(t_2) t_1\wedge h_\alpha \end{aligned} \tag{4.12}$$

■

4.4. **A general reducibility criterion for $V[0]$.** Let Θ be the Chevalley involution of $\mathfrak{g}$ relative to a choice of simple root vectors $e_{\alpha_i}, f_{\alpha_i}$, *i.e.,* the automorphism of $\mathfrak{g}$ defined by

$$\Theta(e_{\alpha_i}) = -f_{\alpha_i} \qquad ,\Theta(f_{\alpha_i}) = -e_{\alpha_i} \qquad \text{and} \qquad \Theta(h_{\alpha_i}) = -h_{\alpha_i} \tag{4.13}$$

If V is a simple, finite–dimensional $\mathfrak{g}$–module, and V^Θ is the module obtained by twisting the action of $\mathfrak{g}$ by Θ, then V^Θ is isomorphic to the dual V^* of V. In particular, if V is self–dual, there exists an involution Θ_V acting on V such that, for any $X\in\mathfrak{g}$,

$$\Theta_V X \Theta_V = \Theta(X) \tag{4.14}$$

Although Θ_V is only unique up to a sign, we shall abusively refer to it as *the* Chevalley involution of V. Since Θ acts as -1 on the Cartan subalgebra $\mathfrak{h}$ and fixes the Casimirs κ_α, Θ_V leaves the zero weight space $V[0]$ invariant and commutes with the action of $\mathcal{C}_\mathfrak{g}$. The following gives a useful criterion to show that Θ_V does not act as a scalar on $V[0]$ and therefore that the latter is reducible under $\mathcal{C}_\mathfrak{g}$.

Proposition 4.7. *Let V be a self–dual $\mathfrak{g}$–module with $V[0]\neq 0$. Let $\mathfrak{r}\subset\mathfrak{g}$ be a reductive subalgebra normalised by $\mathfrak{h}$. Assume that there exists a non–zero vector $v\in V[0]$ lying in a simple $\mathfrak{r}$–module U such that $U\not\cong U^*$. Then, Θ_V does not acts a scalar on $V[0]$ and the latter is reducible under $\mathcal{C}_\mathfrak{g}$.*

Proof. The assumptions imply that Θ leaves $\mathfrak{r}$ invariant and therefore acts as a Chevalley involution on it. Thus, $\Theta_V U\subset V$ is a simple $\mathfrak{r}$–module isomorphic to U^* which has zero intersection with U since $U\not\cong U^*$. In particular, $\Theta_V v$ is not proportional to v ■

We record for later use the following alternative proof of theorem 4.6.

Proposition 4.8. *Let $\mathfrak{g} \ncong \mathfrak{sl}_n$ and let V be the simple, self–dual $\mathfrak{g}$–module $V = \mathrm{Ker}[\cdot,\cdot] \subset \bigwedge^2 \mathfrak{g}$. Then, the Chevalley involution Θ does not act as a scalar on $V[0]$.*

Proof. Θ acts as $+1$ on the subspace $\mathfrak{h} \wedge \mathfrak{h} \subset V[0]$ and as -1 on the span of the vectors $e_\alpha \wedge f_\alpha \in \bigwedge^2 \mathfrak{g}[0]$. The conclusion follows since, as noted in the proof of theorem 4.6, $\mathfrak{h} \wedge \mathfrak{h}$ is a proper subspace of $V[0]$ ■

4.5. **The Gelfand–Zetlin branching rules.** Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$ and $E_{ab}e_c = \delta_{bc}e_a$ the corresponding elementary matrices. Consider the chain of subalgebras

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n \tag{4.15}$$

where each $\mathfrak{gl}_k$ is spanned by the matrices E_{ij}, $1 \leq i, j \leq k$. By the Gelfand–Zetlin branching rules [GZ1, Zh1], the irreducible representation V_λ of $\mathfrak{gl}_k$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$ decomposes under $\mathfrak{gl}_{k-1}$ as

$$\mathrm{res}^{\mathfrak{gl}_{k-1}}_{\mathfrak{gl}_k} V_\lambda = \bigoplus_{\overline{\lambda}} V_{\overline{\lambda}} \tag{4.16}$$

where $V_{\overline{\lambda}}$ is the irreducible $\mathfrak{gl}_{k-1}$–module with highest weight $\overline{\lambda}$ and $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_{k-1}) \in \mathbb{Z}^{k-1}$ ranges over all dominant weights of $\mathfrak{gl}_{k-1}$ satisfying the inequalities

$$\lambda_1 \geq \overline{\lambda}_1 \geq \lambda_2 \geq \cdots \geq \overline{\lambda}_{k-1} \geq \lambda_k \tag{4.17}$$

which we denote by $\lambda \succ \overline{\lambda}$. Since the above decomposition is multiplicity–free, it follows, by restricting in stages from $\mathfrak{gl}_n$ to $\mathfrak{gl}_1$ along (4.15), that any simple $\mathfrak{gl}_n$–module V possesses a basis labelled by Gelfand–Zetlin patterns, *i.e.,* arrays μ of the form

$$\begin{array}{ccccccccc} \mu_1^{(n)} & & \mu_2^{(n)} & & \cdots & & \mu_{n-1}^{(n)} & & \mu_n^{(n)} \\ & \mu_1^{(n-1)} & & & \cdots & & & \mu_{n-1}^{(n-1)} & \\ & & & & & & & & \\ & & \mu_1^{(2)} & & \mu_2^{(2)} & & & & \\ & & & \mu_1^{(1)} & & & & & \end{array} \tag{4.18}$$

where the top row $\mu^{(n)}$ is equal to the highest weight of V and each pair $\mu^{(k)} \in \mathbb{Z}^k, \mu^{(k-1)}\mathbb{Z}^{k-1}$ of consecutive rows satisfies $\mu^{(k)} \succ \mu^{(k-1)}$. Up to a scalar factor, the vector v_μ corresponding to the above pattern is uniquely determined by the requirement that it tranforms under each $\mathfrak{gl}_k \subset \mathfrak{gl}_n$ according to the irreducible representation with highest weight $\mu^{(k)}$. In particular, since the central element $\sum_{i=1}^k E_{ii} \in \mathfrak{gl}_k$ acts in the latter as multiplication by $|\mu^{(k)}| = \sum_{i=1}^k \mu_i^{(k)}$, we find that, for any $1 \leq i \leq n$,

$$E_{ii}v_\mu = \left(|\mu^{(i)}| - |\mu^{(i-1)}|\right) v_\mu \tag{4.19}$$

so that v_μ has weight zero for the action of $\mathfrak{sl}_n$ iff, for any $1 \leq i \leq n$,

$$|\mu^{(i)}| = i|\mu^{(1)}| = i\mu_1^{(1)} \tag{4.20}$$

For later use in §6.1, we shall need the non–vanishing of some of the matrix coefficients for the action of the simple roots vectors of $\mathfrak{gl}_n$ in the above basis. This follows from the explicit formulae for the action of all elementary matrices E_{ij} in a suitably normalised Gelfand–Zetlin basis v_μ which may be found in [GZ1, Zh2].

Theorem 4.9 (Gelfand–Zetlin)**.** *Let μ be a Gelfand–Zetlin pattern, then, for any $1 \leq i \leq n-1$*

$$E_{i\,i+1} v_\mu = \sum_{\mu'} c^i_{\mu,\mu'} v_{\mu'} \tag{4.21}$$

where the sum ranges over all patterns μ' obtained from μ by adding 1 to one of the entries of its ith row and the coefficients $c^i_{\mu,\mu'}$ are non–zero and,

$$E_{i+1\,i} v_\mu = \sum_{\mu'} \widetilde{c}^i_{\mu,\mu'} v_{\mu'} \tag{4.22}$$

where the sum ranges over all patterns μ' obtained from μ by substracting 1 to one of the entries of its ith row and the coefficients $\widetilde{c}^i_{\mu,\mu'}$ are non–zero.

Corollary 4.10. *Let μ be a Gelfand–Zetlin pattern, then*

$$E_{i\,i+1} E_{i+1\,i} v_\mu = \sum_{\mu'} d^i_{\mu,\mu'} v_{\mu'} \tag{4.23}$$

where the sum ranges over all patterns μ' differing from μ by the addition and the substraction of 1 on a pair of (not necessarily distinct) entries of the ith row and $d^i_{\mu,\mu'} \neq 0$ if $\mu \neq \mu'$.

4.6. **The Casimir algebra of $\mathfrak{sl}_n$, $n \geq 4$.**

Theorem 4.11. *If $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 4$, there exists a simple $\mathfrak{g}$–module V such that $V[0]$ is reducible under $\mathcal{C}_\mathfrak{g}$. In particular, $\mathcal{C}_\mathfrak{g}$ and $Z(U\mathfrak{g})$ do not generate $U\mathfrak{g}^\mathfrak{h}$.*

Proof. By proposition 4.7, it suffices to exhibit an irreducible representation V of $\mathfrak{gl}_n$ which is self–dual as $\mathfrak{sl}_n$–module and a Gelfand–Zetlin pattern μ describing a zero–weight vector for $\mathfrak{sl}_n$ in V such that, for some $2 \leq k \leq n-1$, the $\mathfrak{sl}_k$–module U with highest weight $\mu^{(k)} = (\mu_1^{(k)}, \ldots, \mu_k^{(k)})$ is not self–dual. Since the highest weight of U^* is $(-\mu_k^{(k)}, \ldots, -\mu_1^{(k)})$, such a U is self–dual iff the sum $\mu_i^{(k)} + \mu_{k+1-i}^{(k)}$ does not depend upon $i = 1 \ldots k$. The following is a suitable Gelfand–Zetlin pattern μ

$$
\begin{array}{ccccccccccccc}
4 & & 3 & & 2 & & \cdots & & 2 & & 1 & & 0 \\
& \ddots & & \ddots & & \ddots & & \iddots & & \iddots & & \iddots & \\
& & 4 & & 3 & & 2 & & 1 & & 0 & & \\
& & & 4 & & 3 & & 1 & & 0 & & & \\
& & & & 4 & & 1 & & 1 & & & & \\
& & & & & 3 & & 1 & & & & & \\
& & & & & & 2 & & & & & &
\end{array}
\tag{4.24}
$$

since the $\mathfrak{sl}_3$–module with highest weight $(4,1,1)$ isn't self–dual and, for any $n \geq 4$, the $\mathfrak{sl}_n$–module with highest weight

$$
(4,3,\underbrace{2,\ldots,2}_{n-4},1,0)
\tag{4.25}
$$

is self–dual ■

5. A conjecture of Kwon and Lusztig on quantum Weyl groups

5.1. We discuss below some results of Kwon on q–Weyl group actions of Artin's braid group B_n on the zero weight spaces of $U_\hbar\mathfrak{sl}_n$–modules [Kw]. We disprove in particular a conjecture of his and Lusztig's stating the irreducibility of all such representations.

Let $U_\hbar\mathfrak{g}$ be the Drinfeld–Jimbo quantum group corresponding to $\mathfrak{g}$ [Dr1, Ji], which we regard as a Hopf algebra over the ring $\mathbb{C}[\![\hbar]\!]$ of formal power series in the variable $\hbar$. By a *finite–dimensional* representation of $U_\hbar\mathfrak{g}$ we shall mean a $U_\hbar\mathfrak{g}$–module $\mathcal{V}$ which is topologically free and finitely–generated over $\mathbb{C}[\![\hbar]\!]$. The isomorphism class of such a representation is uniquely determined by that of the $\mathfrak{g}$–module $V = \mathcal{V}/\hbar\mathcal{V}$.

Lusztig, and independently Kirillov–Reshetikhin and Soibelman [Lu, KR, So], proved that any such $\mathcal{V}$ carries an action, called the *quantum (or q–)Weyl group action* of the braid group $B_\mathfrak{g}$. Its reduction mod $\hbar$ factors through the Tits extension $\widetilde{W}$ given by the triple exponentials (2.36) with $t_i = 1$. Specifically, this action is given by mapping the generator S_i of $B_\mathfrak{g}$ to the triple q–exponential [Sa]

$$
\begin{aligned}
&\exp_{q_i^{-1}}(q_i^{-1}E_i q_i^{-H_i}) \exp_{q_i^{-1}}(-F_i) \exp_{q_i^{-1}}(q_i^{-1}E_i q_i^{-H_i}) q_i^{H_i(H_i+2)/2} \\
=&\exp_{q_i^{-1}}(-q_i^{-1}F_i q_i^{H_i}) \exp_{q_i^{-1}}(E_i) \exp_{q_i^{-1}}(-q_i^{-1}F_i q_i^{H_i}) q_i^{H_i(H_i+2)/2}
\end{aligned}
\tag{5.1}
$$

where E_i, F_i, H_i are the generators of $U_\hbar\mathfrak{g}$ corresponding to the simple root α_i, $q_i = q^{\langle\alpha_i,\alpha_i\rangle\hbar}$ and the q–exponential is defined by

$$
\exp_q(X) = \sum_{n\geq 0} \frac{q^{n(n-1)/2}}{[n]_q!} X^n
\tag{5.2}
$$

where the q–factorials are given by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q \tag{5.3}$$

Recently, Kwon investigated the q–Weyl group action of Artin's braid group $B_n = B_{\mathfrak{sl}_n}$ on the zero weight space of a simple $U_\hbar\mathfrak{sl}_n$–module $\mathcal{V}$. He gave a general criterion for it to be irreducible [Kw][4] and showed moreover that this criterion holds for all representations of $U_\hbar\mathfrak{sl}_3$. From these findings, he and Lusztig conjectured that the action of B_n on $\mathcal{V}[0]$ is irreducible for any simple $U_\hbar\mathfrak{sl}_n$–module $\mathcal{V}$. We shall prove the following

Theorem 5.1. *The Kwon–Lusztig conjecture is false for any complex, simple Lie algebra $\mathfrak{g}$ not isomorphic to $\mathfrak{sl}_2, \mathfrak{sl}_3$.*

The proof of theorem 5.1 is given in the next subsection and relies on considerations very similar to those of §4.4, namely the use of the quantum Chevalley involution.

5.2. **Classical and Quantum Chevalley involution.** Let $\Theta_\hbar$ be the quantum Chevalley involution, *i.e.,* the algebra automorphism of $U_\hbar\mathfrak{g}$ defined by

$$\Theta_\hbar(E_i) = -F_i, \quad \Theta_\hbar(F_i) = -E_i \quad \text{and} \quad \Theta_\hbar(H_i) = -H_i \tag{5.4}$$

As in the classical case, $\Theta_\hbar$ acts on any self–dual finite–dimensional representation of $U_\hbar\mathfrak{g}$ leaving its zero weight space invariant. Since H_i acts as zero on $\mathcal{V}[0]$, we see from (5.1) that $\Theta_\hbar$ centralises $B_\mathfrak{g}$ on $\mathcal{V}[0]$. Corollary 5.4 below relates the action of $\Theta_\hbar$ on $\mathcal{V}[0]$ to that of the classical Chevalley involution Θ on $V[0]$. We shall need a number of preliminary results. Let

$$\mathcal{U} = 1 + \hbar U\mathfrak{g}[\![\hbar]\!] \tag{5.5}$$

and recall that any element $x \in \mathcal{U}$ is invertible and possesses a unique square root $x^{1/2} \in \mathcal{U}$.

Lemma 5.2. *Let*

$$\mathcal{U}_+ = \{x \in \mathcal{U} | \, \Theta(x) = x\} \quad \text{and} \quad \mathcal{U}_- = \{x \in \mathcal{U} | \, \Theta(x) = x^{-1}\} \tag{5.6}$$

Then, any $x \in \mathcal{U}$ has a unique factorisation as $x = x_+ \cdot x_-$ with $x_\pm \in \mathcal{U}_\pm$.

Proof. Let $x \to x^*$ be the anti–involution of $\mathcal{U}$ defined by $x^* = \Theta(x)^{-1}$. We proceed as in the existence of a polar decomposition. If $x = x_+x_-$ is a factorisation with $x_+^* = x_+^{-1}$ and $x_-^* = x_-$, then

$$x^*x = x_-x_+^{-1}x_+x_- = x_-^2 \tag{5.7}$$

so that $x_- = (x^*x)^{1/2}$ and $x_+ = xx_-^{-1}$ are uniquely determined by x. *Define* now $x_\pm$ by

$$x_- = (x^*x)^{1/2} \quad \text{and} \quad x_+ = xx_-^{-1} = x(x^*x)^{-1/2} \tag{5.8}$$

[4] when coefficients are extended to the field $\mathbb{C}((\hbar))$ of formal Laurent series, which we tacitly assume.

Then, $x = x_+x_-$, $x_-^* = x_-$ and

$$x_+^*x_+ = (x^*x)^{-1/2}x^*x(x^*x)^{-1/2} = 1 \tag{5.9}$$

as claimed ■

Proposition 5.3. *There exists an algebra isomorphism $\Psi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ which is $\mathbb{C}[\![\hbar]\!]$–linear, acts as the identity on $\mathfrak{h}$ and satisfies*

$$\Psi \circ \Theta_\hbar \circ \Psi^{-1} = \Theta \tag{5.10}$$

PROOF. Let $\Phi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ be an algebra isomorphism acting as the identity on $\mathfrak{h}$ [Dr2, Prop. 4.3]. The algebra automorphisms Θ and $\Phi \circ \Theta_\hbar \circ \Phi^{-1}$ of $U\mathfrak{g}[\![\hbar]\!]$ have the same reduction mod $\hbar$. Since $H^1(\mathfrak{g}, U\mathfrak{g}) = 0$, there exists $b \in \mathcal{U} = 1 + \hbar U\mathfrak{g}[\![\hbar]\!]$ such that

$$\Phi \circ \Theta_\hbar \circ \Phi^{-1} = \mathrm{Ad}(b) \circ \Theta \tag{5.11}$$

Note that, since both Θ and $\Phi \circ \Theta_\hbar \circ \Phi^{-1}$ act as -1 on $\mathfrak{h}$, b lies in $\mathcal{U}^\mathfrak{h}$. We wish to find $c \in \mathcal{U}^\mathfrak{h}$ such that $\Psi = \mathrm{Ad}(c) \circ \Phi$ satisfies (5.10). By (5.11), this is equivalent to $\mathrm{Ad}(cb\Theta(c)^{-1}) = 1$, *i.e.,* to

$$cb\Theta(c)^{-1} = z \tag{5.12}$$

for some $z \in \mathcal{Z} = 1 + \hbar Z(U\mathfrak{g})[\![\hbar]\!]$. Using lemma 5.2 to factor b, c and z, the above equation becomes

$$b_+b_- = c^{-1}z\Theta(c) = c_-^{-1}c_+^{-1}zc_+c_-^{-1} = z_+z_-c_-^{-2} \tag{5.13}$$

Since $z_\pm \in \mathcal{Z}$ and $z_-c_-^{-2} \in \mathcal{U}_-$, the solvability of this equation is therefore equivalent to $b_+ \in \mathcal{Z}$. To see that this holds, note that $\mathrm{Ad}(b) \circ \Theta$ is an involution by (5.11). This yields $\mathrm{Ad}(b\Theta(b)) = 1$ and therefore

$$b\Theta(b) = \zeta \in \mathcal{Z} \tag{5.14}$$

Writing this in components, yields

$$b_+b_- = \zeta\Theta(b)^{-1} = \zeta_+\zeta_-b_-b_+^{-1} = \zeta_+b_+^{-1} \cdot \zeta_-b_+b_-b_+^{-1} \tag{5.15}$$

Since $\zeta_+b_+^{-1} \in \mathcal{U}_+$ and $\zeta_-b_+b_-b_+^{-1} \in \mathcal{U}_-$, this implies, by uniqueness of factorisation,

$$b_+ = \zeta_+b_+^{-1} \tag{5.16}$$

whence $b_+ = \zeta_+^{1/2} \in \mathcal{Z}$ as claimed ■

Corollary 5.4. *Let $\mathcal{V}$ be a self–dual, finite–dimensional $U_\hbar\mathfrak{g}$–module and let $V = \mathcal{V}/\hbar\mathcal{V}$ be its reduction mod $\hbar$. Then, $\mathcal{V}$ and $V[\![\hbar]\!]$ are isomorphic as $\mathfrak{h} \rtimes \mathbb{Z}_2$–modules where the generator of $\mathbb{Z}_2$ acts as $\Theta_\hbar$ on $\mathcal{V}$ and as the classical Chevalley involution Θ on V. In particular, $\Theta_\hbar$ acts as a scalar on $\mathcal{V}[0]$ iff it acts as a scalar on $V[0]$.*

PROOF. The isomorphism $\Psi : U\mathfrak{g}[\![\hbar]\!] \to U\mathfrak{g}[\![\hbar]\!]$ given by proposition 5.3 endows $V[\![\hbar]\!]$ with the structure of a $U_\hbar\mathfrak{g}$–module such that the action of $\mathfrak{h} \subset U_\hbar\mathfrak{g}$ coincides with that of $\mathfrak{h} \subset \mathfrak{g}$. Since $\mathcal{V}$ and $V[\![\hbar]\!]$ have the same reduction mod $\hbar$, they are isomorphic as $U_\hbar\mathfrak{g}$, and therefore $\mathfrak{h}$–modules.

Equation (5.10) then guarantees that, under this isomorphism, the Chevalley involution of $\mathcal{V}$ is mapped to the Chevalley involution of V ∎

Proof of theorem 5.1. It follows from proposition 4.8 and the proof of theorem 4.11 that if $\mathfrak{g} \not\cong \mathfrak{sl}_2, \mathfrak{sl}_3$, there exists a simple, self–dual $\mathfrak{g}$–module V such that Θ does not act as a scalar on $V[0]$. By corollary 5.4, $\Theta_\hbar$ does not act as scalar on $V[\![\hbar]\!]$ and the latter is reducible under the q–Weyl group action of $B_\mathfrak{g}$ ∎

We mention in passing the following q–analogue of proposition 3.13

Proposition 5.5. *Let $\mathcal{V}$ be a finite–dimensional representation of $U_\hbar\mathfrak{g}$ with non–trivial zero weight space $\mathcal{V}[0]$. Then, $\mathcal{V}[0]$ is irreducible under $B_\mathfrak{g}$ iff it is irreducible under $P_\mathfrak{g}$.*

Proof. By proposition 1.2.1 of [Sa], S_i acts on the zero–weight space of the indecomposable $U_\hbar\mathfrak{sl}_2^{\alpha_i}$–module of dimension $2n+1$ as multiplication by $(-1)^n q_i^{n(n+1)}$. It follows that the image of S_i in $\mathrm{End}(\mathcal{V}[0])$ is a polynomial in the image of $S_i^2 \in P_\mathfrak{g}$ whence the conclusion ∎

5.3. **A Kohno–Drinfeld theorem for q–Weyl groups.** Let $\mathcal{V}$ be a finite–dimensional representation of $U_\hbar\mathfrak{g}$ and $V = \mathcal{V}/\hbar\mathcal{V}$ its reduction mod $\hbar$. It was conjectured in [TL2], by analogy with the Kohno–Drinfeld theorem, that the q–Weyl group action of $B_\mathfrak{g}$ on $\mathcal{V}$ is equivalent to to the monodromy action of $B_\mathfrak{g}$ on ∇_κ studied in the present paper. This conjecture is proved in [TL3] for a number of pairs $(\mathfrak{g}, V)$ including vector representations of classical Lie algebras and adjoint representations of all simple Lie algebras and in [TL2] for all representations of $\mathfrak{g} = \mathfrak{sl}_n$. More precisely,

Theorem 5.6 ([TL2])**.** *Assume that $\mathfrak{g} \cong \mathfrak{sl}_n$. Let μ be a weight of V and*

$$V^\mu = \bigoplus_{\nu \in W\mu} V[\nu] \tag{5.17}$$

the direct sum of the weight spaces of V corresponding to the Weyl group orbit of μ. Let $\sigma(B_\mathfrak{g}) \subset N(T)$ be a Tits extension and

$$\rho^\sigma : B_\mathfrak{g} \to GL(V^\mu[\![h]\!]) \tag{5.18}$$

the corresponding monodromy representation defined by proposition 2.3 by regarding h as a formal variable. Let $\pi_W : B_\mathfrak{g} \to GL(\mathcal{V}^\mu)$ be the q–Weyl group action. Then, ρ_h and π_W are equivalent for $\hbar = 2\pi i h$.

Combining the above theorem with corollary 3.14, we obtain the following

Proposition 5.7. *Let $\mathcal{V}$ be a finite–dimensional representation of $U_\hbar\mathfrak{sl}_n$ and set $V = \mathcal{V}/\hbar\mathcal{V}$. The following statements are equivalent*

(i) *$\mathcal{V}[0]$ is irreducible under the q–Weyl group action of B_n.*
(ii) *$\mathcal{V}[0]$ is irreducible under the q–Weyl group action of P_n.*
(iii) *$V[0]$ is irreducible under the Casimir algebra $\mathcal{C}_{\mathfrak{sl}_n}$.*

In particular, by theorem 4.1, the q–Weyl group action of P_3 on the zero weight space $\mathcal{V}[0]$ of a $U_\hbar\mathfrak{sl}_3$–module is always irreducible, a slight refinement of a result of Kwon asserting the irreducibility of $\mathcal{V}[0]$ under the full braid group B_3.

6. Irreducible representations of $\mathcal{C}_\mathfrak{g}$

The aim of this section is to show that the connection ∇_κ yields irreducible monodromy representations of $B_\mathfrak{g}$ of arbitrarily large dimension. For $\mathfrak{g}$ classical, we show for example in subsections 6.1–6.3 that, with V the adjoint representation if $\mathfrak{g} \cong \mathfrak{sl}_n$ and the defining vector one otherwise, the weight spaces of all Cartan powers of V are irreducible under the Casimir algebra $\mathcal{C}_\mathfrak{g}$. For $\mathfrak{g}$ of exceptional type, we obtain in §6.4 a slightly weaker result : for every $p \in \mathbb{N}$, the zero weight space of the pth Cartan power of $\mathrm{ad}(\mathfrak{g})$ has a subspace K_p which is irreducible under $\mathcal{C}_\mathfrak{g}$ and such that $\lim_{p\to+\infty} \dim K_p = +\infty$.

6.1. Irreducible representations of $\mathcal{C}_{\mathfrak{sl}_n}$.

Theorem 6.1. *For any $p, q \in \mathbb{N}$, the action of $\mathcal{C}_{\mathfrak{sl}_n}$ on the weight spaces of the simple $\mathfrak{sl}_n$–module of highest weight $(p, 0, \ldots, 0, -q)$ is irreducible.*

Proof. For any $2 \leq k \leq n$ and $a, b \in \mathbb{N}$, set $\lambda^{(k)}_{a,b} = (a, 0, \ldots, 0, -b) \in \mathbb{Z}^k$ so that the Gelfand–Zetlin basis of the simple $\mathfrak{gl}_n$–module V with highest weight $\lambda^{(n)}_{p,q}$ is parametrised by patterns of the form

$$\lambda = \begin{pmatrix} \lambda^{(n)}_{p_n,q_n} \\ \vdots \\ \lambda^{(k)}_{p_k,q_k} \\ \vdots \\ \lambda^{(2)}_{p_2,q_2} \\ r \end{pmatrix} \tag{6.1}$$

where the $p_k = p_k(\lambda)$, $q_k = q_k(\lambda)$ and $r = r(\lambda)$ are integers satisfying

$$p = p_n \geq p_{n-1} \geq \cdots \geq p_2 \geq 0 \tag{6.2}$$

$$q = q_n \geq q_{n-1} \geq \cdots \geq q_2 \geq 0 \tag{6.3}$$

$$p_2 \geq r \geq -q_2 \tag{6.4}$$

The vectors of a given weight $\mu = (\mu_1, \ldots, \mu_n)$ correspond to patterns satisfying in addition

$$r = \mu_1 \quad \text{and, for any } 2 \leq k \leq n, \quad p_k - q_k = M_k \tag{6.5}$$

where $M_k = \sum_{i=1}^k \mu_k$. We claim that the commuting Casimir operators $C_{\mathfrak{gl}_k}$, $2 \leq k \leq n$, have joint simple spectrum on the weight space $V[\mu]$, with corresponding diagonal basis given by Gelfand–Zetlin vectors. Indeed, $C_{\mathfrak{gl}_k}$ acts on $v_\lambda \in V[\mu]$, with λ of the form (6.1), as multiplication by

$$\begin{aligned}\langle \lambda^{(k)}_{p_k,q_k}, \lambda^{(k)}_{p_k,q_k} + 2\rho^{(k)}\rangle &= p_k^2 + q_k^2 + (k-1)(p_k+q_k)\\ &= 2q_k^2 + 2q_k(M_k+k-1) + M_k(M_k+k-1)\end{aligned} \tag{6.6}$$

where $2\rho^{(k)} = \sum_{i=1}^k \theta_i(k-2i+1)$ is the sum of the positive roots of $\mathfrak{gl}_k$. Since $2q_k + M_k = q_k + p_k \geq 0$ and the the right–hand side of (6.6) is a parabola with vertex at

$$q_k^0 = -\frac{1}{2}(M_k+k-1) < -\frac{M_k}{2} \tag{6.7}$$

the $C_{\mathfrak{gl}_k}$–eigenvalue of a pattern λ of form (6.1) and weight μ determines $p_k(\lambda)$ and $q_k(\lambda)$ uniquely as claimed.

We claim now that if $K \subseteq V[\mu]$ is a non–zero subspace invariant under $\mathcal{C}_{\mathfrak{sl}_n}$, then $K = V[\mu]$. To see this, it suffices to show that, for any given pattern λ of the form (6.1) and weight μ, there exists a Gelfand–Zetlin vector lying in K such that the corresponding pattern has the same $(n-1)$ row $\lambda^{(n-1)}_{p_{n-1},q_{n-1}}$ as λ, for then a descending induction on n shows that K contains all Gelfand–Zetlin vectors of weight μ. Since the Casimirs $C_{\mathfrak{gl}_k}$ have simple spectrum on $V[\mu]$, K contains at least one Gelfand–Zetlin vector $v_{\lambda'}$. Let $\lambda^{(n-1)}_{p'_{n-1},q'_{n-1}}$ be the $n-1$ row of the corresponding pattern. If $p'_{n-1} = p_{n-1}$, then,

$$q'_{n-1} = -M_{n-1} + p'_{n-1} = -M_{n-1} + p_{n-1} = q_{n-1} \tag{6.8}$$

and we are done. If $p'_{n-1} < p_{n-1}(\lambda)$, we may further assume that $p'_{n-1} = \max_{\widetilde{\lambda}} p_{n-1}(\widetilde{\lambda})$, where the maximum is taken over all patterns $\widetilde{\lambda}$ such that $v_{\widetilde{\lambda}} \in K$ and $p_{n-1}(\widetilde{\lambda}) \leq p_{n-1}$. Note then that

$$q'_{n-1} = -\sum_{i=1}^{n-1}\mu_i + p'_{n-1} < -\sum_{i=1}^{n-1}\mu_i + p_{n-1} = q_{n-1} \leq q_n \tag{6.9}$$

It therefore follows by corollary 4.10 that

$$\begin{aligned}\kappa_{\theta_{n-1}-\theta_n} v_{\lambda'} &= (2E_{n-1,n}E_{n,n-1} + (E_{n-1,n-1} - E_{n,n}))v_{\lambda'}\\ &= a v_{\lambda' + \varepsilon_1^{(n-1)} - \varepsilon_{n-1}^{(n-1)}} + b v_{\lambda' - \varepsilon_1^{(n-1)} + \varepsilon_{n-1}^{(n-1)}} + c v_{\lambda'}\end{aligned} \tag{6.10}$$

for some $a, b, c \in \mathbb{C}$ with $a \neq 0$. Hence, K contains $v_{\lambda' + \varepsilon_1^{(n-1)} - \varepsilon_{n-1}^{(n-1)}}$ in contradiction with the maximality of p'_{n-1}. The case $p'_{n-1} > p_{n-1}$ follows similarly ■

Remark. The same method of proof shows for example that the Casimir algebra $C_{\mathfrak{sl}_n}$ acts irreducibily on all weight spaces of the irreducible representations with highest weight of the form $(p, q, 0, \dots, 0)$ where p, q are any integers satisfying $p \geq q \geq 0$. More generally, one can show that if the commuting Casimirs $C_{\mathfrak{gl}_k}$, $k = 2 \dots n$, have joint simple spectrum on the weight space $V[\mu]$ of a simple $\mathfrak{sl}_n$–module V, then $\mathcal{C}_{\mathfrak{sl}_n}$ acts irreducibly on $V[\mu]$. The proof is similar to that of theorem 6.1 but somewhat more involved technically and will be given in a future publication.

6.2. **Irreducible representations of $\mathcal{C}_{\mathfrak{so}_m}$.** Let $\mathfrak{g} = \mathfrak{so}_m$, with $m = 2n$ or $m = 2n+1$, and identify $\mathfrak{h}^*$ and $\mathbb{C}^n$ with basis $\theta_1, \ldots, \theta_n$ so that the roots of $\mathfrak{g}$ are the $\theta_i \pm \theta_j$, $1 \leq i \neq j \leq n$ if $m = 2n$ and $\theta_i \pm \theta_j$, $1 \leq i \neq j \leq n$ and $\pm\theta_i$, $1 \leq i \leq n$ if $m = 2n+1$. Recall that the defining representation V of $\mathfrak{g}$ has highest weight θ_1. The aim of this subsection is to prove the following

Theorem 6.2. *For any $p \in \mathbb{N}$, the action of $\mathcal{C}_{\mathfrak{so}_m}$ on the weight spaces of the simple $\mathfrak{so}_m$–module $V_{p\theta_1}$ of highest weight $p\theta_1$ is irreducible.*

As a corollary, the monodromy action of $P_{\mathfrak{so}_m}$ on the weight spaces of $V_{p\theta_1}$, and of $B_{\mathfrak{so}_m}$ on its zero weight space, is generically irreducible. The proof of theorem 6.2 for the case $m = 2n$ is given in §6.2.1 and follows readily from theorem 6.1 and the fact that $V_{p\theta_1}$ decomposes, when restricted to the equal rank subalgebra $\mathfrak{gl}_n \subset \mathfrak{so}_{2n}$, as a direct sum of irreducible representations with highest weight of the form $(p-q, 0, \ldots, 0, -q)$, with $q = 0 \ldots p$, in such a way that each $\mathfrak{so}_{2n}$–weight space of $V_{p\theta_1}$ is contained in only one of these. The case $m = 2n+1$ requires a little more work since the restriction to $\mathfrak{so}_{2n}$ of the $\mathfrak{so}_{2n+1}$–module $V_{p\theta_1}$ is multiplicity–free but the corresponding weight spaces do not possess such a nice property. Still, each decomposes as a sum of weight spaces for the simple $\mathfrak{so}_{2n}$–summands which are readily seen to be inequivalent, and by the previous discussion irreducible, representations of $\mathcal{C}_{\mathfrak{so}_{2n}}$. The proof is then completed by showing that the short root Casimirs κ_{θ_i}, $i = 1 \ldots n$ of $\mathfrak{so}_{2n+1}$ define non–zero maps between these weight spaces. This fact requires an explicit description of the operators κ_{θ_i} which is obtained in §6.2.2 by realising $V_{p\theta_1}$ as the space of homogeneous harmonic functions on V^* of degree p.

6.2.1. *Even orthogonal Lie algebras.* Let $V \cong \mathbb{C}^{2n}$ be an even–dimensional complex vector space endowed with a non–degenerate, symmetric, bilinear form $(\cdot, \cdot)$. Split V as the direct sum $U \oplus U^*$ of two maximal isotropic subspaces for $(\cdot, \cdot)$ which we identify with each other's dual and consider the equal rank embedding $\mathfrak{gl}(U) \to \mathfrak{so}(V)$ given by $X \to X \oplus -X^t$. Under the corresponding restriction, the Koike–Terada branching rules [KT] yield[5]

$$\operatorname{res}^{GL(U)}_{SO(V)} V_{p\theta_1} = \bigoplus_{q=0}^{p} V_{(p-q)\theta_1 - q\theta_n} \tag{6.11}$$

Let $\mu = \mu_1\theta_1 + \cdots + \mu_n\theta_n$ be a weight of $V_{p\theta_1}$. Since $1 \in \mathfrak{gl}(U)$ acts on the weight space $V_{p\theta_1}[\mu]$ as multiplication by $|\mu| = \mu_1 + \cdots + \mu_n$ and on $V_{(p-q)\theta_1 - q\theta_n}$ as multiplication by $p - 2q$, we deduce from (6.11) that

$$V_{p\theta_1}[\mu] = V_{(p+|\mu|)/2\cdot\theta_1 - (p-|\mu|)/2\cdot\theta_n}[\mu] \tag{6.12}$$

[5]these rules are reviewed in more detail in subsection 7.4.

Theorem 6.2 for $m = 2n$ now follows from theorem 6.1 ■

6.2.2. *Odd orthogonal Lie algebras.* Let now $V \cong \mathbb{C}^{2n+1}$, $n \geq 2$, be an odd–dimensional orthogonal vector space with bilinear form $(\cdot, \cdot)$. Choose a basis $e_{-n}, \ldots, e_{-1}, e_0, e_1, \ldots, e_n$ such that $(e_i, e_j) = \delta_{i,-j}$ and denote the corresponding coordinate functions by $x_{-n}, \ldots, x_{-1}, z, x_1, \ldots, x_n$. Let $\overline{V} \subset V$ be the orthogonal complement of e_0 and

$$\mathfrak{so}_{2n} \cong \mathfrak{so}(\overline{V}) \subset \mathfrak{so}(V) \cong \mathfrak{so}_{2n+1} \tag{6.13}$$

the corresponding orthogonal Lie algebra. Identifying $\overline{V}$ with $\overline{V}^*$ and V with V^* using $(\cdot, \cdot)$, we denote by $S^p\overline{V}$ and S^pV the spaces of homogeneous polynomials of degree p on $\overline{V}$ and V respectively, and by $\mathcal{H}^p\overline{V}$, $\mathcal{H}^pV$ their subspaces of harmonic functions for the Laplacians

$$\overline{\Delta} = 2\sum_{i=1}^{n} \partial_i \partial_{-i} \quad \text{and} \quad \Delta = \overline{\Delta} + \partial_z^2 \tag{6.14}$$

Let $\overline{\rho} = 2\sum_{i=1}^n x_i x_{-i} \in S^2\overline{V}$ be the squared norm function on $\overline{V}$.

Proposition 6.3.

(i) *Any $f \in S^pV$ may be uniquely written as*

$$f = \sum_{\substack{k,l \geq 0 \\ 2k+l \leq p}} h_{k,l} \overline{\rho}^k z^l \tag{6.15}$$

where $h_{k,l} \in \mathcal{H}^{p-(2k+l)}\overline{V}$.

(ii) *The map $f \longrightarrow (h_{0,p}, h_{0,p-1}, \ldots, h_{0,1})$ restricts to an $\mathfrak{so}_{2n}$–equivariant isomorphism*

$$\mathcal{H}^pV \cong \bigoplus_{q=0}^{p} \mathcal{H}^q\overline{V} \tag{6.16}$$

(iii) *Under this isomorphism, the action of the short root Casimirs κ_{θ_i} of $\mathfrak{so}_{2n+1}$ has homogeneous components $\kappa^d_{\theta_i}$ of degrees $d = -2, 0, +2$ only with respect to the $\mathbb{N}$–grading on $\mathcal{H}^pV$ given by* (6.16). *Moreover, if $f \in \mathcal{H}^q\overline{V}$*

$$\kappa^{-2}_{\theta_i} f = -2\partial_i \partial_{-i} f \tag{6.17}$$

$$\kappa^{+2}_{\theta_i} f = \begin{cases} -2(\epsilon+2)(\epsilon+1)P_{\mathcal{H}}(x_i x_{-i} f) & \text{if } q \leq p-2 \\ 0 & \text{otherwise} \end{cases} \tag{6.18}$$

where $P_{\mathcal{H}}(g)$ denotes the projection of $g \in S\overline{V}$ onto the subspace of harmonic functions and ϵ is 0 or 1 according to whether $p - q$ is even or odd.

Proof. (i) follows by expanding $f = \sum_{l=0}^p f_l z^l$ in powers of z and then each homogeneous component $f_l \in S^{p-l}\overline{V}$ in terms of spherical harmonics. Such expansions are well–known to be unique (see, *e.g.*, [Ho, §4]).

(ii) Since, for any $g \in S\overline{V}$,

$$\overline{\Delta}(\overline{\rho}g) = 4ng + 4\overline{E}g + \overline{\rho}\overline{\Delta}g \tag{6.19}$$

where $\overline{E} = \sum_{i=1}^n x_i\partial_{-i} + x_{-i}\partial_i$ is the Euler operator giving the $\mathbb{N}$–grading on $S\overline{V}$, we get, for any $k \in \mathbb{N}$,

$$\overline{\Delta}(\overline{\rho}^k g) = 4k(n+k-1)\overline{\rho}^{k-1}g + 4k\overline{\rho}^{k-1}\overline{E}g + \overline{\rho}^k\overline{\Delta}g \tag{6.20}$$

so that if g is harmonic of degree q, then $\overline{\Delta}(\overline{\rho}^k g) = 4k(n+k-1+q)\overline{\rho}^{k-1}g$. It readily follows that the function $f = \sum_{k,l} h_{k,l}\overline{\rho}^k z^l \in S^pV$ is harmonic if, and only if, for any $k \geq 1$ and $l \geq 0$

$$4k(n-1+p-k-l)h_{k,l} = -(l+1)(l+2)h_{k-1,l+2} \tag{6.21}$$

This shows that the harmonic coefficients $h_{k,l}$ of f are uniquely determined by $(h_{0,p},\ldots,h_{0,0})$ and, conversely, that any sequence $(g_0,\ldots,g_p)$ with $g_q \in \mathcal{H}^q\overline{V}$ determines recursively the harmonic coefficients of a harmonic function f with $h_{0,q} = g_{p-q}$ for any $q = 0\ldots p$. In fact, for $m \geq 2$ and $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$

$$h_{k,m-2k} = \frac{(-1)^k}{4^k k!}\frac{m!(n+p-m-1)!}{(m-2k)!(n+p-m+k-1)!}\,h_{0,m} \tag{6.22}$$

(iii) Identifying $\mathfrak{so}_{2n+1}$ with $V \wedge V$, the root vectors e_{θ_i} and f_{θ_i} may be chosen as $\sqrt{2}\cdot e_0 \wedge e_i$ and $\sqrt{2}\cdot e_{-i} \wedge e_0$ respectively [FH] and therefore act on SV by

$$e_{\theta_i} = \sqrt{2}(z\partial_{-i} - x_i\partial_z) \tag{6.23}$$

$$f_{\theta_i} = \sqrt{2}(x_{-i}\partial_z - z\partial_i) \tag{6.24}$$

Thus, $\kappa_{\theta_i} = 1/2(e_{\theta_i}f_{\theta_i} + f_{\theta_i}e_{\theta_i})$ acts on SV by

$$-2z^2\partial_i\partial_{-i} + 2(1 + x_{-i}\partial_{-i} + x_i\partial_i)z\partial_z + (x_i\partial_i + x_{-i}\partial_{-i}) - 2x_ix_{-i}\partial_z^2 \tag{6.25}$$

and therefore, when restricted to $\mathcal{H}^pV$, possesses only homogeneous components $\kappa_{\theta_i}^d$ of even degree d with $|d| \leq 2$. Let now $h \in \mathcal{H}^q\overline{V}$ be a harmonic function of degree q and regard it as an element of $\mathcal{H}^pV$ of the form $\sum_{2k+l=p-q} h_{k,l}\overline{\rho}^k z^l$ where the $h_{k,l}$ are determined by (6.22) with $h_{0,p-q} = h$. Then, up to terms of strictly lower order in z, $\kappa_{\theta_i}h = -2\partial_i\partial_{-i}hz^{p-q+2}$ which yields (6.17) since $\partial_i\partial_{-i}h$ is harmonic. Similarly, let $\epsilon = p - q - 2\lfloor\frac{p-q}{2}\rfloor \in \{0,1\}$ so that the expansion of h is of the form

$$h_{0,p-q}z^{p-q} + \cdots + h_{\lfloor\frac{p-q}{2}\rfloor-1,\epsilon+2}\,\overline{\rho}^{\lfloor\frac{p-q}{2}\rfloor-1}z^{\epsilon+2} + h_{\lfloor\frac{p-q}{2}\rfloor,\epsilon}\,\overline{\rho}^{\lfloor\frac{p-q}{2}\rfloor}z^{\epsilon} \tag{6.26}$$

Then, up to terms of strictly higher order in $\overline{\rho}$, the coefficient of z^ϵ in the expansion of $\kappa_{\theta_i}h$ is equal to

$$-2(\epsilon+2)(\epsilon+1)P_{\mathcal{H}}(x_ix_{-i}h_{\lfloor\frac{p-q}{2}\rfloor-1,\epsilon+2})\overline{\rho}^{\lfloor\frac{p-q}{2}\rfloor-1} \tag{6.27}$$

Since this should be of the form $\widetilde{h}_{\lfloor\frac{p-q}{2}\rfloor-1,\epsilon}\overline{\rho}^{\lfloor\frac{p-q}{2}\rfloor-1}$ where $\widetilde{h}_{\lfloor\frac{p-q}{2}\rfloor-1,\epsilon}$ is a harmonic coefficient in the expansion of some $\widetilde{h} \in \mathcal{H}^{q-2}\overline{V}$, it readily follows that $\widetilde{h} = -2(\epsilon+2)(\epsilon+1)P_{\mathcal{H}}(x_ix_{-i}h)$ and therefore that (6.18) holds ■

Lemma 6.4. *Let $\mu = \mu_1\theta_1 + \cdots + \mu_n\theta_n$ be a weight of $\mathcal{H}^q\overline{V}$ and set $|\mu| = \sum_i |\mu_i|$. Then,*

(i) $|\mu| \leq q$ *and* $|\mu| = q$ *mod 2.*

(ii) μ *is also a weight of* $\mathcal{H}^{q+2}\overline{V}$ *and, provided* $q \leq p-2$*, any* $\kappa^{+2}_{\theta_i}$ *restricts to a non–zero map* $\mathcal{H}^q\overline{V}[\mu] \longrightarrow \mathcal{H}^{q+2}\overline{V}[\mu]$.

(iii) *If* $|\mu| < q$*, then* μ *is also a weight of* $\mathcal{H}^{q-2}\overline{V}$ *and, for some* $1 \leq i \leq n$, $\kappa^{-2}_{\theta_i}$ *restricts to a non–zero map* $\mathcal{H}^q\overline{V}[\mu] \longrightarrow \mathcal{H}^{q-2}\overline{V}[\mu]$.

Proof. (i) A function $f \in S\overline{V}$ is of weight μ iff it is a linear combination of monomials of the form

$$f = \sum_{m_{\pm 1},\ldots,m_{\pm n}} \lambda_{m_1,m_{-1},\ldots,m_n,m_{-n}} x_1^{m_1} x_{-1}^{m_{-1}} \cdots x_n^{m_n} x_{-n}^{m_{-n}} \tag{6.28}$$

where the $m_{\pm i} \in \mathbb{N}$ satisfy $m_i - m_{-i} = \mu_i$ for any $1 \leq i \leq n$. If f is homogeneous of degree q, then, for each of the monomials involved,

$$q = \sum_{i=1}^n (m_i + m_{-i}) = \sum_{i=1}^n (2\min(m_i, m_{-i}) + |\mu_i|) \tag{6.29}$$

(ii) It is a simple consequence of the decomposition into spherical harmonics that a function $f \in S\overline{V}$ has a zero harmonic projection only if f is divisible by $\overline{\rho}$. Since $\dim \overline{V} \geq 4$, $\overline{\rho}$ is an irreducible polynomial and it follows from (6.18) that, if $f \in \mathcal{H}^q\overline{V}$ with $q \leq p-2$, $\kappa^{+2}_{\theta_i} f$ is zero iff f itself is divisible by $\overline{\rho}$. Since f is harmonic however, $f = P_{\mathcal{H}}(f) = 0$. Thus, if $q \leq p-2$, $\kappa^{+2}_{\theta_i}$ restricts to an injective map on $\mathcal{H}^q\overline{V}[\mu]$ the image of which lies in $\mathcal{H}^{q+2}\overline{V}[\mu]$ since κ_{θ_i} is of weight 0.

(iii) Expanding $f \in S^q\overline{V}[\mu]$ as in (6.28) shows that $\partial_i\partial_{-i} f = 0$ iff only monomials with $m_i m_{-i} = 0$ are involved. Thus, if f is harmonic and lies in the joint kernel of all $\kappa^{-2}_{\theta_i}$, it follows from (6.17) and (6.29) that $q = |\mu|$. As a consequence, if $|\mu| < q$, at least one $\kappa^{+2}_{\theta_i}$ restricts to a non–zero map on $\mathcal{H}^q\overline{V}[\mu]$ which, because κ_{θ_i} is of weight zero, maps into $\mathcal{H}^{q-2}\overline{V}[\mu]$ ■

Proof of theorem 6.2 for $m = 2n+1$. Let μ be a weight of $V_{p\theta_1} \cong \mathcal{H}^p V$. Regarding μ as an $\mathfrak{so}_{2n}$–weight and using the decomposition (6.16) and lemma 6.4, we find

$$\mathcal{H}^p V[\mu] = \bigoplus_{\substack{|\mu| \leq q \leq p \\ q = |\mu| \mod 2}} \mathcal{H}^q\overline{V}[\mu] \tag{6.30}$$

By §6.2.1, the above summands are irreducible representation of $\mathcal{C}_{\mathfrak{so}_{2n}}$ which are moreover inequivalent since the Casimir operator of $\mathfrak{so}_{2n}$ acts on $\mathcal{H}^q\overline{V} \cong V_{q\theta_1}$ as multiplication by $q(q + 2(n-1))$. Thus, if $\{0\} \neq K \subseteq \mathcal{H}^p V[\mu]$ is a subspace invariant under $\mathcal{C}_{\mathfrak{so}_{2n+1}}$ it must contain one of the $\mathcal{H}^q\overline{V}$ appearing in (6.30). Applying (ii) and (iii) of lemma 6.4, we see however that K contains them all and therefore that $K = \mathcal{H}^p V[\mu]$ ■

6.3. **Irreducible representations of $\mathcal{C}_{\mathfrak{sp}_{2n}}$.** Let $V \cong \mathbb{C}^{2n}$ be a symplectic vector space and $\mathfrak{sp}(V) \cong \mathfrak{sp}_{2n}$ the corresponding symplectic Lie algebra. Let $\langle \cdot, \cdot \rangle$ be the symplectic form on V and choose a basis $e_{\pm 1}, \ldots, e_{\pm n}$ of V satisfying $\langle e_i, e_j \rangle = \text{sign}(i)\delta_{i+j,0}$. Consider as Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sp}(V)$ the span of the diagonal matrices $D_i = E_{ii} - E_{-i,-i}$, $i = 1 \ldots n$, where $E_{ab}e_c = \delta_{bc}e_a$ are the elementary matrices in the basis $e_{\pm i}$ and let $\{\theta_i\}$ be the basis of $\mathfrak{h}^* \cong \mathbb{C}^n$ dual to $\{D_i\}$. Then, the long roots of $\mathfrak{sp}(V)$ are $\pm 2\theta_i$, $1 \le i \le n$ and the short ones $\theta_i \pm \theta_j$, $1 \le i \ne j \le n$. Note that, as a simple $\mathfrak{sp}_{2n}$–module, V has highest weight θ_1.

Theorem 6.5. *For any $p \in \mathbb{N}$, the action of $\mathcal{C}_{\mathfrak{sp}_{2n}}$ on the weight spaces of the simple $\mathfrak{sp}_{2n}$–module $V_{p\theta_1}$ of highest weight $p\theta_1$ is irreducible.*

Proof. We proceed, in spirit, as in the proof of theorem 6.1 for $\mathfrak{g} = \mathfrak{sl}_n$ with the Gelfand–Zetlin Casimirs $C_{\mathfrak{gl}_k}$, $k = 1 \ldots n$, replaced by the (commuting) long root Casimirs $\kappa_{2\theta_i}$. We claim that the operators $\kappa_{2\theta_i}$ have joint simple spectrum on any weight space of $V_{p\theta_1}$. To see this, realise $V_{p\theta_1}$ as the space S^pV of homogeneous functions of degree p on V^*, which we identify with V by means of the symplectic form $\langle \cdot, \cdot \rangle$. Let $x_{\pm 1}, \ldots, x_{\pm n}$ be the coordinate functions corresponding to $e_{\pm 1}, \ldots, e_{\pm n}$ and consider the weight basis of S^pV given by the monomials

$$x^{\alpha,\beta} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_{-1}^{\beta_1} \cdots x_{-n}^{\beta_n} \tag{6.31}$$

where $\alpha, \beta \in \mathbb{N}^n$ are multi–indices satisfying $|\alpha| + |\beta| = p$. The $\mathfrak{sl}_2$–triple $e_{2\theta_i}, f_{2\theta_i}$ and $h_{2\theta_i}$ acts on V as $E_{i,-i}$, $E_{-i,i}$ and $E_{ii} - E_{-i-i}$ respectively [FH, §16.1], and therefore on SV as

$$e_{2\theta_i} = x_i\partial_{-i}, \quad f_{2\theta_i} = x_{-i}\partial_i \quad \text{and} \quad h_{2\theta_i} = x_i\partial_i - x_{-i}\partial_{-i} \tag{6.32}$$

so that

$$e_{2\theta_i}\, x^{\alpha,\beta} = \beta_i x^{\alpha+\varepsilon_i,\beta-\varepsilon_i} \tag{6.33}$$

$$f_{2\theta_i}\, x^{\alpha,\beta} = \alpha_i x^{\alpha-\varepsilon_i,\beta+\varepsilon_i} \tag{6.34}$$

$$h_{2\theta_i}\, x^{\alpha,\beta} = (\alpha_i - \beta_i) x^{\alpha,\beta} \tag{6.35}$$

where ε_i is the ith basis vector of $\mathbb{Z}^n$. It follows that the monomial $x^{\alpha,\beta}$ lies in the $(\alpha_i - \beta_i)$ weight space of an irreducible $\mathfrak{sl}_2^{2\theta_i}$ of highest weight $\alpha_i + \beta_i$ and therefore that

$$C_{2\theta_i}\, x^{\alpha,\beta} = \frac{1}{2}(\alpha_i + \beta_i)(\alpha_i + \beta_i + 2) x^{\alpha,\beta} \tag{6.36}$$

For any given $\mu \in \mathbb{Z}^n$, the weight space $S^pV[\mu]$ is spanned by all $x^{\alpha,\beta}$ with $\alpha - \beta = \mu$. Since the function $t \to t(t+2)/2$ is injective on $t \ge 0$, the $C_{2\theta_i}$–eigenvalue of a monomial $x^{\alpha,\beta} \in S^pV[\mu]$ determines $\alpha_i + \beta_i$ and therefore α_i, β_i uniquely since $\alpha_i - \beta_i = \mu_i$. It follows that the long root Casimirs $\kappa_{2\theta_i}$ have simple spectrum on $S^pV[\mu]$.

We turn now to the action of the short root Casimirs $\kappa_{\theta_i-\theta_j}$ and $\kappa_{\theta_i+\theta_j}$ on S^pV. The root vectors $e_{\theta_i-\theta_j}$ and $f_{\theta_i-\theta_j}$ act on V as $E_{ij}-E_{-j-i}$ and $E_{ji}-E_{-i-j}$ respectively [FH, §16.1] and therefore on SV as

$$e_{\theta_i-\theta_j} = x_i\partial_j - x_{-j}\partial_{-i} \tag{6.37}$$

$$f_{\theta_i-\theta_j} = x_j\partial_i - x_{-i}\partial_{-j} \tag{6.38}$$

from which it readily follows that

$$\begin{aligned}\kappa_{\theta_i-\theta_j}\, x^{\alpha,\beta} =& \frac{1}{2}(\alpha_i(\alpha_j+1)+(\alpha_i+1)\alpha_j+\beta_i(\beta_j+1)+(\beta_i+1)\beta_j)x^{\alpha,\beta}\\ &-\alpha_i\beta_i x^{\alpha-\varepsilon_i+\varepsilon_j,\beta-\varepsilon_i+\varepsilon_j}\\ &-\alpha_j\beta_j x^{\alpha+\varepsilon_i-\varepsilon_j,\beta+\varepsilon_i-\varepsilon_j}\end{aligned} \tag{6.39}$$

Similarly, $e_{\theta_i+\theta_j}$ and $f_{\theta_i+\theta_j}$ act on V by $E_{i-j}+E_{j-i}$ and $E_{-ji}+E_{-ij}$ respectively and therefore on SV as

$$e_{\theta_i+\theta_j} = x_i\partial_{-j} + x_j\partial_{-i} \tag{6.40}$$

$$f_{\theta_i+\theta_j} = x_{-i}\partial_j + x_{-j}\partial_i \tag{6.41}$$

from which it readily follows that

$$\begin{aligned}\kappa_{\theta_i+\theta_j}\, x^{\alpha,\beta} =& \frac{1}{2}(\alpha_i(\beta_j+1)+(\alpha_i+1)\beta_j+\alpha_j(\beta_i+1)+(\alpha_j+1)\beta_i)x^{\alpha,\beta}\\ &+\alpha_i\beta_i x^{\alpha-\varepsilon_i+\varepsilon_j,\beta-\varepsilon_i+\varepsilon_j}\\ &+\alpha_j\beta_j x^{\alpha+\varepsilon_i-\varepsilon_j,\beta+\varepsilon_i-\varepsilon_j}\end{aligned} \tag{6.42}$$

Let now $\mu\in\mathbb{Z}^n$ be a weight of S^pV. Since the long root Casimirs $\kappa_{2\theta_i}$ generate the algebra of diagonal matrices in the monomial basis $\{x^{\alpha,\beta}\}$ of $S^pV[\mu]$ and the first terms in the right–hand sides of (6.39) and (6.42) are diagonal in that basis, the irreducibility of $S^pV[\mu]$ under $\mathcal{C}_{\mathfrak{g}}$ will follow if we can show that any subspace $K\subseteq S^pV[\mu]$ containing at least one monomial and invariant under the shift operators S_{ij}, $1\leq i\neq j\leq n$ given by

$$S_{ij}\, x^{\alpha,\beta} = \alpha_i\beta_i x^{\alpha-\varepsilon_i+\varepsilon_j,\beta-\varepsilon_i+\varepsilon_j} \tag{6.43}$$

is equal to $S^pV[\mu]$. Let $x^{\alpha,\beta}\in S^pV[\mu]$ be a fixed monomial. We wish to prove by induction on $i=1\ldots n-1$ that there exists a monomial $x^{\alpha^{(i)},\beta^{(i)}}\in K$ such that

$$\alpha^{(i)}_j = \alpha_j \quad\text{and}\quad \beta^{(i)}_j = \beta_j \tag{6.44}$$

for any $1\leq j\leq i$. It then follows easily from this that $x^{\alpha,\beta}\in K$ since

$$\alpha_n - \beta_n = \mu_n = \alpha^{(n-1)}_n - \beta^{(n-1)}_n \tag{6.45}$$

and

$$\begin{aligned}\alpha_n + \beta_n &= p - \sum_{j=1}^{n}(\alpha_j + \beta_j) = p - \sum_{j=1}^{n}(\alpha_j^{(n-1)} + \beta_j^{(n-1)}) \\ &= \alpha_n^{(n-1)} + \beta_n^{(n-1)}\end{aligned} \tag{6.46}$$

so that $\alpha = \alpha^{(n-1)}$ and $\beta = \beta^{(n-1)}$. Let

$$K_i^{\alpha,\beta} = \{(\widetilde{\alpha}, \widetilde{\beta}) | x^{\widetilde{\alpha},\widetilde{\beta}} \in K \text{ and } \widetilde{\alpha}_j = \alpha_j,\ \widetilde{\beta}_j = \beta_j,\ \forall 1 \leq j \leq i-1\} \tag{6.47}$$

a non–empty set by induction. If there exists a pair $(\widetilde{\alpha}, \widetilde{\beta}) \in K_i^{\alpha,\beta}$ with $\widetilde{\alpha}_i > \alpha_i$, then $\widetilde{\beta}_i > \beta_i$ since $\widetilde{\alpha}_i - \widetilde{\beta}_i = \mu_i = \alpha_i - \beta_i$ and

$$\begin{aligned}S_{ii+1}^{\widetilde{\alpha}_i - \alpha_i}\, x^{\widetilde{\alpha},\widetilde{\beta}} &= \widetilde{\alpha}_i(\widetilde{\alpha}_i - 1)\cdots(\alpha_i + 1)\cdot\widetilde{\beta}_i(\widetilde{\beta}_i - 1)\cdots(\beta_i + 1) \\ &\quad \cdot x^{\widetilde{\alpha} - (\widetilde{\alpha}_i - \alpha_i)(\varepsilon_i - \varepsilon_{i+1}), \widetilde{\beta} - (\widetilde{\beta}_i - \beta_i)(\varepsilon_i - \varepsilon_{i+1})}\end{aligned} \tag{6.48}$$

implies that $x^{\widetilde{\alpha} - (\widetilde{\alpha}_i - \alpha_i)(\varepsilon_i - \varepsilon_{i+1}), \widetilde{\beta} - (\widetilde{\beta}_i - \beta_i)(\varepsilon_i - \varepsilon_{i+1})} \in K$ so that $K_{i+1}^{\alpha,\beta}$ is non–empty. We may therefore assume that $\widetilde{\alpha}_i \leq \alpha_i$ for any pair $(\widetilde{\alpha}, \widetilde{\beta}) \in K_i^{\alpha,\beta}$. Choose $(\widetilde{\alpha}, \widetilde{\beta}) \in K_i^{\alpha,\beta}$ with $\widetilde{\alpha}_i$ maximal. If $\widetilde{\alpha}_i = \alpha_i$, we are done. Otherwise, note that, for any $j = 1 + 1 \ldots n$,

$$S_{ji} x^{\widetilde{\alpha},\widetilde{\beta}} = \widetilde{\alpha}_j \widetilde{\beta}_j x^{\widetilde{\alpha} + \varepsilon_i - \varepsilon_j, \widetilde{\beta} + \varepsilon_i - \varepsilon_j} \tag{6.49}$$

implies that $x^{\widetilde{\alpha} + \varepsilon_i - \varepsilon_j, \widetilde{\beta} + \varepsilon_i - \varepsilon_j} \in K$, thus violating the maximality of $\widetilde{\alpha}_i$, unless $\widetilde{\alpha}_j \widetilde{\beta}_j = 0$. Thus, for any such j, $\min(\widetilde{\alpha}_j, \widetilde{\beta}_j) = 0$ whence

$$\begin{aligned}p &= |\widetilde{\alpha}| + |\widetilde{\beta}| \\ &= |\mu| + 2\sum_{k=1}^{n} \min(\widetilde{\alpha}_k, \widetilde{\beta}_k) \\ &= |\mu| + 2\sum_{k=1}^{i-1} \min(\widetilde{\alpha}_k, \widetilde{\beta}_k) + \min(\widetilde{\alpha}_i, \widetilde{\beta}_i) \\ &< |\mu| + 2\sum_{k=1}^{i} \min(\alpha_k, \beta_k) \\ &\leq |\mu| + 2\sum_{k=1}^{n} \min(\alpha_k, \beta_k) \\ &= p\end{aligned} \tag{6.50}$$

and therefore a contradiction ■

Remark. Since the highest weights of the adjoint representations of $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$ are $(1, 0, \ldots, 0, -1)$ and $(2, 0, \ldots, 0)$ respectively, theorems 6.1 and 6.5 might lead one to conjecture that the weight spaces of the pth Cartan power of the adjoint representation of any simple Lie algebra $\mathfrak{g}$ are irreducible under the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$. We will prove in §7.7 that this is true for the zero weight space when $p = 1, 2$ and show in §7.4 that this fails for $\mathfrak{g} = \mathfrak{so}_m$ if $p \geq 3$.

6.4. **Irreducible representations of $\mathcal{C}_\mathfrak{g}$, $\mathfrak{g}$ exceptional.** Let $\mathfrak{g}$ be a complex, simple Lie algebra of exceptional type and let θ be the highest root of $\mathfrak{g}$.

Theorem 6.6. *For any $p \in \mathbb{N}$, there exists a subspace K_p of the zero weight space of the simple $\mathfrak{g}$–module with highest weight $p\theta$ which is irreducible under $\mathcal{C}_\mathfrak{g}$ and such that $\lim_{p\to\infty} \dim K_p = \infty$.*

Proof. We shall need the following simple

Lemma 6.7. *Let R be a root system and $\alpha \neq \pm\beta \in R$ two long roots which are not orthogonal. Then, $R \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ is a root system of type A_2.*

Proof. Since $\|\alpha\| = \|\beta\|$, one has $\langle \alpha, \beta^\vee \rangle = \pm 1$. Replacing β by $-\beta$ if necessary, we may assume that $\langle \alpha, \beta^\vee \rangle = -1$. Thus, $\pm\alpha, \pm\beta, \pm(\alpha + \beta) = \pm\sigma_\beta\alpha \in R$ and it is easy to check that these are the only $\mathbb{Z}$–linear combinations of α, β which lie in R since any other has norm strictly larger than $\|\alpha\|$ ■

An inspection of the tables in [Bo] shows that if $\mathfrak{g}$ is of exceptional type, there is a unique simple root α of $\mathfrak{g}$ which is not orthogonal to θ and is, moreover, long. Applying the above lemma to the pair (α, θ) yields a subalgebra

$$\mathfrak{l} = \mathbb{C}h_\alpha \oplus \mathbb{C}h_\theta \bigoplus_{\gamma \in R(\mathfrak{g}) \cap (\mathbb{Z}\alpha + \mathbb{Z}\theta), \gamma \succ 0} \mathbb{C}e_\gamma \oplus \mathbb{C}f_\gamma \subset \mathfrak{g} \tag{6.51}$$

which is isomorphic to $\mathfrak{sl}_3$ and has as highest root vector e_θ. Choose $\mathfrak{h}_\mathfrak{l} = \mathbb{C}h_\alpha \oplus \mathbb{C}h_\theta$ as Cartan subalgebra of $\mathfrak{l}$ and denote by $\mathfrak{l}^{*p}$ (resp. $\mathfrak{g}^{*p}$) the irreducible representation of $\mathfrak{l}$ (resp. $\mathfrak{g}$) with highest weight $p\theta$. Since $\mathfrak{l}^{*p}$ is generated by $e_\theta^{\otimes p}$ inside $\mathfrak{l}^{\otimes p} \subset \mathfrak{g}^{\otimes p}$, it follows that $\mathfrak{g}^{*p}$ contains $\mathfrak{l}^{*p}$ as $\mathfrak{l}$–submodule. This inclusion induces one of weight spaces $\mathfrak{l}^{*p}[0] \subset \mathfrak{g}^{*p}[0]$ since $\mathfrak{h} = \mathfrak{h}_\mathfrak{l} \oplus \mathfrak{h}_\mathfrak{l}^\perp$ and $\mathfrak{h}_\mathfrak{l}^\perp$ centralises $\mathfrak{l}$. By theorem 6.1, $\mathfrak{l}^{*p}[0]$ is irreducible under $\mathcal{C}_\mathfrak{l}$. Let $U = \mathcal{C}_\mathfrak{g}\mathfrak{l}^{*p}[0]$ be the $\mathcal{C}_\mathfrak{g}$–submodule of $\mathfrak{g}^{*p}[0]$ generated by $\mathfrak{l}^{*p}[0]$ and decompose it as a sum $\bigoplus_i U_i$ of irreducible summands with projections p_i. By Schur's lemma, the restriction of each p_i to $\mathfrak{l}^{*p}[0]$ is either zero or injective. Thus, the dimension of at least one of the U_i's is greater or equal to that of $\mathfrak{l}^{*p}[0]$ and therefore tends to infinity with p ■

7. Zero weight spaces of self–dual $\mathfrak{g}$–modules

7.1. The aim of this section is to show that, when $\mathfrak{g} \ncong \mathfrak{sl}_2, \mathfrak{sl}_3$ is classical or $\mathfrak{g}_2$, the zero weight spaces of most *self–dual*, simple $\mathfrak{g}$–modules are reducible under the Casimir algebra of $\mathfrak{g}$, thus strengthening the results of section 4.

Our results are perhaps more appealing for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g}_2$, where we give a complete classification of those self–dual V for which $V[0]$ is irreducible in subsections 7.2 and 7.3 respectively. For $\mathfrak{g} = \mathfrak{sl}_n$, these are the Cartan powers of the adjoint representation considered in §6.1, another infinite series if $n = 4$, and the representations with highest weight of the form

$$\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{k}), \qquad 0 \leq k \leq n/2$$

For $\mathfrak{g} = \mathfrak{g}_2$, only finitely many V turn out to have an irreducible zero weight space, namely the first and second Cartan powers of the fundamental representations.

The reducibility of $V[0]$ is obtained by showing that the Chevalley involution of $\mathfrak{g}$ which, as pointed out in §4.4, acts on any self–dual $\mathfrak{g}$–module and centralises the Casimirs κ_α, does not act as a scalar on $V[0]$. In most cases, this is achieved by using proposition 4.7, that is finding a reductive subalgebra $\mathfrak{r} \subset \mathfrak{g}$ such that the restriction of V to $\mathfrak{r}$ contains a non self–dual summand with non–trivial intersection with $V[0]$. For $\mathfrak{g} = \mathfrak{sl}_n$, we use $\mathfrak{r} = \mathfrak{gl}_k$ for some $2 \leq k \leq n-1$, and for $\mathfrak{g} = \mathfrak{g}_2$, $\mathfrak{r} = \mathfrak{sl}_3$. The corresponding restrictions are computed by using the Gelfand–Zetlin and Perroud branching rules respectively [GZ1, Pe].

The method of proof is very similar for the cases where $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$, which are treated in subsections 7.4–7.6. The reductive subalgebra in this case is $\mathfrak{r} = \mathfrak{gl}_n$ and restriction to it is computed by using the Koike–Terada branching rules [KT]. These however are combinatorial, in that they express the branching as a sum of $\mathfrak{gl}_n$–modules with manifestly positive multiplicities, only when the highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the $\mathfrak{g}$–module is such that $\lambda_i = 0$ for $i > n/2$ and we restrict to this range for technical simplicity. Within it, we give a complete classification of all simple, self–dual V for which $V[0]$ is irreducible under $\mathcal{C}_\mathfrak{g}$. The general case, and that of the Lie algebras of types E and F will be dealt with in a future publication.

The analysis of the zero weight spaces of the small (*i.e.,* first and second) Cartan powers of the adjoint representation of $\mathfrak{g}$, which is needed to complete the above classification results, is relegated to a separate subsection 7.7 since they turn out to be irreducible for any $\mathfrak{g}$. Finally, in §7.8, we systematise our findings by conjecturing that, for any $\mathfrak{g}$ and self–dual $\mathfrak{g}$–module V, $V[0]$ is irreducible under $\mathcal{C}_\mathfrak{g}$ iff the Chevalley involution acts as a scalar. We note also that the classification sketched above proves this conjecture for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g}_2$, as well as for the irreducible representations of $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}, \mathfrak{sp}_{2n}$ with highest weight λ such that $\lambda_i = 0$ for $i > n/2$. We also give a conjecturally complete list of all self–dual, irreducible representations of these Lie algebras for which $V[0]$ is irreducible under $\mathcal{C}_\mathfrak{g}$.

REMARK. Since the reducibility of the self–dual zero weight spaces $V[0]$ we consider is always obtained by showing that the Chevalley involution does not act as a scalar on $V[0]$, our results also imply, by corollary 5.4, that the quantum Weyl group action of $B_\mathfrak{g}$ on the zero weight spaces of most self–dual $U_\hbar\mathfrak{g}$–modules is reducible.

7.2. Self–dual representations of $\mathfrak{sl}_n$.

Theorem 7.1. *Let V be a simple, self–dual $\mathfrak{sl}_n$–module with non–trivial zero weight space $V[0]$. Then, $V[0]$ is irreducible under the Casimir algebra $\mathcal{C}_{\mathfrak{sl}_n}$ if the highest weight λ of V is of one of the following forms*

(i) $\lambda = (p, 0, \ldots, 0, -p)$, $p \in \mathbb{N}$.

(ii) $\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{k})$, *for some* $0 \leq k \leq n/2$

(iii) $\lambda = (p, p, -p, -p)$, $p \in \mathbb{N}$.

Conversely, if λ is of none of the above forms, then, for some $k < n$, V contains a simple $\mathfrak{gl}_k$–summand U with $U \not\cong U^*$ and $U \cap V[0] \neq \{0\}$. In particular, $V[0]$ is reducible under $\mathcal{C}_{\mathfrak{sl}_n}$ by proposition 4.7.

Since the case (i) follows from theorem 6.1, the "if" part of theorem 7.1 is settled by the following two lemmas.

Lemma 7.2. *If $V_{n,k}$ is the simple $\mathfrak{sl}_n$–module with highest weight*

$$\lambda_k^{(n)} = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{k}) \tag{7.1}$$

then $V_{n,k}[0]$ is irreducible under $\mathcal{C}_{\mathfrak{sl}_n}$.

PROOF. We claim that the Casimirs $C_{\mathfrak{gl}_m}$, $m = 2, \ldots n$, have joint simple spectrum on $V_{n,k}[0]$. Indeed, the mth row of a zero weight Gelfand–Zetlin pattern corresponding to $V_{n,k}$ is of the form $\lambda_l^{(m)}$, for some $0 \leq l \leq m/2$. Since $C_{\mathfrak{gl}_m}$ acts as multiplication by $2l(m-l+1)$ on the representation with highest weight $\lambda_l^{(m)}$ and the function $f(x) = 2x(m-x+1)$ is injective on the interval $[0, (m+1)/2]$, the $C_{\mathfrak{gl}_m}$–eigenvalue of a zero weight Gelfand–Zetlin pattern determines its mth row uniquely, as claimed. The proof is now completed as in theorem 6.1 ∎

Lemma 7.3. *For any $p \in \mathbb{N}$, let V_p be the simple $\mathfrak{sl}_4$–module with highest weight $(p, p, -p, -p)$. Then, $V_p[0]$ is irreducible under $\mathcal{C}_{\mathfrak{sl}_4}$.*

PROOF. The Gelfand–Zetlin patterns corresponding to $V_p[0]$ are of the form

$$\begin{array}{ccccccc} p & & p & & -p & & -p \\ & p & & 0 & & -p & \\ & & q & & -q & & \\ & & & 0 & & & \end{array} \tag{7.2}$$

for some $0 \leq q \leq p$, so that they are separated by the action of the Casimir of $\mathfrak{sl}_2 \subset \mathfrak{sl}_n$. The proof is now completed as in theorem 6.1 ■

Remark. Note that V_p is the $2p$th Cartan power of the second exterior power $\bigwedge^2 \mathbb{C}^4$ of the vector representation of $\mathfrak{sl}_4$. Under the isomorphism $\mathfrak{sl}_4 \cong \mathfrak{so}_6$, the latter becomes the vector representation of $\mathfrak{so}_6$ so that lemma 7.3 is consistent with theorem 6.2.

Proof of theorem 7.1. Assume that the highest weight λ of V is not of the form (i)–(iii) and let

$$s(\lambda) = |\{i = 1 \ldots n-1 | \lambda_i - \lambda_{i+1} > 0\}| \tag{7.3}$$

be the number of steps in the corresponding Young diagram. Noting that $s(\lambda) = 2$ for the highest weights of the form (i)–(iii), we begin by proving that $V[0]$ is reducible under $\mathcal{C}_{\mathfrak{sl}_n}$ if $s(\lambda) \geq 3$. Suppose first that $n = 2k+1$ is odd. Since $V[0] \neq \{0\}$, we may assume that the sum $|\lambda|$ of the entries in λ is zero so that, by self–duality, λ is of the form

$$\lambda = (a_1, \ldots, a_l, 0, \ldots, 0, -a_l, \ldots, -a_1) \tag{7.4}$$

for some $a_1 \geq \cdots \geq a_l > 0$, with at least one middle zero. Since $s(\lambda) \geq 3$, there exists some $1 \leq i \leq l-1$ such that $a_i > a_{i+1}$. Let μ be the $\mathfrak{sl}_{n-1}$–weight obtained by replacing a_i by $a_i - 1$ in the ith position, $-a_l$ by $-a_l + 1$ in the $n-l+1$th position and by omitting the middle zero. Then, $|\mu| = \sum_{i=1}^{n-1} \mu_i = |\lambda| = 0$ and

$$\mu_i + \mu_{n-i} = -1 \neq 1 = \mu_l + \mu_{n-l} \tag{7.5}$$

so that μ is a non–self dual weight of $\mathfrak{sl}_{n-1}$ such that $V_\mu \cap V[0] \neq \{0\}$.

Consider now the case where $n = 2k$ is even. Assuming again that $|\lambda| = 0$, we find that

$$\lambda = (a_1, \ldots, a_l, \underbrace{0, \ldots, 0}_{n_0}, -a_l, \ldots, -a_1) \tag{7.6}$$

where the number n_0 of zeroes is even. If $n_0 > 0$, the $\mathfrak{sl}_{n-1}$ weight μ obtained by replacing the two middle zeroes by a single one in λ is self–dual and satisfies $|\mu| = 0$ and $s(\mu) = s(\lambda)$. By our previous analysis, there therefore exists a non–self dual $\mathfrak{sl}_{n-2}$ weight $\nu \prec \mu \prec \lambda$ such that $|\nu| = 0$ and $V_\nu \cap V[0] \neq 0$. If, on the other hand, $n_0 = 0$, then

$$\lambda = (a_1, \cdots, a_{n/2}, -a_{n/2}, \cdots, -a_1) \tag{7.7}$$

with $a_{n/2} > 0$. Let $1 \leq i \leq n/2 - 1$ be such that $a_i > a_{i+1}$. Then, the non–self dual $\mathfrak{sl}_{n-1}$–weight $\mu \prec \lambda$ obtained by replacing a_i by $a_i - 1$ in the ith position and the pair $a_{n/2}, -a_{n/2}$ allows to conclude.

Consider now the case $s(\lambda) \leq 2$. By self–duality, we may take λ of the form

$$\lambda = (\underbrace{p, \ldots, p}_{k}, 0, \ldots, 0, \underbrace{-p, \ldots, -p}_{k}) \tag{7.8}$$

for some $0 \leq k \leq n/2$. By assumption, $k \geq 2$ and $p > 1$ since λ is not of the forms (i)–(ii). If n is odd, replacing the innermost pair $(p, -p)$ by $(p-1, -(p-1))$ and suppressing the middle zero yields an $\mathfrak{sl}_{n-1}$ weight μ with $s(\mu) \geq 3$ and our previous analysis allows to conclude. If n is even and there are two or more middle zeroes, we suppress one of them to obtain an $\mathfrak{sl}_{n-1}$–weight ν of of the form treated in the previous paragraph. If there are no middle zeroes, then by assumption $n \geq 6$. We change the innermost pair $(p, -p)$ to 0 and proceed as above ■

7.3. **Representations of $\mathfrak{g}_2$.** Recall that, for $\mathfrak{g} = \mathfrak{g}_2$, every $\mathfrak{g}$–module V is self–dual and has a non–trival zero weight space. The aim of this subsection is to prove the following.

Theorem 7.4. *The zero weight space $V[0]$ of a simple $\mathfrak{g}_2$–module V is irreducible under the Casimir algebra $\mathcal{C}_{\mathfrak{g}_2}$ iff V is a trivial or fundamental representation, or its second Cartan power.*

The proof of the theorem is given in the next three propositions. We begin by reviewing Perroud's branching rules for the equal rank inclusion $\mathfrak{sl}_3 \subset \mathfrak{g}_2$ [Pe]. Let α_1, α_2 be the long and short simple roots of $\mathfrak{g}_2$ respectively and ϖ_1, ϖ_2 the corresponding fundamental weights[6]. Let V_λ be the simple $\mathfrak{g}_2$–module with highest weight $\lambda = m_1\varpi_1 + m_2\varpi_2$. Consider the set of Gelfand–Zetlin patterns $\mu(a, b, c)$ of the form

$$\begin{array}{ccccc} m_1+m_2 & & m_2 & & 0 \\ & a & & b & \\ & & c & & \end{array} \tag{7.9}$$

Then,

$$\operatorname{res}^{\mathfrak{sl}_3}_{\mathfrak{g}_2} V_\lambda = \bigoplus_{\mu(a,b,c)} V_{(m_1+c, a-m_2+b, 0)} \tag{7.10}$$

Proposition 7.5. *The zero weight spaces of the fundamental representations of $\mathfrak{g}$ and of their second Cartan powers are irreducible under $\mathcal{C}_{\mathfrak{g}}$.*

Proof. By theorem 7.24, $V[0]$ is irreducible if V is the first or second Cartan power of V_{ϖ_1} since the latter is the adjoint representation of $\mathfrak{g}$. Taking now $V = V_{\varpi_2}, V_{2\varpi_2}$, we obtain, from Perroud's branching rules

$$\operatorname{res}^{\mathfrak{sl}_3}_{\mathfrak{g}_2} V_{\varpi_2} = \mathbb{C} \oplus R_1 \tag{7.11}$$

$$\operatorname{res}^{\mathfrak{sl}_3}_{\mathfrak{g}_2} V_{2\varpi_2} = \operatorname{ad}(\mathfrak{sl}_3) \oplus R_2 \tag{7.12}$$

where R_1, R_2 are reducible $\mathfrak{sl}_3$–modules with trivial zero weight spaces, and the irreducibility of $V[0]$ follows from theorem 4.1 ■

[6] we follow here Perroud's convention [Pe] which are the opposite of the usual ones [Bo, FH]

Proposition 7.6. *Let V be a simple $\mathfrak{g}_2$–module with highest weight $\lambda = m_1\varpi_1 + m_2\varpi_2$. If $m_1 + m_2 \geq 3$, the zero weight space of V is reducible under $\mathcal{C}_{\mathfrak{g}_2}$.*

Proof. By proposition 4.7, it suffices to prove that the restriction of V to $\mathfrak{sl}_3$ contains an irreducible summand U with $U[0] \neq \{0\}$ and $U \not\cong U^*$. We begin by treating the special cases $m_1 = 0$ and $m_2 = 0$. Assume first that $m_2 = 0$ so that $b = 0$ in (7.9). If $m_1 = 0 \mod 3$, then setting $a = c = 0$ in (7.10) yields $\operatorname{res}^{\mathfrak{sl}_3}_{\mathfrak{g}_2} V \supset V_{(m_1,0,0)}$. Similarly, if $m_1 = 1 \mod 3$, with $m_1 > 1$, taking $a = 1$ and $c = 1$ yields $\operatorname{res}^{\mathfrak{sl}_3}_{\mathfrak{g}_2} \supset V_{(m_1+1,1,0)}$. Finally, if $m_1 = 2 \mod 3$, $m_1 > 2$, $a = 1, c = 0$ yields $\operatorname{res}^{\mathfrak{sl}_3}_{\mathfrak{g}_2} \supset V_{(m_1,1,0)}$ as required. Assume now that $m_1 = 0$ and $m_2 \geq 3$ so that $a = m_2$ in (7.9). Then, taking $b = 0, c = 3$, we find $\operatorname{res}^{\mathfrak{sl}_3}_{\mathfrak{g}_2} \supset V_{(3,0,0)}$.

Consider now the case $m_1, m_2 > 0$. The values of (a, b, c) corresponding to the Gelfand–Zetlin patterns (7.9) are readily seen to span the integral points of a convex polytope in $\mathbb{R}^3$ with vertices given by

$$(m_2, 0, 0), (m_2, 0, m_2), (m_2, m_2, m_2), \tag{7.13}$$
$$(m_1 + m_2, 0, 0), (m_1 + m_2, 0, m_1 + m_2), \tag{7.14}$$
$$(m_1 + m_2, m_2, m_2), (m_1 + m_2, m_2, m_1 + m_2) \tag{7.15}$$

The image $P(m_1, m_2) \subset \mathbb{R}^3$ of this polytope under the Perroud map $\pi : (a, b, c) \to (m_1 + c, a - m_2 + b, 0)$ is the convex hull of the images of the above points, namely

$$(m_1, 0, 0), (m_1 + m_2, 0, 0), (m_1 + m_2, m_2, 0), \tag{7.16}$$
$$(m_1, m_1, 0), (2m_1 + m_2, m_1, 0), \tag{7.17}$$
$$(m_1 + m_2, m_1 + m_2, 0), (2m_1 + m_2, m_1 + m_2, 0) \tag{7.18}$$

and is readily seen to be described by the following inequalities in the plane $(\mu_1, \mu_2, 0) \subset \mathbb{R}^3$

$$m_1 \leq \mu_1 \leq 2m_1 + m_2 \tag{7.19}$$
$$0 \leq \mu_2 \leq m_1 + m_2 \tag{7.20}$$
$$0 \leq \mu_1 - \mu_2 \leq m_1 + m_2 \tag{7.21}$$

Moreover,

$$\operatorname{res}^{\mathfrak{sl}_3}_{\mathfrak{g}_2} V = \bigoplus_{\mu \in P(m_1,m_2) \cap \mathbb{N}^3} V_\mu \otimes \mathbb{C}^{|\pi^{-1}(\mu)|} \tag{7.22}$$

We seek to derive a contradiction from the assumption that all summands in (7.22) with non–trivial zero weight space are self–dual. Let U be a summand with $U[0] \neq \{0\}$ and $U \cong U^*$ so that its highest weight is of the form $\mu = (2k, k, 0)$ for some $k \in \mathbb{N}$. Let $\tau : \mathbb{Z}^3 \to \mathbb{Z}^3$ be defined by

$$\tau(\nu_1, \nu_2, \nu_3) = (\nu_1 - 1, \nu_2 + 1, \nu_3) \tag{7.23}$$

so that, if $\tau(\mu) \in P(m_1,m_2)$ (resp. $\tau^{-1}(\mu) \in P(m_1,m_2)$) then $V_{\tau(\mu)} \subset V$ (resp. $V_{\tau^{-1}(\mu)} \subset V$) is a non self–dual $\mathfrak{sl}_3$–summand with non–trivial zero weight space. We shall need the following

Lemma 7.7. *Assume that $m_1, m_2 \neq 0$ and that $\mu = (2k,k,0) \in P(m_1,m_2)$.*

(i) *If $\tau(\mu) \notin P(m_1,m_2)$, then $k \in \{1, m_1/2\}$.*
(ii) *If $\tau^{-1}(\mu) \notin P(m_1,m_2)$, then $k \in \{m_1+m_2-1, m_1+m_2/2\}$.*

PROOF. (i) By assumption, $\tau(\mu)$ violates at least one of the inequalities (7.19)–(7.21), so that at least one of the following conditions holds

$$\mu_1 = m_1 \tag{7.24}$$

$$\mu_2 = m_1 + m_2 \tag{7.25}$$

$$\mu_1 - \mu_2 \in \{0,1\} \tag{7.26}$$

The condition $k = \mu_1 - \mu_2 = 0$ is ruled out by the fact that $(0,0,0) \notin P(m_1,m_2)$ if $m_1 > 0$. Similarly, $k = \mu_2 = m_1+m_2$ leads to $(2(m_1+m_2), m_1+m_2, 0) \in P(m_1,m_2)$ which violates (7.19) since $m_2 > 0$. We are therefore left with $\mu_1 = m_1$ or $\mu_1 - \mu_2 = 1$ which lead to $k = m_1/2, 1$ respectively.
(ii) Similarly, $\tau^{-1}(m) \notin P(m_1,m_2)$ iff at least one of the following equations holds

$$\mu_1 = 2m_1 + m_2 \tag{7.27}$$

$$\mu_2 = 0 \tag{7.28}$$

$$\mu_1 - \mu_2 \in \{m_1+m_2-1, m_1+m_2\} \tag{7.29}$$

$\mu_2 = 0$ and $\mu_1 - \mu_2 = m_1 + m_2$ imply that $\mu = (0,0,0)$ and $\mu = (2(m_1+m_2), m_1+m_2)$ respectively both of which are ruled out by $m_1, m_2 > 0$. Thus, $\mu_1 = 2m_1 + m_2$ or $\mu_1 - \mu_2 = m_1 + m_2 - 1$ hold yielding $k \in \{m_1+m_2/2, m_1+m_2-1\}$ ■

Returning to our main argument, if all $\mathfrak{sl}_3$–summands in V with non–trivial zero weight spaces are self–dual then $\tau(\mu), \tau^{-1}(\mu) \notin P(m_1,m_2)$ for any $\mu \in P(m_1,m_2)$ of the form $(2k,k,0)$. By the above lemma, this implies

$$\{1, m_1/2\} \cap \{m_1+m_2-1, m_1+m_2/2\} \neq \emptyset \tag{7.30}$$

so that at least one of the following equations holds

$$m_1 + m_2 = 2 \tag{7.31}$$

$$m_1 + m_2/2 = 1 \tag{7.32}$$

$$m_1/2 + m_2 = 1 \tag{7.33}$$

$$m_1/2 + m_2/2 = 0 \tag{7.34}$$

contradicting the fact that $m_1 + m_2 \geq 3$ ■

Proposition 7.8. *Let U be the Cartan product of the two fundamental representations of $\mathfrak{g}_2$. Then, $U[0]$ is reducible under $\mathcal{C}_{\mathfrak{g}_2}$.*

PROOF. It suffices to show that the Chevalley involution of $\mathfrak{g} = \mathfrak{g}_2$ does not act as a scalar on $U[0]$. Let α_1, α_2 be the short and long simple roots respectively [7] and label the positive roots by

$$\alpha_i = (i-2)\alpha_1 + \alpha_2,\ \ 2 \leq i \leq 5 \qquad \text{and} \qquad \alpha_6 = 3\alpha_1 + 2\alpha_2 \tag{7.35}$$

so that the highest root is $\theta = \alpha_6$. The corresponding fundamental weights ϖ_1, ϖ_2 of $\mathfrak{g}$ are

$$\varpi_1 = 2\alpha_1 + \alpha_2 = \alpha_4 \qquad \text{and} \qquad \varpi_2 = 3\alpha_1 + 2\alpha_2 = \alpha_6 \tag{7.36}$$

so that $V_{\varpi_2} \cong \mathrm{ad}(\mathfrak{g}_2)$ and $V = V_{\varpi_1} \cong \mathbb{C}^7$ has weights 0 and $\pm\alpha_i$, $i = 1, 3, 4$ [FH, §22.1]. Choose a Cartan–Weyl basis $e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_1}, h_{\alpha_2}$ of $\mathfrak{g}$ and a weight basis $v_{\pm\alpha_i}$, $i = 1, 3, 4$ and v_0 of V_{ϖ_1} where v_β has weight β. The highest weight vector in $U \subset \mathfrak{g}_2 \otimes V$ is $e_{\alpha_6} \otimes v_{\alpha_4}$ so that

$$u_0 = f_{\alpha_4} f_{\alpha_6}\, e_{\alpha_6} \otimes v_{\alpha_4} \in U[0] \tag{7.37}$$

Computing u_0 explicitly yields

$$\begin{aligned} u_0 &= f_{\alpha_4}\left(-h_{\alpha_6} \otimes v_{\alpha_4} + a e_{\alpha_6} \otimes v_{-\alpha_3}\right) \\ &= -\alpha_4(h_{\alpha_6}) f_{\alpha_4} \otimes v_{\alpha_4} + b h_{\alpha_6} \otimes v_0 + c e_{\alpha_3} \otimes v_{-\alpha_3} \end{aligned} \tag{7.38}$$

where the constants a, b, c depend upon the choices of the basis of V_{ϖ_i}, $i = 1, 2$ and are not zero by elementary sl_2–representation theory. On the other hand, if Θ is the Chevalley involution of $\mathfrak{g}$, then $\Theta\, h_{\alpha_i} = -h_{\alpha_i}$ and, up to multiplicative constants

$$\Theta\, e_{\alpha_i} = f_{\alpha_i} \qquad \Theta\, f_{\alpha_i} = e_{\alpha_i} \qquad \Theta\, v_{\alpha_i} = v_{-\alpha_i} \tag{7.39}$$

so that Θu_0 is not proportional to u_0 ■

7.4. **Self–dual representations of orthogonal and symplectic Lie algebras.** Let $\mathfrak{g}$ be one of $\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}, \mathfrak{sp}_{2n}$ and retain the notation of subsections 6.2 and 6.3. Let V be a simple $\mathfrak{g}$–module with highest weight $\lambda = \sum_{i=1}^n \lambda_i \theta_i$. Recall that, if $\mathfrak{g} \cong \mathfrak{so}_{2n}$, V is self–dual iff $\lambda_n = 0$ and that V is always self–dual if $\mathfrak{g}$ is isomorphic to $\mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$. The zero weight space $V[0]$ of V is non–trivial iff $|\lambda| = \sum_i \lambda_i \in 2\mathbb{N}$ if $\mathfrak{g} \cong \mathfrak{sp}_{2n}$, iff $\lambda_i \in \mathbb{N}$ for any $1 \leq i \leq n$ if $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$ and iff $\lambda_i \in \mathbb{Z}$ for any $1 \leq i \leq n$ and $|\lambda| \in 2\mathbb{N}$ for $\mathfrak{g} \cong \mathfrak{so}_{2n}$.

Assume now that $V \cong V^*$, $V[0] \neq \{0\}$ and that λ satisfies

$$\lambda_i = 0 \quad \text{for} \quad i > n/2 \tag{7.40}$$

The aim of this subsection is to prove the following

[7] we adhere now to the standard notation [FH]

Proposition 7.9. *If $\lambda \neq 0$ is of none of the following forms*

$$\lambda = (p, 0, 0, \dots, 0), \quad p \in \mathbb{N} \tag{7.41}$$

$$\lambda = (2, 1, 0, \dots, 0) \tag{7.42}$$

$$\lambda = (2, 2, 0, \dots, 0) \tag{7.43}$$

$$\lambda = (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0), \quad 1 \leq k \leq n \tag{7.44}$$

then, for some $k \leq n$, V contains a simple, non–self dual $\mathfrak{gl}_k$–summand U with $U \cap V[0] \neq \{0\}$. In particular, $V[0]$ is reducible under the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$ by proposition 4.7.

The (ir)reducibility of $V[0]$ under $\mathcal{C}_{\mathfrak{g}}$ for λ of one of the forms (7.41)–(7.44) will be treated in subsections 7.5 and 7.6. The proof of proposition 7.9 relies on the branching rules for the equal rank inclusion $\mathfrak{gl}_n \subset \mathfrak{g}$ obtained by Koike and Terada [KT, thm. A1] which we begin by reviewing. Let V_a, V_b, V_c be the simple $\mathfrak{gl}_n$–modules with highest weights $a, b, c \in \mathbb{Z}^n$ respectively and denote by

$$\mathrm{LR}^c_{a,b} = \dim \mathrm{Hom}_{\mathfrak{gl}_n}(V_c, V_a \otimes V_b) \tag{7.45}$$

the corresponding tensor product multiplicity given by the Littlewood–Richardson rules [FH]. Let

$$[b, c] = (b_1, \dots, b_n) - (c_n, \dots, c_1) \tag{7.46}$$

be the highest weight of $V_b \otimes V_c^*$. Then, for any highest weight λ satisfying (7.40),

$$\mathrm{res}^{\mathfrak{gl}_n}_{\mathfrak{so}_{2n+1}} V_\lambda = \bigoplus_{\beta,\kappa \in \mathcal{P}_n} \mathrm{LR}^\lambda_{\beta,\kappa} \bigoplus_{\mu,\nu \in \mathcal{P}_n} \mathrm{LR}^\beta_{\mu,\nu} \ V_{[\mu,\nu]} \tag{7.47}$$

$$\mathrm{res}^{\mathfrak{gl}_n}_{\mathfrak{sp}_{2n}} V_\lambda = \bigoplus_{\beta,\kappa \in \mathcal{P}_n} \mathrm{LR}^\lambda_{\beta,2\kappa} \bigoplus_{\mu,\nu \in \mathcal{P}_n} \mathrm{LR}^\beta_{\mu,\nu} \ V_{[\mu,\nu]} \tag{7.48}$$

$$\mathrm{res}^{\mathfrak{gl}_n}_{\mathfrak{so}_{2n}} V_\lambda = \bigoplus_{\beta \in \mathcal{P}_n, \kappa \in \mathcal{P}^{n/2}} \mathrm{LR}^\lambda_{\beta,(2\kappa)'} \bigoplus_{\mu,\nu \in \mathcal{P}_n} \mathrm{LR}^\beta_{\mu,\nu} \ V_{[\mu,\nu]} \tag{7.49}$$

In (7.47)–(7.49), $\mathcal{P}_n \subset \mathbb{N}^n$ is the set of partitions with at most n parts, $\mathcal{P}^m \subset \mathbb{N}^\infty$ is the set of partitions whose parts are at most equal to m and, for $\rho \in \mathcal{P}^m$, $\rho' \in \mathcal{P}_m$ is the conjugate partition. Note that, by the Littelwood–Richardson rules, the β, μ, ν involved in the above sums all satisfy (7.40). In particular,

$$[\mu, \nu] = \begin{cases} (\mu_1, \dots, \mu_{n/2}, -\nu_{n/2}, \dots, -\nu_1) & \text{if } n \text{ is even} \\ (\mu_1, \dots, \mu_{(n-1)/2}, 0, -\nu_{(n-1)/2}, \dots, -\nu_1) & \text{if } n \text{ is odd} \end{cases} \tag{7.50}$$

so that $V_{[\mu,\nu]}$ is self–dual iff $\mu = \nu$ since $V^*_{[\mu,\nu]} = V_{[\nu,\mu]}$.

We will use the above branching formulae mostly with $\beta = \lambda$ and $\kappa = 0$ by showing the existence of $\mu, \nu \in \mathcal{P}_n$ with

$$\mathrm{LR}^{\lambda}_{\mu,\nu} \neq 0, \tag{7.51}$$

$$\mu \neq \nu \qquad \text{and} \qquad |\mu| = |\nu| \tag{7.52}$$

where $|\mu| = \sum_i \mu_i$, so that $V_{[\mu,\nu]} \subset V_\lambda$ isn't self–dual but has a non–trivial zero weight space. To this end, we need an effective way to check (7.51). This is provided by the Parthasarthy–Ranga Rao–Varadarajan (PRV) conjecture [PRV] proved by Kumar [Ku]. Following [Ku], we denote by $\overline{\mu}$ the unique dominant weight in the Weyl group orbit of a $\mathfrak{gl}_n$–weight μ.

Theorem 7.10 (Kumar)**.** *Let λ, μ, ν be integral $\mathfrak{gl}_n$–weights with λ dominant and*

$$\lambda = \mu + \nu \tag{7.53}$$

Then, $\mathrm{LR}^{\lambda}_{\overline{\mu},\overline{\nu}} \neq 0$.

In the light of the above theorem, it is sufficient to find weights $\mu', \nu' \in \mathbb{N}^n$ such that $\lambda = \mu' + \nu'$ and $|\mu'| = |\nu'|$ and then take $\mu = \overline{\mu'}$, $\nu = \overline{\nu'}$, provided $\mu \neq \nu$. When this last requirement cannot be met, we will find a non self–dual $\mathfrak{gl}_k$–summand of the self–dual $V_{[\mu,\nu]}$ by resorting to theorem 7.1.

Proof of proposition 7.9. Assume first that $|\lambda| = \sum_{i=1}^n \lambda_i$ is even. Let

$$I_o = \{i |\ \lambda_i \in 2\mathbb{N} + 1\} \tag{7.54}$$

and set

$$\lambda_e = \sum_{i \notin I_o} \lambda_i e_i \qquad \text{and} \qquad \lambda_o = \sum_{i \in I_o} \lambda_i e_i \tag{7.55}$$

where e_i is the canonical basis of $\mathbb{Z}^n$, so that $\lambda = \lambda_e + \lambda_o$. Since $|\lambda| \in 2\mathbb{N}$, $|I_o|$ is even. Partition I_o as $I_o^+ \sqcup I_o^-$ with $|I_o^\pm| = |I_o|/2$ and set

$$\lambda_o^\pm = \sum_{i \in I_o^\pm} \frac{\lambda_i + 1}{2} e_i + \sum_{i \in I_o^\mp} \frac{\lambda_i - 1}{2} e_i \in \mathbb{N}^n \tag{7.56}$$

Let

$$\mu^\pm = \lambda_e/2 + \lambda_o^\pm \in \mathbb{N}^n \tag{7.57}$$

so that

$$\lambda = \mu^+ + \mu^- \qquad \text{with} \qquad |\mu^+| = |\mu^-| \tag{7.58}$$

Choosing $\beta = \lambda$ and $\kappa = 0$ in the Koike–Terada branching formulae and using the PRV conjecture we readily obtain that the restriction of V to $\mathfrak{gl}_n$ contains a summand with highest weight $[\overline{\mu^+}, \overline{\mu^-}]$. If $\overline{\mu^+} \neq \overline{\mu^-}$ this summand isn't self–dual and the theorem follows. If, on the other hand, $\overline{\mu^+} = \overline{\mu^-}$ but neither are of the form

$$(p, 0, \ldots, 0), \qquad (p, p, 0, \ldots, 0) \qquad \text{or} \qquad (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \tag{7.59}$$

for some $p \in \mathbb{N}$ and $k \leq n/2$, then $V_{[\overline{\mu^+},\overline{\mu^-}]}$ is a self–dual $\mathfrak{gl}_n$–module but, by theorem 7.1 contains a non–self dual $\mathfrak{gl}_k$–summand U with $U \cap V[0] \neq \{0\}$.

We need therefore only consider the case where $\overline{\mu^+} = \overline{\mu^-}$ and both are of the form (7.59). If, under this assumption, $\lambda_o \neq 0$, then all its non–zero entries must be 1's or else each $\mu^{\pm}$, and therefore $\overline{\mu_{\pm}}$, would have two distinct non–zero entries. In particular $\lambda_e \neq 0$ or λ would be of the form (7.44), so that $\lambda_1 \in 2\mathbb{N}+2$ since λ is dominant. Pick $j \in I_o^+$ and set

$$\widetilde{\mu}^{\pm} = \mu^{\pm} \pm e_1 \mp e_j \in \mathbb{N}^n \tag{7.60}$$

Then again $\lambda = \widetilde{\mu}^+ + \widetilde{\mu}^-$,

$$|\widetilde{\mu}^+| = |\widetilde{\mu}^-| \qquad \text{but} \qquad \overline{\widetilde{\mu}^+} \neq \overline{\widetilde{\mu}^-} \tag{7.61}$$

If, on the other hand, $\lambda_o = 0$, then

$$\lambda = (\underbrace{2a, \dots, 2a}_{l}, 0, \dots, 0) \tag{7.62}$$

for some $a \geq 1$ and $2 \leq l \leq n/2$. If $a = 1$, then $l \geq 3$ since λ is not of the form (7.43) and we may take instead $\beta = \lambda$, $\kappa = 0$ and

$$\mu = (2, 0, \underbrace{1, \dots, 1}_{l-2}, 0, \dots, 0) \tag{7.63}$$

$$\nu = (0, 2, \underbrace{1, \dots, 1}_{l-2}, 0, \dots, 0) \tag{7.64}$$

In this case $\overline{\mu} = \overline{\nu}$ but, by theorem 7.1, the simple $\mathfrak{gl}_n$–module with highest weight

$$[\overline{\mu}, \overline{\nu}] = (2, \underbrace{1, \dots, 1}_{l-2}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{l-2}, -2) \tag{7.65}$$

contains a simple $\mathfrak{gl}_k$–summand which isn't self–dual and intersects $V[0]$ non–trivially. If, on the other hand, $a > 1$, then $l \geq 2$ since λ is not of the form (7.41) and we take $\beta = \lambda$, $\kappa = 0$ and

$$\mu = (2a-1, 1, \underbrace{a, \dots, a}_{l-2}, 0, \dots, 0) \tag{7.66}$$

$$\nu = (1, 2a-1, \underbrace{a, \dots, a}_{l-2}, 0, \dots, 0) \tag{7.67}$$

and again conclude via theorem 7.1.

Assume now $|\lambda| \in 2\mathbb{N}+1$, so that $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$. With $l = \min\{i | \lambda_j = 0, \forall j > i\}$, set

$$\beta = \lambda - e_l = (\lambda_1, \dots, \lambda_{l-1}, \lambda_l - 1, 0, \dots, 0) \tag{7.68}$$

and $\kappa = e_1 = \overline{e_l}$ so that $L^{\lambda}_{\beta,\kappa} \neq 0$ by the PRV conjecture. Since $|\beta| \in 2\mathbb{N}$, we may apply the first part of the proof to β to conclude unless the latter is of one of the forms (7.41)–(7.44). (7.42) is ruled out by the fact that $|\beta| \in 2\mathbb{N}$

and (7.44) by the fact that λ itself is not of the form (7.44). If, on the other hand, β is of the form (7.41) or (7.43) then $\lambda_l = 1$ and either $l = 2$ or $l = 3$ and $\lambda = (2, 2, 1, 0, \ldots, 0)$. If $l = 2$, then $\lambda = (p, 1, 0, \ldots, 0)$ where, in view of the theorem's assumptions, $p \geq 3$. In that case, we take $\kappa = (p-3, 0, \ldots, 0)$ and

$$\beta = (3, 1, 0, \ldots, 0) \tag{7.69}$$
$$\mu = (2, 0, 0, \ldots, 0) \tag{7.70}$$
$$\nu = (1, 1, 0, \ldots, 0) \tag{7.71}$$

If, on the other hand, $\lambda = (2, 2, 1, 0, \ldots, 0)$, we choose $\kappa' = (0, 1, 0, \ldots, 0)$, $\kappa = \overline{\kappa'}$ and

$$\beta = (2, 1, 1, \ldots, 0) \tag{7.72}$$
$$\mu = (2, 0, 0, \ldots, 0) \tag{7.73}$$
$$\nu = (0, 1, 1, \ldots, 0) \tag{7.74}$$

■

7.5. **Self–dual representations of $\mathfrak{so}_m$.**

Theorem 7.11. *Let $\mathfrak{g} = \mathfrak{so}_m$, with $m = 2n, 2n+1$, and let V be a simple, self–dual $\mathfrak{g}$–module with $V[0] \neq \{0\}$. Then, $V[0]$ is irreducible under the Casimir algebra $\mathcal{C}_\mathfrak{g}$ if its highest weight λ has one of the following forms,*

(i) $\lambda = (p, 0, \ldots, 0)$, $p \in \mathbb{N}$.
(ii) $\lambda = (2, 2, 0, \ldots, 0)$.
(iii) $\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$, $1 \leq k \leq n$.
(iv) $\lambda = (\underbrace{1, \ldots, 1}_{n-1}, -1)$.

Conversely, if $\lambda \neq 0$ is of none of the above forms and satisfies $\lambda_i = 0$ for $i > n/2$, then $V[0]$ is reducible under $\mathcal{C}_\mathfrak{g}$.

Proof. Case (i) is the contents of theorem 6.2. Case (ii) follows from theorem 7.24 since the highest weight of the adjoint representation of $\mathfrak{g}$ is $(1, 1, 0, \ldots, 0)$. Cases (iii) and (iv) follow from proposition 7.12 below since the corresponding λ's are the highest weights of the exterior powers of the vector representation $U \cong \mathbb{C}^m$ of $\mathfrak{g}$ or, if $m = 2n$ and $\lambda = (1, \ldots, 1, \pm 1)$ of the eigenspaces of the Hodge star operator inside $\bigwedge^n U$. Finally, the converse follows from proposition 7.13 below, which gives the reducibility of $V[0]$ if $\lambda = (2, 1, 0, \ldots, 0)$ and proposition 7.9 which deals with all other cases ■

Let $U \cong \mathbb{C}^m$ be the vector representation of $\mathfrak{g} = \mathfrak{so}_m$ and recall that the exterior powers $\bigwedge^k U$, $0 \leq k < m/2$ are simple $\mathfrak{g}$–modules and that, for m even the eigenspaces $\bigwedge^{m/2}_{\pm} U \subset \bigwedge^{m/2} U$ of the Hodge $*$–operator are also irreducible under $\mathfrak{g}$. For m odd, all exterior powers of U have non–trivial zero weight spaces while, for m even, only the even exterior powers do.

Proposition 7.12.

(i) *If $m = 2n+1$ is odd, the zero weight spaces $\bigwedge^k U[0]$, $0 \leq k \leq n$, are irreducible under the Casimir algebra of $\mathfrak{g}$.*

(ii) *If $m = 2n$ is even, the zero weight spaces $\bigwedge^{2k} U[0]$, $0 \leq k \leq n/2-1$, and, if n is even, the zero weight spaces $\bigwedge^n_{\pm} U[0]$ are irreducible under the Casimir algebra of $\mathfrak{g}$.*

Proof. By proposition 3.13, it suffices to show that the above zero weight spaces are irreducible under the Weyl group of $\mathfrak{g}$. This is proved by a simple, direct calculation in [Re, section 3] ■

Proposition 7.13. *Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and let U be the simple $\mathfrak{g}$–module with highest weight $\lambda = (2, 1, 0, \ldots, 0)$. Then, $U[0]$ is reducible under the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$ of $\mathfrak{g}$.*

Proof. We claim that the Chevalley involution Θ of $\mathfrak{g}$ does not act as a scalar on the zero weight space $U[0]$. Let $v_{2\theta_1 \pm \theta_2} \in U$ be the highest weight vectors in U so that

$$v_0 = f_{\theta_1} f_\theta\, v_{2\theta_1+\theta_2} \tag{7.75}$$

where $\theta = \theta_1 + \theta_2$ is the highest root of $\mathfrak{g}$, lies in $U[0]$. It suffices to show that Θv_0 is not proportional to v_0. Let for this purpose $V = V_{\theta_1} \cong \mathbb{C}^{2n+1}$ be the defining representation of $\mathfrak{g}$ and realise U as the highest weight component of $V \otimes \mathrm{ad}(\mathfrak{g})$. We may then take

$$v_{2\theta_1+\theta_2} = e_1 \otimes e_\theta \tag{7.76}$$

where $\{e_{\pm i}, e_0\}$ is the standard weight basis of V with $e_{\pm i}$ of weight $\pm\theta_i$ and e_0 of weight 0 so that e_1 is the highest weight vector. This yields,

$$\begin{aligned} v_0 &= f_{\theta_1}(a e_{-2} \otimes e_\theta - e_1 \otimes h_\theta) \\ &= b e_{-2} \otimes e_{\theta_2} + c e_0 \otimes h_\theta - e_1 \otimes f_{\theta_1} \end{aligned} \tag{7.77}$$

where a, b, c are some (non–zero) constants. Since, up to a sign

$$\Theta\, e_i = e_{-i}, \quad \Theta\, e_0 = e_0, \quad \Theta\, e_\alpha = -f_\alpha \quad \text{and} \quad \Theta\, h_\theta = -h_\theta \tag{7.78}$$

Θv_0 is not proportional to v_0 ■

7.6. Representations of $\mathfrak{sp}_{2n}$.

Theorem 7.14. *Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ and let V be a simple $\mathfrak{g}$–module with $V[0] \neq \{0\}$. Then, $V[0]$ is irreducible under the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$ if its highest weight λ has one of the following forms,*

(i) $\lambda = (2p, 0, \ldots, 0)$, $p \in \mathbb{N}$.

(ii) $\lambda = (2, 2, 0, \ldots, 0)$.

(iii) $\lambda = (\underbrace{1, \ldots, 1}_{2k}, 0, \ldots, 0)$, $1 \leq k \leq n/4$.

Conversely, if $\lambda \neq 0$ is not of the above form and satisfies $\lambda_i = 0$ for $i > n/2$, then $V[0]$ is reducible under $\mathcal{C}_{\mathfrak{g}}$.

Proof. The irreducibility of $V[0]$ if λ is of the form (i), (ii) or (iii) follows from theorem 6.5 and propositions 7.16 and 7.15 below respectively. The converse follows from proposition 7.9 ■

Proposition 7.15. *Let V_k be the simple $\mathfrak{sp}_{2n}$–module with highest weight*

$$\lambda_k = (\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k}) \tag{7.79}$$

with $1 \leq k \leq n/2$. If k is even, so that $V_k[0] \neq \{0\}$, then $V_k[0]$ is irreducible under $\mathcal{C}_{\mathfrak{sp}_{2n}}$.

Proof. By (7.48)

$$\operatorname{res}^{\mathfrak{gl}_n}_{\mathfrak{sp}_{2n}} V_k = \bigoplus_{l=0}^{k} V_{l,k} \tag{7.80}$$

where $V_{l,k}$ is the simple $\mathfrak{gl}_n$–module with highest weight

$$\mu_{l,k} = (\underbrace{1, \ldots, 1}_{l}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{k-l}) \tag{7.81}$$

Thus, $V_k[0] = V_{k/2,k}[0]$ and the claim follows from lemma 7.2 ■

The rest of this subsection will be devoted to the proof of the following

Proposition 7.16. *The zero weight space of the simple $\mathfrak{g}$–module with highest weight $2\theta_1 + 2\theta_2$ is irreducible under the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$ of $\mathfrak{g}$.*

We shall need an explicit description of $V_{2\theta_1+2\theta_2}$. The latter is the second Cartan power of the representation with highest weight $\theta_1 + \theta_2$, which may in turn be realised as the subspace $\bigwedge_0^2 V \subset \bigwedge^2 V$ of vectors whose pairing with the symplectic form is zero.

Lemma 7.17. *Let $e : S^2 \bigwedge_0^2 V \longrightarrow \bigwedge^4 V$ be the exterior multiplication given by*

$$e(u_1 \wedge v_1 \cdot u_2 \wedge v_2) = u_1 \wedge v_1 \wedge u_2 \wedge v_2 \tag{7.82}$$

Then e is surjective and

$$\operatorname{Ker}(e) \cong V_{2\theta_1+2\theta_2} \tag{7.83}$$

Proof. The highest weight vector of $\bigwedge_0^2 V$ is $v_{\theta_1+\theta_2} = e_1 \wedge e_2$. Since e is $\mathfrak{g}$–equivariant and maps the highest weight vector of $V_{2\theta_1+2\theta_2}$, namely $v_{\theta_1+\theta_2} \cdot v_{\theta_1+\theta_2}$, to zero, one has

$$V_{2\theta_1+2\theta_2} \subset \operatorname{Ker}(e) \tag{7.84}$$

On the other hand, a simple application of the Weyl dimension formula yields

$$\dim V_{2\theta_1+2\theta_2} = \frac{n(n-1)(2n-1)(2n+3)}{3} \tag{7.85}$$

which is readily seen to be equal to

$$\dim S^2 \bigwedge_0^2 V - \dim \bigwedge^4 V \tag{7.86}$$

It therefore suffices to show that e is surjective to prove (7.83). Recall first that, for any $2 \leq k \leq n$, the irreducible representation with highest $\lambda_k = \theta_1 + \cdots + \theta_k$ may be realised as the kernel of the surjective map $\tau_k : \bigwedge^k V \to \bigwedge^{k-2} V$ given by contracting with the symplectic form $(\cdot,\cdot)$. Explicitly,

$$\tau_k(v_1 \wedge \cdots \wedge v_k) = \sum_{1\leq i<j\leq k} (-1)^{i+j-1}(v_i, v_j) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k \tag{7.87}$$

In particular, if $n \geq 4$, which we henceforth assume,

$$\bigwedge^4 V = \bigwedge_0^4 V \oplus \bigwedge_0^2 V \oplus \mathbb{C} \tag{7.88}$$

as $\mathfrak{g}$–modules. Consider now the image of the exterior multiplication e in $\bigwedge^4 V$. It contains the highest weight vector

$$v_{\theta_1+\cdots+\theta_4} = e_1 \wedge e_2 \wedge e_3 \wedge e_4 = e(e_1 \wedge e_2 \cdot e_3 \wedge e_4) \tag{7.89}$$

of $\bigwedge_0^4 V$ and therefore contains $\bigwedge_0^4 V$. Similarly, the image of $\tau_4 \circ e$ contains the highest weight vector

$$v_{\theta_1+\theta_2} = e_1 \wedge e_2 = \tau_4 \circ e(e_1 \wedge e_3 \cdot e_{-3} \wedge e_2) \tag{7.90}$$

of $\bigwedge_0^2 V$ and therefore contains $\bigwedge_0^2 V$. Finally

$$\tau_2 \circ \tau_4 \circ e : S^2 \bigwedge_0^2 \to \mathbb{C} \tag{7.91}$$

is readily seen to be non–zero and therefore surjective from which it follows that e itself is surjective if $n \geq 4$. The remaining cases are treated by a simple variant of the above argument by noting that, if $n = 3$,

$$\bigwedge^4 V \cong \bigwedge^2 V \tag{7.92}$$

via the Hodge $*$–operator and, if $n = 2$, $\bigwedge^4 V \cong \mathbb{C}$ ■

Remark. It follows from the previous lemma that

$$\begin{aligned}\dim(V_{2\theta_1+2\theta_2}[0]) &= \dim S^2 \bigwedge_0^2 V[0] - \dim \bigwedge^4 V[0] \\ &= n(n-1)\end{aligned} \tag{7.93}$$

Consider now the following zero weight vectors in $S^2 \bigwedge^2 V$

$$a_{ij} = e_i \wedge e_{-j} \cdot e_j \wedge e_{-i} \tag{7.94}$$

$$b_{ij} = e_i \wedge e_j \cdot e_{-j} \wedge e_{-i} \tag{7.95}$$

$$c_{ij} = e_i \wedge e_{-i} \cdot e_j \wedge e_{-j} \tag{7.96}$$

where $1 \leq i < j \leq n$ for a_{ij} and b_{ij} and $1 \leq i \leq j \leq n$ for c_{ij}, and set

$$\begin{aligned} e_{ij} &= c_{ii} + c_{jj} - 2c_{ij} \\ &= (e_i \wedge e_{-i} - e_j \wedge e_{-j}) \cdot (e_i \wedge e_{-i} - e_j \wedge e_{-j}) \\ &\in S^2 \bigwedge\nolimits_0^2 V \end{aligned} \tag{7.97}$$

Lemma 7.18.

(i) *For $1 \leq i < j \leq n$, the vectors*

$$v_{ij} = a_{ij} + b_{ij} \tag{7.98}$$

$$h_{ij} = e_{ij} - a_{ij} + b_{ij} \tag{7.99}$$

form a basis of the zero weight space of $V_{2\theta_1+2\theta_2}$.

(ii) *One has*

$$\kappa_{2\theta_k}\, v_{ij} = -4\delta_{k\in\{i,j\}} v_{ij} \tag{7.100}$$

$$\kappa_{2\theta_k}\, h_{ij} = 0 \tag{7.101}$$

PROOF. The vectors v_{ij}, h_{ij} lie in the kernel of the exterior multiplication e and are readily seen to be linearly independent. By the remark following lemma 7.17 they therefore are a basis of $V_{2\theta_1+2\theta_2}[0]$. A straightforward calculation, using the fact that $e_{2\theta_k}$ and $f_{2\theta_k}$ may be chosen to act as the elementary matrices $E_{k,-k}, -E_{-k,k}$ on V respectively [FH, §16.1], yields

$$f_{\theta_k} e_{\theta_k}\, a_{ij} = -\delta_{k\in\{i,j\}}(a_{ij} + b_{ij}) \tag{7.102}$$

$$f_{\theta_k} e_{\theta_k}\, b_{ij} = -\delta_{k\in\{i,j\}}(a_{ij} + b_{ij}) \tag{7.103}$$

$$f_{\theta_k} e_{\theta_k}\, c_{ij} = 0 \tag{7.104}$$

and therefore (7.100)–(7.101) since κ_α acts as $\langle\alpha,\alpha\rangle f_\alpha e_\alpha$ on zero weight vectors ■

Thus, the (commuting) long root Casimirs $\kappa_{2\theta_k}$ are diagonal in the basis v_{ij}, h_{ij}. We turn now to the action of the short root Casimirs $\kappa_{\theta_k\pm\theta_l}$ in this basis.

Lemma 7.19. *One has*

$$\begin{aligned} -\kappa_{\theta_k+\theta_l}\, v_{ij} &= \delta_{\{i,j\},\{k,l\}}\,(e_{kl} + 2b_{kl}) \\ &\quad + \delta_{|\{i,j\}\cap\{k,l\}|,1}\,(v_{ij} + v_{\{i,j\}\Delta\{k,l\}}) \end{aligned} \tag{7.105}$$

$$\begin{aligned} -\kappa_{\theta_k+\theta_l}\, h_{ij} &= (5\delta_{\{i,j\},\{k,l\}} + \delta_{|\{i,j\}\cap\{k,l\}|,1})\,(e_{kl} + 2b_{kl}) \\ &\quad + \delta_{|\{i,j\}\cap\{k,l\}|,1}\,(h_{ij} - h_{\{i,j\}\Delta\{k,l\}}) \end{aligned} \tag{7.106}$$

PROOF. A straightfoward calculation, using the fact that $e_{\theta_k+\theta_l}$ and $f_{\theta_k+\theta_l}$ may be chosen to act on V as the elementary matrices $E_{k,-l} + E_{l,-k}$ and

$-(E_{-k,l}+E_{-l,k})$ respectively [FH, §16.1], shows that, for $1\leq i<j\leq n$,

$$-\kappa_{\theta_k+\theta_l}\,a_{ij}=\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(a_{ij}+b_{\{i,j\}\Delta\{k,l\}}) \tag{7.107}$$

$$\begin{aligned}-\kappa_{\theta_k+\theta_l}\,b_{ij}&=\delta_{\{i,j\},\{k,l\}}\,(e_{ij}+2b_{ij})\\&\quad+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(b_{ij}+a_{\{i,j\}\Delta\{k,l\}})\end{aligned} \tag{7.108}$$

$$\begin{aligned}-\kappa_{\theta_k+\theta_l}\,c_{ij}&=\delta_{\{i,j\},\{k,l\}}\,(-e_{ij}-2b_{ij})\\&\quad+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(c_{ij}-c_{\{i,j\}\Delta\{k,l\}})\end{aligned} \tag{7.109}$$

$$\begin{aligned}-\kappa_{\theta_k+\theta_l}\,c_{ii}&=2\delta_{ik}\,(c_{ii}-c_{il}+b_{il})\\&\quad+2\delta_{il}\,(c_{ii}-c_{ik}+b_{ik})\end{aligned} \tag{7.110}$$

where $\{i,j\}\Delta\{k,l\}$ is the symmetric difference $\{i,j\}\setminus\{k,l\}\sqcup\{k,l\}\setminus\{i,j\}$. Assembling these results, one finds

$$\begin{aligned}-\kappa_{\theta_k+\theta_l}\,e_{ij}&=4\delta_{\{i,j\},\{k,l\}}\,(e_{ij}+2b_{ij})\\&\quad+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(e_{ij}-e_{\{i,j\}\Delta\{k,l\}})\\&\quad+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(e_{kl}+2b_{kl})\end{aligned} \tag{7.111}$$

and therefore the formulae (7.105)–(7.106) ■

Lemma 7.20. *One has*

$$\begin{aligned}\kappa_{\theta_k-\theta_l}\,v_{ij}&=\delta_{\{i,j\},\{k,l\}}\,(-e_{kl}+2a_{kl})\\&\quad+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(v_{ij}-v_{\{i,j\}\Delta\{k,l\}})\end{aligned} \tag{7.112}$$

$$\begin{aligned}\kappa_{\theta_k-\theta_l}\,h_{ij}&=(5\delta_{\{i,j\},\{k,l\}}+\delta_{|\{i,j\}\cap\{k,l\}|,1})\,(e_{kl}-2a_{kl})\\&\quad+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(h_{ij}-h_{\{i,j\}\Delta\{k,l\}})\end{aligned} \tag{7.113}$$

Proof. A straightfoward calculation, using the fact that $e_{\theta_k-\theta_l}$ and $f_{\theta_k-\theta_l}$ may be chosen to act on V as the elementary matrices $-E_{k,l}+E_{-l,-k}$ and $-E_{l,k}+E_{-k,-l}$ respectively [FH, §16.1], shows that,

$$\begin{aligned}\kappa_{\theta_k-\theta_l}\,a_{ij}&=\delta_{\{i,j\},\{k,l\}}\,(-e_{ij}+2a_{ij})\\&\quad+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(a_{ij}-a_{\{i,j\}\Delta\{k,l\}})\end{aligned} \tag{7.114}$$

$$\kappa_{\theta_k-\theta_l}\,b_{ij}=\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(b_{ij}-b_{\{i,j\}\Delta\{k,l\}}) \tag{7.115}$$

$$\begin{aligned}\kappa_{\theta_k-\theta_l}\,c_{ij}&=\delta_{\{i,j\},\{k,l\}}\,(-e_{ij}+2a_{ij})\\&\quad+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(c_{ij}-c_{\{i,j\}\Delta\{k,l\}})\end{aligned} \tag{7.116}$$

$$\begin{aligned}\kappa_{\theta_k-\theta_l}\,c_{ii}&=2\delta_{ik}\,(c_{ii}-c_{il}-a_{il})\\&\quad+2\delta_{il}\,(c_{ii}-c_{ik}-a_{ik})\end{aligned} \tag{7.117}$$

Assembling these results, one finds

$$\begin{aligned}\kappa_{\theta_k-\theta_l}\, e_{ij} &= 4\delta_{\{i,j\},\{k,l\}}\,(e_{ij}-2a_{ij})\\ &+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(e_{ij}-e_{\{i,j\}\Delta\{k,l\}})\\ &+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(e_{kl}-2a_{kl})\end{aligned}\tag{7.118}$$

and therefore formulae (7.112)–(7.113) ■

Let us summarise the findings of the previous two lemmas.

Corollary 7.21. *For any* $1\leq k<l\leq n$*, set*

$$P_{kl}=-\frac{\kappa_{\theta_k+\theta_l}+\kappa_{\theta_k-\theta_l}}{2}\qquad\text{and}\qquad M_{kl}=\frac{\kappa_{\theta_k+\theta_l}-\kappa_{\theta_k-\theta_l}}{2}\tag{7.119}$$

Then,

$$P_{kl}\,v_{ij}=\delta_{\{i,j\},\{k,l\}}\,h_{kl}+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,v_{\{i,j\}\Delta\{k,l\}}\tag{7.120}$$

$$P_{kl}\,h_{ij}=(5\delta_{\{i,j\},\{k,l\}}+\delta_{|\{i,j\}\cap\{k,l\}|,1})\,v_{kl}\tag{7.121}$$

$$M_{kl}\,v_{ij}=\delta_{\{i,j\}\cap\{k,l\}\neq\phi}\,v_{ij}\tag{7.122}$$

$$M_{kl}\,h_{ij}=5\delta_{\{i,j\},\{k,l\}}\,h_{kl}+\delta_{|\{i,j\}\cap\{k,l\}|,1}\,(h_{ij}-h_{\{i,j\}\Delta\{k,l\}}+h_{kl})\tag{7.123}$$

Proof of Proposition 7.16. Let $U\subseteq V_{2\theta_1+2\theta_2}[0]$ be a subspace invariant under $\mathcal{C}_{\mathfrak{g}}$. By lemma 7.18, it decomposes as $(U\cap V)\oplus(U\cap H)$ where V (resp. H) is the span of the v_{ij} (resp. h_{ij}). If $U\cap V$ is non–zero, it must contain at least one v_{ij} since, by lemma 7.18, $1/16\ \kappa_{2\theta_i}\kappa_{2\theta_j}$ acts on $V_{2\theta_1+2\theta_2}[0]$ as the projector onto v_{ij}. A repeated application of (7.120) then shows that U contains all of V. Since $P_{ij}v_{ij}=h_{ij}$ by (7.120), $U\supseteq\sum_{i,j}P_{ij}V\supseteq H$ whence $U=V[0]$. If, on the other hand, $U\cap V=\{0\}$, then, by (7.121) $P_{kl}u=0$ for any $1\leq k<l\leq n$ and $u\in U$. Thus, if $u=\sum u_{ij}h_{ij}\in U=U\cap H$, then, for any $1\leq k<l\leq n$,

$$0=P_{kl}u=(5u_{kl}+\sum_{i\neq k,l}u_{il}+\sum_{j\neq k,l}u_{kj})v_{kl}=(3u_{kl}+\sum_{i\neq l}u_{il}+\sum_{j\neq k}u_{kj})v_{kl}\tag{7.124}$$

Regarding u as a symmetric, $n\times n$ matrix with zero diagonal entries, we may rewrite the above system of equations as

$$-3u=Q(Pu+uP)=Q(l(P)+r(P))Qu\tag{7.125}$$

where P is the $n\times n$ matrix with all entries equal to 1, $l(P),r(P)\in\mathrm{End}(M_n(\mathbb{C}))$ are the operators of left and right multiplication by P and Q is the projection onto the subspace of matrices with zero diagonal in $M_n(\mathbb{C})$. This implies that $u=0$ since $l(P),r(P),Q$, and therefore $Q(l(P)+r(P))Q$ are positive semi–definite. It follows that $U=\{0\}$ ■

7.7. **Small Cartan powers of the adjoint representation.** We prove in this subsection that, for any simple $\mathfrak{g}$, the zero weight spaces of the first and second Cartan powers of the adjoint representaton of $\mathfrak{g}$ are irreducible under the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$.

Proposition 7.22. *For any complex, simple Lie algebra* $\mathfrak{g}$, $\mathfrak{h} = \mathrm{ad}(\mathfrak{g})[0]$ *is irreducible under* $\mathcal{C}_{\mathfrak{g}}$.

Proof. The result follows at once from proposition 3.13 because $\mathfrak{h}$ is irreducible under W ■

Let now θ be the highest root of $\mathfrak{g}$ and e_θ a corresponding root vector. Clearly, $e_\theta \cdot e_\theta$ lies in $C^2\mathfrak{g} \subset S^2\mathfrak{g}$ and therefore so does

$$\mathrm{ad}(f_\theta)^2\, e_\theta \cdot e_\theta = 2h_\theta^2 - 4f_\theta \cdot e_\theta \tag{7.126}$$

Lemma 7.23. *If* $\mathfrak{g}$ *is simply–laced, the vectors*

$$v_\alpha = e_\alpha \cdot f_\alpha - \frac{1}{2}h_\alpha^2 \in S^2\mathfrak{g} \tag{7.127}$$

corresponding to the positive roots of $\mathfrak{g}$ *form a basis of the zero weight space of* $C^2\mathfrak{g}$.

Proof. Since v_θ lies in $C^2\mathfrak{g}[0]$, so do all the v_α since the Weyl group of $\mathfrak{g}$ transitively permutes the roots of $\mathfrak{g}$. Since the v_α are linearly independent, it suffices to show that the dimension of $C^2\mathfrak{g}[0]$ is equal to the number of positive roots of $\mathfrak{g}$, which will be proved case–by–case. For $\mathfrak{g} = \mathfrak{sl}_n$, $C^2\mathfrak{g}$ has highest weight $(2, 0, \ldots, 0, -2)$. A simple computation using Gelfand–Zetlin patterns then shows that $\dim(C^2\mathfrak{g}[0]) = n(n-1)/2$ which is the number of positive roots of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{so}_{2n}$, we may assume that $n \geq 4$ since $\mathfrak{so}_6 \cong \mathfrak{sl}_4$. The highest weight of the adjoint representation is $(1, 1, 0, \ldots, 0)$ and the Koike–Terada branching formulae (7.49) yield

$$\mathrm{res}^{\mathfrak{gl}_n}_{\mathfrak{so}_{2n}}\, C^2\mathfrak{g} = \mathbb{C} \oplus V_{(1,0,\ldots,0,-1)} \oplus V_{(2,0,\ldots,0,-2)} \oplus V_{(1,1,0,\ldots,0,-1,-1)} \oplus R \tag{7.128}$$

where R has trivial zero–weight space. Since $V_{(1,0,\ldots,0,-1)}$ is the adjoint representation of $\mathfrak{gl}_n$, the zero–weight spaces of the first three summands have dimensions $1, n-1, n(n-1)/2$ respectively and it therefore suffices to show that $\dim V_{(1,1,0,\ldots,0,-1,-1)}[0] = n(n-3)/2$ for then $\dim(C^2\mathfrak{g}[0]) = n(n-1)$ which is the number of positive roots of $\mathfrak{so}_{2n}$.

To compute $\dim V_{(1,1,0,\ldots,0,-1,-1)}[0]$, let U be the vector representation of $\mathfrak{gl}_n$ and recall that the exterior powers of U are minuscule representations of $\mathfrak{gl}_n$ *i.e.,* their weights lie on a single orbit under the Weyl group. Moreover, a straightforward application of the Weyl character formula shows that for a minuscule representation V_λ with highest weight λ and any other irreducible representation V_μ with highest weight μ, one has

$$V_\lambda \otimes V_\mu = \bigoplus_{\lambda'} V_{\lambda'+\mu} \tag{7.129}$$

where the sum ranges over the weights λ' in the Weyl group orbit of λ such that $\lambda'+\mu$ is dominant (see, *e.g.*, [TL1, prop. II.2.2.2]). For $V_\lambda = \bigwedge^2 U$ and $V_\mu = \bigwedge^2 U^*$, this yields,

$$\bigwedge^2 U \otimes \bigwedge^2 U^* = \mathbb{C} \oplus V_{(1,0,\ldots,0,-1)} \oplus V_{(1,1,0,\ldots,0,-1,-1)} \tag{7.130}$$

Equating the dimensions of the zero weight spaces of each side yields

$$\dim V_{(1,1,0,\ldots,0,-1,-1)}[0] = n(n-3)/2 \tag{7.131}$$

as required.

Finally, for $\mathfrak{g}$ of exceptional type, the program LiE [LiE] yields

$$\dim(C^2\,\mathrm{ad}(E_6)[0]) = 36 \tag{7.132}$$

$$\dim(C^2\,\mathrm{ad}(E_7)[0]) = 63 \tag{7.133}$$

$$\dim(C^2\,\mathrm{ad}(E_8)[0]) = 120 \tag{7.134}$$

which are the number of positive roots of E_6, E_7 and E_8 respectively ■

Theorem 7.24. *If $\mathfrak{g}$ is a complex, simple Lie algebra, the zero weight spaces of the first and second Cartan powers of the adjoint representation of $\mathfrak{g}$ are irreducible under $\mathcal{C}_{\mathfrak{g}}$.*

Proof. Assume first that $\mathfrak{g}$ is simply–laced and fix a positive root α. We claim that the vector $v_\alpha \in C^2\mathfrak{g}[0]$ given by (7.127) is the unique eigenvector for C_α in $\mathfrak{g}\otimes\mathfrak{g}[0]$, and *a fortiori* in $C^2\mathfrak{g}[0]$, with eigenvalue corresponding to the Casimir eigenvalue of the 5–dimensional representation of $\mathfrak{sl}_2^\alpha$. To see this, one readily checks that the restriction of $\mathrm{ad}(e_\alpha)^3$ to $\mathfrak{g}\otimes\mathfrak{g}[0]$ is zero and therefore that any $v\in\mathfrak{g}\otimes\mathfrak{g}[0]$ decomposes uniquely as

$$v = v_0^\alpha + v_1^\alpha + v_2^\alpha \tag{7.135}$$

where v_i^α lies in an irreducible $\mathfrak{sl}_2^\alpha$ summand of dimension $2i+1$. Another straightforward computation shows that the restriction of $\mathrm{ad}(e_\alpha^2)$ to $\mathfrak{g}\otimes\mathfrak{g}[0]$ maps this space onto $\mathbb{C}\cdot e_\alpha\otimes e_\alpha$. Thus, the v_2^α in (7.135), which is necessarily proportional to $\mathrm{ad}(f_\alpha)^2\,\mathrm{ad}(e_\alpha)^2 v$, is a multiple of v_α as claimed. Let now $U\subseteq C^2\mathfrak{g}[0]$ be invariant under $\mathcal{C}_{\mathfrak{g}}$. By what precedes, and lemma 7.23, U necessarily contains one v_α for it cannot be orthogonal to all of them. Since U is also invariant under the Weyl group by proposition 3.13, it follows that U contains all v_α's and is therefore equal to $C^2\mathfrak{g}[0]$.

If $\mathfrak{g}=\mathfrak{sp}_{2n}$, the adjoint representation has highest weight $(2,0,\ldots,0)$ and the result follows from theorem 6.5.

The remaining cases, namely $\mathfrak{g}=\mathfrak{so}_{2n+1},\mathfrak{g}_2,\mathfrak{f}_4$ will be treated by restricting $C^2\mathfrak{g}$ to the following simple, simply–laced subalgebras of equal rank $\mathfrak{r}\subset\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{so}_{2n+1}\supset\mathfrak{so}_{2n}=\mathfrak{r} \tag{7.136}$$

$$\mathfrak{g}=\mathfrak{g}_2\supset\mathfrak{sl}_3=\mathfrak{r} \tag{7.137}$$

$$\mathfrak{g}=\mathfrak{f}_4\supset\mathfrak{so}_9=\mathfrak{r} \tag{7.138}$$

In all three cases, one finds that

$$\operatorname{res}^{\mathfrak{r}}_{\mathfrak{g}} C^2\operatorname{ad}(\mathfrak{g})=C^2\operatorname{ad}(\mathfrak{r})\oplus V\oplus R \tag{7.139}$$

where V is a simple $\mathfrak{r}$–module with non–trivial zero weight space and R is a possibly reducible $\mathfrak{r}$–module with trivial zero weight space. Specifically, for $\mathfrak{g}=\mathfrak{so}_{2n+1}$, the Gelfand–Zetlin rules for the ortogonal groups [GZ2], [Zh2, p. 103] yield

$$V=V_{(2,0,\ldots,0)}\cong C^2\mathbb{C}^{2n} \tag{7.140}$$

$$R=V_{(2,1,0,\ldots,0)} \tag{7.141}$$

while for $\mathfrak{g}=\mathfrak{g}_2$, Perroud's branching rules (7.10) give

$$V=V_{(2,1,0)}\cong\operatorname{ad}(\mathfrak{sl}_3) \tag{7.142}$$

$$R=V_{(3,2,0)}\oplus V_{(3,1,0)}\oplus V_{(2,2,0)}\oplus V_{(2,0,0)} \tag{7.143}$$

Finally, for $\mathfrak{g}=\mathfrak{f}_4$ one gets, from the program LiE [LiE] that

$$V=V_{(1,1,1,1)}\cong\bigwedge^4\mathbb{C}^9 \tag{7.144}$$

$$R=V_{1/2(3,3,1,1)} \tag{7.145}$$

One readily checks in each case that $C^2\operatorname{ad}(\mathfrak{r})$ and V are distinguished by the Casimir eigenvalue of $\mathfrak{r}$ so that, by our previous analysis and theorems 7.11 for $\mathfrak{r}=\mathfrak{so}_{2n}$ and $\mathfrak{r}=\mathfrak{so}_9$ and theorem 4.1 for $\mathfrak{r}=\mathfrak{sl}_3$, the zero weight spaces $C^2\operatorname{ad}(\mathfrak{r})[0]$ and $V[0]$ are irreducible and inequivalent representations of $\mathcal{C}_{\mathfrak{r}}$. Since $C^2\operatorname{ad}(\mathfrak{g})[0]=C^2\operatorname{ad}(\mathfrak{r})[0]\oplus V[0]$ it suffices to show that $C^2\operatorname{ad}(\mathfrak{r})[0]$ is not invariant under $\mathcal{C}_{\mathfrak{g}}$. Let θ be the highest root of $\mathfrak{g}$ and $e_\theta,f_\theta,h_\theta$ a corresponding $\mathfrak{sl}_2^\theta$ triple. In all cases, this triple lies in $\mathfrak{r}$ so that

$$v_\theta=-1/4\operatorname{ad}(f_\theta)\operatorname{ad}(e_\theta)e_\theta^2=e_\theta\cdot f_\theta-\frac{1}{2}h_\theta^2\in C^2\operatorname{ad}(\mathfrak{r})\subset C^2\operatorname{ad}(\mathfrak{g}) \tag{7.146}$$

where we are realising $C^2\operatorname{ad}(\mathfrak{g})$ as the highest weight component of $S^2\mathfrak{g}$. Choose in each case a positive, short root α of $\mathfrak{g}$ such that $\langle\theta^\vee,\alpha\rangle=1$ so that the α–string through θ is of the form $\theta-2\alpha,\theta-\alpha,\theta$. It is easy to see that such an α exists by consulting the tables in [Bo]. Then, a simple computation using a Chevalley basis of $\mathfrak{g}$ yields

$$-1\frac{1}{2}\operatorname{ad}(f_\alpha)\operatorname{ad}(e_\alpha)\,h_\theta^2=e_\alpha\cdot f_\alpha-h_\alpha\cdot h_\theta \tag{7.147}$$

$$\operatorname{ad}(f_\alpha)\operatorname{ad}(e_\alpha)\,e_\theta\cdot f_\theta=\pm e_{\theta-\alpha}\cdot f_{\theta-\alpha}\pm 2e_\theta\cdot f_\theta \tag{7.148}$$

where the signs depend on the choice of the root vectors. Thus,

$$\frac{1}{\langle\alpha,\alpha\rangle}C_\alpha v_\theta = \operatorname{ad}(f_\alpha)\operatorname{ad}(e_\alpha)v_\theta = e_\alpha\cdot f_\alpha \pm e_{\theta-\alpha}\cdot f_{\theta-\alpha} \pm 2e_\theta\cdot f_\theta - h_\alpha\cdot h_\theta \quad (7.149)$$

which does not lie in $C^2\operatorname{ad}(\mathfrak{r})$ since $\mathfrak{r}$ is simply–laced, and α is short ■

7.8. **Some conjectures.** Let us record the following corollary of the proofs of theorems 7.1, 7.4, 7.11 and 7.14.

Theorem 7.25. *Let $\mathfrak{g}$ be a classical Lie algebra or $\mathfrak{g}_2$ and let V be a simple, self–dual $\mathfrak{g}$–module with $V[0]\neq\{0\}$. If $\mathfrak{g}=\mathfrak{so}_{2n+1},\mathfrak{so}_{2n},\mathfrak{sp}_{2n}$, assume in addition that the highest weight λ of V satisfies $\lambda_i=0$ for all $i>n/2$. Then, if $V[0]$ is reducible under the Casimir algebra of $\mathfrak{g}$, the Chevalley involution does not act as a scalar on $V[0]$.*

In other words, for the above representations V, the failure of the Chevalley involution to act as a scalar on $V[0]$ is the only mechanism which causes $V[0]$ to be reducible under the Casimir algebra $\mathcal{C}_\mathfrak{g}$. It is therefore natural to make the following

Conjecture 7.26. *Let $\mathfrak{g}$ be a complex, simple Lie algebra and let V be a simple $\mathfrak{g}$–module which is self–dual and has a non–trivial zero weight space $V[0]$. Then $V[0]$ is irreducible under the Casimir algebra of $\mathfrak{g}$ iff the Chevalley involution of $\mathfrak{g}$ acts as a scalar on $V[0]$.*

We hope to return to this conjecture in a future publication. We note also that it would be interesting to be able to give a more precise formulation to our observation that the zero weight space of 'most' self–dual $\mathfrak{g}$–modules is reducible under the Casimir algebra. For $\mathfrak{g}=\mathfrak{sl}_n$ and $\mathfrak{g}_2$, theorems 7.1 and 7.4 give a complete list of those V for which $V[0]$ is irreducible. For $\mathfrak{g}=\mathfrak{so}_{2n+1},\mathfrak{so}_{2n},\mathfrak{sp}_{2n}$, we make the following

Conjecture 7.27. *Let $\mathfrak{g}$ be one of $\mathfrak{so}_{2n+1},\mathfrak{so}_{2n},\mathfrak{sp}_{2n}$ and V a self–dual, simple $\mathfrak{g}$–module with $V[0]\neq\{0\}$. If the highest weight λ of V is not of one of the forms listed in theorems 7.11 and 7.14 then $V[0]$ is reducible under $\mathcal{C}_\mathfrak{g}$.*

Still, it would be highly desirable to be able to formulate what 'most' means in a way independent of the Lie type of $\mathfrak{g}$. At the very least, for example, we conjecture

Conjecture 7.28. *Let V be a simple, self–dual $\mathfrak{g}$–module with $V[0]\neq 0$. If the highest weight of V is regular, then $V[0]$ is reducible under $\mathcal{C}_\mathfrak{g}$.*

The above conjecture is true for $\mathfrak{g}=\mathfrak{sl}_n$ and $\mathfrak{g}=\mathfrak{g}_2$ by theorems 7.1 and 7.4. It also holds for $\mathfrak{g}=\mathfrak{so}_{2n},\mathfrak{so}_{2n+1},\mathfrak{sp}_{2n}$. The proof in this case will be given in a future publication.

8. Appendix : The centraliser of the Casimir algebra

The results in this section are due to P. Etingof [Et] to whom we are grateful for allowing us to reproduce them here. Our aim is to prove the following

Theorem 8.1. *The centraliser in $U\mathfrak{g}$ of the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$ of $\mathfrak{g}$ is generated by the Cartan subalgebra $\mathfrak{h}$ and the centre $Z(U\mathfrak{g})$ of $U\mathfrak{g}$.*

The proof of theorem 8.1 rests on Knop's calculation of the centre of the subalgebra $U_{\hbar}\mathfrak{g} \subset U\mathfrak{g}$ of $\mathfrak{h}$–invariants and on the following result which is of independent interest

Theorem 8.2. *For any positive linear combination $\beta \in \bigoplus \alpha_i \cdot \mathbb{N}$ of simple roots, there exists a Zariski open set $O_\beta \subset \mathfrak{h}^*$ such that, for any $\mu \in O_\beta$, the $\mu - \beta$–weight space $M_\mu[\mu - \beta]$ of the Verma module with highest weight μ is irreducible under the action of the Casimir algebra $\mathcal{C}_{\mathfrak{g}}$.*

Proof of theorem 8.1. Assume that $x \in U\mathfrak{g}$ lies in the centraliser of $\mathcal{C}_{\mathfrak{g}}$. Since x commutes with $\mathfrak{h}$, it lies in the $\mathfrak{h}$–invariant subalgebra $U\mathfrak{g}^{\mathfrak{h}} \subset U\mathfrak{g}$. We claim that x lies in fact in the centre of $U\mathfrak{g}^{\mathfrak{h}}$. Indeed, if $y \in U\mathfrak{g}^{\mathfrak{h}}$, both x and y leave the weight spaces $M_\mu[\mu - \beta]$ invariant, where $\mu \in \mathfrak{h}^*$ and β is a fixed positive linear combination of simple roots. By theorem 8.2, the commutator $[x, y]$ acts as zero on $M_\mu[\mu - \beta]$ generically in μ, and therefore for all μ. Since this holds for any β, $[x, y]$ acts as zero on all Verma modules and is therefore zero since these separate elements in $U\mathfrak{g}$. Thus, $x \in Z(U\mathfrak{g}^{\mathfrak{h}})$ as claimed. Theorem 8.1 now follows from the fact that the centre of $U\mathfrak{g}^{\mathfrak{h}}$ is $U\mathfrak{h} \otimes Z(U\mathfrak{g})$ [Knp, thm. 10.1] ■

We need a preliminary result to prove theorem 8.2. Fix a weight $\lambda \in \mathfrak{h}^*$ and let $M_{t^2\lambda}$ be the Verma module with highest weight $t^2\lambda$, where $t \in \mathbb{C}^*$ is some non–zero complex number. Consider the standard identifications

$$M_{t^2\lambda} \xrightarrow{\imath} U\mathfrak{n}_- \xrightarrow{\sigma^{-1}} S\mathfrak{n}_- \tag{8.1}$$

where σ is the symmetrisation map. The corresponding isomorphism $M_{t^2\lambda} \cong S\mathfrak{n}_-$ is one of $\mathfrak{h}$–modules provided the adjoint action of $\mathfrak{h}$ on $S\mathfrak{n}_-$ is tensored by the character $t^2\lambda$. Denoting the generators of $S\mathfrak{n}_-$ by x_α and transporting the action of $\mathfrak{g}$ on $M_{t^2\lambda}$ to $S\mathfrak{n}_-$, we have the following

Lemma 8.3. *Let d be the grading operator on $S\mathfrak{n}_-$. Then, for any $t \in \mathbb{C}^*$*

$$t^d e_\alpha t^{-d} = t \cdot \langle \lambda, \alpha^\vee \rangle \partial_\alpha + O(1) \tag{8.2}$$

$$t^d f_\alpha t^{-d} = t \cdot x_\alpha + O(1) \tag{8.3}$$

$$t^d h_\alpha t^{-d} = t^2 \cdot \langle \lambda, \alpha^\vee \rangle + O(1) \tag{8.4}$$

where the terms $O(1)$ have a finite limit for $t \longrightarrow \infty$.

PROOF. (8.2) Let $v_{t^2\lambda} \in M_{t^2\lambda}$ be the highest weight vector. Then, for any sequence of positive roots $\beta_1, \ldots, \beta_k$, we have

$$\begin{aligned} & t^d e_\alpha t^{-d}\, x_{\beta_1} \cdots x_{\beta_k} = \\ & \quad t^{d-k}\sigma^{-1}\imath\left(\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}k}\sum_{i=1}^{k} f_{\beta_{\sigma(1)}}\cdots f_{\beta_{\sigma(i-1)}}[e_\alpha, f_{\beta_{\sigma(i)}}]f_{\beta_{\sigma(i+1)}}\cdots f_{\beta_{\sigma(k)}}\, v_{t^2\lambda}\right) \end{aligned} \tag{8.5}$$

The term corresponding to a fixed $\sigma \in \mathfrak{S}k$ and $i = 1 \ldots k$ clearly vanishes unless $\alpha - \beta_{\sigma(i)}$ is a root or zero. If $\alpha - \beta_{\sigma(i)}$ is a negative root, the corresponding term in $S\mathfrak{n}_-$ is of degree $\leq k$ and its total contribution an $O(1)$. On the other hand, the total contribution of the terms for which

$$\sigma(i) \in I_\alpha = \{j = 1 \ldots k | \beta_j = \alpha\} \tag{8.6}$$

is

$$\begin{aligned} & t^{d-k}\sigma^{-1} \\ & \left(\frac{1}{k!}\sum_{i,\sigma:\sigma(i)\in I_\alpha}\langle t^2\lambda - \beta_{\sigma(i+1)}\cdots - \beta_{\sigma(k)}, \alpha^\vee\rangle f_{\beta_{\sigma(1)}}\cdots f_{\beta_{\sigma(i-1)}} f_{\beta_{\sigma(i+1)}}\cdots f_{\beta_{\sigma(k)}}\right) \\ & = t^{d+2-k}\langle\lambda, \alpha^\vee\rangle\sigma^{-1}\left(\frac{|I_\alpha|}{k}\sum_{i=1}^{k}\sigma(\frac{x_{\beta_1}\cdots x_{\beta_k}}{x_\alpha})\right) + t^{-1}O(1) \\ & = t\langle\lambda, \alpha^\vee\rangle\partial_\alpha x_{\beta_1}\cdots x_{\beta_k} + t^{-1}O(1) \end{aligned} \tag{8.7}$$

Finally, if $\alpha - \beta_{\sigma(i)}$ is a positive root, a repetition of the above argument shows that the net contribution is an O(1). (8.3) We have,

$$\begin{aligned} t^d f_\alpha t^{-d}\, x_{\beta_1}\cdots x_{\beta_k} &= t^{d-k}\sigma^{-1}(f_\alpha\sigma(x_{\beta_1}\cdots x_{\beta_k})) \\ &= t^{d-k}\sigma^{-1}(\sigma(x_\alpha x_{\beta_1}\cdots x_{\beta_k}) + r) \\ &= t x_\alpha x_{\beta_1}\cdots x_{\beta_k} + O(1) \end{aligned} \tag{8.8}$$

for some $r \in U\mathfrak{n}_-$ of degree $\leq k$, where we used $f_\alpha = \sigma(x_\alpha)$ and

$$\sigma(p\cdot q) = \sigma(p)\cdot\sigma(q) + r' \tag{8.9}$$

for any $p, q \in S\mathfrak{n}_-$ where the remainder $r' \in U\mathfrak{n}_-$ is of degree $\leq \deg(p) + \deg(q) - 1$. (8.4) follows from the fact that the eigenvalues of h_α on $M_{t^2\lambda}$ lie in $t^2\langle\lambda, \alpha^\vee\rangle + \mathbb{Z}$ and that h_α commutes with d ■

PROOF OF THEOREM 8.2. Since the action of the Casimirs C_α on $M_\mu[\mu-\beta]$ depends polynomially on μ and irreducibility is an open condition, it suffices to show that the set of μ for which the C_α act irreducibly on $M_\mu[\mu-\beta]$ is non–empty. Let $\lambda \in \mathfrak{h}^*$ be a regular weight and choose μ of the form $t^2\lambda$, where $t \in \mathbb{C}^*$. Using the notation of lemma 8.3, it is sufficient to show that the operators

$$t^{-2}\,\mathrm{Ad}(t^d) f_\alpha e_\alpha \qquad \text{and} \qquad t^{-3}\,\mathrm{Ad}(t^d)[f_\alpha e_\alpha, f_\beta e_\beta] \tag{8.10}$$

act irreducibly on any subspace of $S\mathfrak{n}_-$ of fixed weight. Since this is again an open condition in t and, as will be shown below, the operators at hand have a finite limit as $t \to \infty$, it suffices to prove this for $t = \infty$. By lemma 8.3,

$$\lim_{t\to\infty} t^{-2}\,\mathrm{Ad}(t^d) f_\alpha e_\alpha = \langle \lambda, \alpha^\vee \rangle x_\alpha \partial_\alpha \tag{8.11}$$

Let now $\alpha \neq \beta$ be positive roots. Denoting by $R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$, we have

$$\begin{aligned}[f_\alpha e_\alpha, f_\beta e_\beta] &= f_\alpha e_\alpha f_\beta e_\beta - f_\beta e_\beta f_\alpha e_\alpha \\ &= f_\alpha f_\beta e_\alpha e_\beta - f_\beta f_\alpha e_\beta e_\alpha \\ &+ \delta_{\alpha-\beta\in R}\left(c_{\alpha,\beta} f_\alpha \varepsilon_{\alpha-\beta} e_\beta - c_{\beta,\alpha} f_\beta \varepsilon_{\beta-\alpha} e_\alpha\right)\end{aligned} \tag{8.12}$$

where $\varepsilon_\gamma = e_\gamma$ or f_γ according to whether the root γ is positive or negative and the $c_{\cdot,\cdot}$ are non–zero constants. Since

$$f_\alpha f_\beta e_\alpha e_\beta - f_\beta f_\alpha e_\beta e_\alpha = [f_\alpha, f_\beta] e_\alpha e_\beta + f_\beta f_\alpha [e_\alpha, e_\beta] \tag{8.13}$$

we find that

$$\begin{aligned}[f_\alpha e_\alpha, f_\beta e_\beta] = \delta_{\alpha+\beta\in R}&\left(c'_{\alpha,\beta} f_{\alpha+\beta} e_\alpha e_\beta + c'_{\beta,\alpha} f_\beta f_\alpha e_{\alpha+\beta}\right) \\ + \delta_{\alpha-\beta\in R}&\left(c_{\alpha,\beta} f_\alpha \varepsilon_{\alpha-\beta} e_\beta - c_{\beta,\alpha} f_\beta \varepsilon_{\beta-\alpha} e_\alpha\right)\end{aligned} \tag{8.14}$$

for some non–zero constants $c'_{\cdot,\cdot}$. It therefore follows from lemma 8.3 that

$$\begin{aligned}\lim_{t\to\infty} t^{-3}\,\mathrm{Ad}(t^d)[f_\alpha e_\alpha, f_\beta e_\beta] &= \delta_{\alpha+\beta\in R}\left(\widetilde{c}'_{\alpha,\beta} x_{\alpha+\beta} \partial_\alpha \partial_\beta + \widetilde{c}'_{\beta,\alpha} x_\beta x_\alpha \partial_{\alpha+\beta}\right) \\ &+ \delta_{\alpha-\beta\in R}\left(\widetilde{c}_{\alpha,\beta} x_\alpha \overline{\varepsilon}_{\alpha-\beta} \partial_\beta - \widetilde{c}_{\beta,\alpha} x_\beta \overline{\varepsilon}_{\beta-\alpha} \partial_\alpha\right)\end{aligned} \tag{8.15}$$

where $\overline{\varepsilon}_\gamma$ is now the operator ∂_γ or x_γ according to whether γ is positive or negative and the $\widetilde{c}_{\cdot,\cdot}, \widetilde{c}'_{\cdot,\cdot}$ are non–zero constants. Since the summands in the above expression have distinct homogeneity degrees with respect to the commuting Euler operators $x_\alpha \partial_\alpha$, it is sufficient to show that the weight spaces of $S\mathfrak{n}_-$, *i.e.,* the subspaces spanned by the monomials $\prod_{\alpha\succ 0} x_\alpha^{m_\alpha}$ with $\sum_{\alpha\succ 0} m_\alpha \alpha$ fixed, are irreducible under the operators

$$x_\alpha \partial_\alpha, \quad x_{\alpha+\beta} \partial_\alpha \partial_\beta, \quad x_\beta x_\alpha \partial_{\alpha+\beta}, \quad x_\alpha \overline{\varepsilon}_{\alpha-\beta} \partial_\beta, \quad x_\beta \overline{\varepsilon}_{\beta-\alpha} \partial_\alpha \tag{8.16}$$

which is a simple enough exercise ■

References

[Ar] M. Artin, *On Solutions of Analytic Equations*, Invent. Math. **5** (1968), 277–291.

[Bo] N. Bourbaki, *Groupes et Algèbres de Lie.* Chapitres IV,V,VI. Hermann, Paris 1968.

[Br] E. Brieskorn, *Die Fundamentalgruppe des Raumes der Regulären Orbits einer Endlichen Komplexen Spiegelungsgruppe*, Invent. Math. **12** (1971), 57–61.

[BMR] M. Broué, G. Malle, R. Rouquier, *Complex Reflection Groups, Braid Groups, Hecke Algebras*, J. Reine Angew. Math. **500** (1998), 127–190.

[Ch] E. Chirka , *Complex Analytic Sets.* Kluwer, Boston, 1985.

[Dr1] V. G. Drinfeld, *Quantum Groups*, Proceedings of the International Congress of Mathematicians, Berkeley 1986, 798–820.
[Dr2] V. G. Drinfeld, *On Almost Cocommutative Hopf Algebras*, Leningrad Math. J. **1** (1990), 321–342.
[Dr3] V. G. Drinfeld, *Quasi–Hopf Algebras*, Leningrad Math. J. **1** (1990), 1419–57.
[Dr4] V. G. Drinfeld, *On Quasitriangular Quasi–Hopf Algebras and on a Group that is closely connected with* $\mathrm{Gal}(\overline{Q}/Q)$, Leningrad Math. J. **2** (1991), 829–860.
[Et] P. Etingof, emails to V. Toledano Laredo, September–October 2001.
[FMTV] G. Felder, Y. Markov, V. Tarasov, A. Varchenko, *Differential Equations Compatible with KZ Equations*, Math. Phys. Anal. Geom. **3** (2000), 139–177.
[Fi] G. Fischer, *Complex Analytic Geometry*. Lecture Notes in Math. **538**, Springer–Verlag, New York, 1994.
[FH] W. Fulton, J. Harris, *Representation Theory. A First Course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.
[Ga] K. Gawędzki, *Lectures on Conformal Field Theory*, in *Quantum Fields and Strings: a Course for Mathematicians*, Vol. 1, 727–805, AMS, Providence, 1999.
[GZ1] I. M. Gelfand, M. L. Zetlin, *Finite–Dimensional Representations of the Group of Unimodular Matrices*. (Russian) Doklady Akad. Nauk SSSR (N.S.) **71**, (1950). 825–828.
[GZ2] I. M. Gelfand, M. L. Zetlin, *Finite–Dimensional Representations of Groups of Orthogonal Matrices*, (Russian) Doklady Akad. Nauk SSSR (N.S.) **71**, (1950). 1017–1020.
[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. Birkhauser, Boston, 1985.
[Ha] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Mathematics No. 52, Springer–Verlag, New York–Heidelberg, 1977.
[Ho] R. Howe, *Remarks on Classical Invariant Theory*. Trans. Amer. Math. Soc. **313** (1989), 539–570.
[Ji] M. Jimbo, *A q-Difference Analogue of* $U(\mathfrak{g})$ *and the Yang–Baxter Equation*. Lett. Math. Phys. **10** (1985), 63–69.
[KM] M. Kapovich, J. J. Millson, *Quantization of Bending Deformations of Polygons in* E^3*, Hypergeometric Integrals and the Gassner Representation*, Canad. Math. Bull. **44** (2001), 36–60.
[KL] D. Kazhdan, G. Lusztig, *Tensor Structures Arising from Affine Lie Algebras. I,II,III,IV*, J. Amer. Math. Soc. **6** (1993), 905–947, 949–1011 and **7** (1994), 335–381, 383–453.
[KR] A. N. Kirillov, N. Reshetikhin, *q-Weyl Group and a Multiplicative Formula for Universal R-Matrices*, Comm. Math. Phys. **134** (1990), 421–431.
[KZ] V. G. Knizhnik, A. B. Zamolodchikov, *Current Algebra and Wess–Zumino Model in Two Dimensions*. Nuclear Phys. B **247** (1984), 83–103.
[Knp] F. Knop, *A Harish–Chandra Homomorphism for Reductive Group Actions*, Ann. of Math. **140** (1994), 253–288.
[Kn] A. Knutson, personal communication to J. Millson, May 1999.
[Ko1] T. Kohno, *Quantized Enveloping Algebras and Monodromy of Braid Groups*, preprint, 1988.
[Ko2] T. Kohno, *Integrable Connections Related to Manin and Schechtman's Higher Braid Groups*, Illinois J. Math. **34** (1990), 476–484.
[KT] K. Koike, I. Terada, *Young Diagrammatic Methods for the Restriction of Representations of Complex Classical Lie Groups to Reductive Subgroups of Maximal Rank*, Adv. Math. **79** (1990), 104–135.
[Ku] S. Kumar, *Proof of the Parthasarathy–Ranga Rao–Varadarajan Conjecture*, Invent. Math. **93** (1988), 117–130.

[Kz] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry.* John Wiley and Sons, New York, 1978.

[Kw] O. K. Kwon, *Irreducible Representations of Braid Groups via Quantized Enveloping Algebras*, J. Algebra **183** (1996), 898–912.

[LiE] M. A. A. van Leeuwen, A. M. Cohen, B. Lisser, *LiE, a Package for Lie Group Computations*, CAN, Amsterdam, 1992, available at http://www.mathlabo.univ-poitiers.fr/~maavl/LiE/.

[Lu] G. Lusztig, *Introduction to Quantum Groups.* Progress in Mathematics, 110. Birkhäuser Boston, 1993.

[MMP] D. Marker, M. Messmer, A. Pillay, *Model Theory of Fields.* Lecture Notes in Logic, vol. **5**, Springer, 1996.

[Mu] D. Mumford, *Algebraic Geometry. I. Complex Projective Varieties.* Corrected reprint. Grundlehren der Mathematischen Wissenschaften 221, Springer–Verlag, 1981.

[PRV] K. R. Parthasarathy, R. Ranga Rao, V. S. Varadarajan, *Representations of Complex Semi–Simple Lie Groups and Lie Algebras*, Ann. of Math. **85** (1967), 383–429.

[Pe] M. Perroud, *On the Irreducible Representations of the Lie Algebra Chain* $G_2 \supset A_2$, J. Mathematical Phys. **17** (1976), 1998–2006.

[Ra] T. R. Ramadas, *The "Harder–Narasimhan Trace" and Unitarity of the KZ/Hitchin Connection : Genus 0*, February 2003 preprint.

[Re] M. Reeder, *Zero Weight Spaces and the Springer Correspondence*, Indag. Math. (N.S.) **9** (1998), 431–441.

[Sa] Y. Saito, *PBW Basis of Quantized Universal Enveloping Algebras*, Publ. Res. Inst. Math. Sci. **30** (1994), 209–232.

[So] Y. S. Soibelman, *Algebra of Functions on a Compact Quantum Group and its Representations*, Leningrad Math. J. **2** (1991), 161–178.

[Ti] J. Tits, *Normalisateurs de Tores. I. Groupes de Coxeter Etendus*, J. Algebra **4** (1966), 96–116.

[TL1] V. Toledano Laredo, *Fusion of Positive Energy Representations of* $LSpin_{2n}$. Ph.D. dissertation, University of Cambridge, 1997.

[TL2] V. Toledano Laredo, *A Kohno–Drinfeld Theorem for Quantum Weyl Groups*, Duke Math. Journal **112** (2002), 421–451.

[TL3] V. Toledano Laredo, *Flat Connections and Quantum Groups*, Acta Appl. Math. **73** (2002), 155–173.

[Tu] V. G. Turaev, *Quantum Invariants of Knots and 3–Manifolds.* de Gruyter Studies in Mathematics, 18, Walter de Gruyter & Co., Berlin, 1994.

[Wal] R. J. Walker, *Algebraic Curves.* Princeton University Press, 1950.

[Wa] A. J. Wassermann, *Operator Algebras and Conformal Field Theory. III. Fusion of Positive Energy Representations of* $\mathrm{LSU}(N)$ *using Bounded Operators*, Invent. Math. **133** (1998), 467–538.

[Zh1] D. P. Zhelobenko, *Compact Lie Groups and their Representations.*, Translations of Mathematical Monographs, Vol. 40. American Mathematical Society, 1973.

[Zh2] D. P. Zhelobenko, *On Gelfand–Zetlin Bases for Classical Lie Algebras*, in Representations of Lie Groups and Lie Algebras (Budapest, 1971), 79–106, Akad. Kiado, Budapest, 1985.

Mathematics Department, University of Maryland, College Park, MD 20742–4015
E-mail address: jjm@math.umd.edu

Institut de Mathematiques de Jussieu, Universite Pierre et Marie Curie, UMR 7586, Case 191, 175 rue du Chevaleret, F–75013 Paris
E-mail address: toledano@math.jussieu.fr