Math 411 problems F20

The following are some practice problems for Math 411. Many are meant to challenge rather that be solved right away. Some could be discussed in class, and some are similar to hard exam questions.

1) Let $f : \mathbb{R} \to \mathbb{R}$ continuous. Prove that the graph $G = \{(x, f(x)) | x \in \mathbb{R}\}$ is closed.

2) Is the converse to 1) true?

3) Let $f : [0, 1] \to \mathbb{R}$ continuous. Prove that the graph $G = \{(x, f(x)) | x \in [0, 1]\}$ is sequentially compact.

4) Is the converse to 3) true?

5) Let $X = \{f \in C([0, \frac{1}{2}], \mathbb{R}) | 0 \leq f(x)| \leq 4 \, \forall x \in [0, \frac{1}{2}]\}$, and let $T : X \to C([0, \frac{1}{2}], \mathbb{R})$ be defined by $(T(f))(x) = 1 + \int_0^x f(t)dt$.

a) Prove or disprove: T maps X into X. Start by stating if this is true or false.

b) Prove or disprove: $T : X \to X$ is a contraction. Start by stating if this is true or false.

c) Write down explicitly a fixed point of T. Your answer should be an element $f \in X$.

6) Let $C \subseteq \mathbb{R}^n$ be set with the property that any $f : C \to \mathbb{R}$ which is continuous has to be uniformly continuous.

a) Prove or disprove: C must be a bounded set. Start by stating if this is true or false.

b) Prove or disprove: C must be a closed set. Start by stating if this is true or false.

7. Let X be a (general, abstract) metric space which is sequentially compact. Prove X must be complete.

8 (see also 38-40 on the same topic). Let $X = \{f \in C([0, 1], \mathbb{R}) || f(x)| \leq 1 \forall x \in [0, 1]\}$, and let $T : X \to C([0, 1], \mathbb{R})$ be defined by $(T(f))(x) = \int_0^x \cos(f(t))dt$. You can assume without proof that T maps X into X.

a) Prove or disprove: $T : X \to X$ is a contraction.

b) Assume f is a fixed point of T. Write down a differential equation and initial condition ($f(0) =?$) satisfied by f.

9. Let A be a closed subset of $\mathbb{R}$, and regard A as a metric space in its own right. Let F be a (relatively) closed subset of the metric space A. Carefully write down what it means for F to be relatively closed in A, and then prove that F is in fact a closed subset of $\mathbb{R}$.

10. Let $F : \mathbb{R}^n \to \mathbb{R}^m$.

a) Prove that $F^{-1}(V)$ is open for every open set V in $\mathbb{R}^m$ if and only if F is continuous. This is proved in the textbook, please repeat the proof.

b) Prove that $F^{-1}(C)$ is closed for every closed set C in $\mathbb{R}^m$ if and only if F is continuous.

c) Consider the following function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if} x = 0 \end{cases}$$

Decide if the sets $f^{-1}((1,2))$ and $f^{-1}((-1,1))$ are open, and justify your answer.

11. Let $f : \mathbb{R} \to \mathbb{R}$. Assume that $f^{-1}(K)$ is closed for every (sequentially) compact set K in $\mathbb{R}$. Prove or disprove: f has to be continuous. Start by stating is this is true or false.

12. Let O be an open set, and $\mathbf{x} \in O$. Let

$$A = \{\mathbf{y} \in O | \text{there exists a parametrized path from } \mathbf{x} \text{ to } \mathbf{y}\}$$

and

$$B = \{\mathbf{y} \in O | \text{there does not exist a parametrized path from } \mathbf{x} \text{ to } \mathbf{y}\}$$

a) Prove that both A and B are open.

b) Prove that if O is open and connected, then it is path connected.

13) For each of the following functions, determine **all** unit vectors $\mathbf{p} \in \mathbb{R}^\mathbf{2}$, $\|\mathbf{p}\| = \mathbf{1}$ for which the directional derivative $\frac{\partial f}{\partial \mathbf{p}}(0,0)$ exists. No proof required. Recall $\frac{\partial f}{\partial \mathbf{p}}(0,0) = \lim_{t \to 0} \frac{f(t\mathbf{p}) - \mathbf{f(0,0)}}{t}$.

a)

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

b)

$$f(x,y) = \begin{cases} \frac{x^2y^4}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

14) Let $f : \mathbb{R}^2 \to \mathbb{R}$, of class C^2, be given. Assume its Hessian matrix is

$$\nabla^2 f(0,0) = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

Define $\phi(t) = f(t,t)$. Find $\phi''(0)$, and justify your answer.

15) Let $f : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable, and define $\phi : (0,\infty) \times (0, 2\pi) \to \mathbb{R}$ by $\phi(r,\theta) = f(r\cos\theta, r\sin\theta)$. We know that for any $(x,y) \in \mathbb{R}^2$, $y \notin [0,\infty)$, there exists a unique (r,θ) as above such that $(x,y) = (r\cos\theta, r\sin\theta)$. Express $(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y))$ in terms of $(\frac{\partial \phi}{\partial r}(r,\theta), \frac{\partial \phi}{\partial \theta}(r,\theta))$. Show your work. It is easiest to do this with matrices, but any method is acceptable.

16) Let $\mathbf{F}(\mathbf{x},\mathbf{y}) = (\mathbf{f}(\mathbf{x},\mathbf{y}), \mathbf{g}(\mathbf{x},\mathbf{y}))$ where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. Assume that the derivative matrix $\mathbf{DF}(\mathbf{x},\mathbf{y})$ is invertible for all $(x,y) \in \mathbb{R}^2$. Define $h(x,y) = f^2(x,y) + g^2(x,y)$. Prove that if (x_0,y_0) is a global minimizer of h, then $h(x_0,y_0) = 0$.

17) Let $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^2, and assume $\mathbf{x} \in \mathbb{R}^2$ is such that

$$\lim_{\mathbf{h}\to(0,0)} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - < \nabla f(\mathbf{x}), \mathbf{h} >}{\|\mathbf{h}\|^2} = 0$$

a) Prove

$$\lim_{\mathbf{h}\to(0,0)} \frac{< \nabla^2 f(\mathbf{x})\mathbf{h}, \mathbf{h} >}{\|\mathbf{h}\|^2} = 0$$

b) Is the previous statement true without taking limits? Is

$$\frac{< \nabla^2 f(\mathbf{x})\mathbf{h}, \mathbf{h} >}{\|\mathbf{h}\|^2} = 0$$

for all $\mathbf{h} \neq (0,0)$ true? Justify your answer.

18) Let $f : \mathbb{R}^2 \to \mathbb{R}$ with continuous second order derivatives. Assume $f(x,y) = 0$ if $x^2 + y^2 = 1$, and

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) + \frac{\partial f}{\partial y}(x,y) = f(x,y)$$

if $x^2 + y^2 < 1$. Circle the statement(s) which must be true (10 pts), and prove it (them)

$f(x,y) \le 0$ for all (x,y), $x^2 + y^2 < 1$.

$f(x,y) \ge 0$ for all (x,y), $x^2 + y^2 < 1$.

19) Compute, if possible

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^2}{x^2+y^4}$$

Prove your claim.

20) Let $f:\mathbb{R}^2\to\mathbb{R}$ such that $|f(x,y)|\le x^2+y^2$ for all x,y. Prove that f has first order partial derivatives with respect to x, y at $(0,0)$.

21) Throughout this problem, let $\mathbf{F}:\mathbb{R}^n\to\mathbb{R}^n$ of class C^1, $\mathbf{F}(\mathbf{0})=\mathbf{0}$.
a) Prove that if $\|\mathbf{DF}(\mathbf{0})\mathbf{h}\|\ge\|\mathbf{h}\|$ for all $\mathbf{h}\in\mathbb{R}^n$ then there exists $\delta>0$ such that $\|\mathbf{F}(\mathbf{h})\|\ge\frac{1}{2}\|\mathbf{h}\|$ for all $\|\mathbf{h}\|\le\delta$.

b) Prove or disprove : if there exists $\delta>0$ such that $\|\mathbf{F}(\mathbf{h})\|\ge\|\mathbf{h}\|$ for all $\|\mathbf{h}\|\le\delta$, then $\|\mathbf{DF}(\mathbf{0})\mathbf{h}\|\ge\|\mathbf{h}\|$ for all $\mathbf{h}\in\mathbb{R}^n$. Start by stating if this is true or false.

22) Let $f:\mathbb{R}^2\to\mathbb{R}$ be of class C^2, and assume $\nabla f(\mathbf{0})=0$
a) Assume $\frac{\partial^2 f}{\partial x^2}(\mathbf{0})>0$ and $\frac{\partial^2 f}{\partial y^2}(\mathbf{0})>0$. Does it follow that $\mathbf{0}$ is a local minimizer of f? State if this is true or false, and prove your claim or give a counterexample.
b) Assume that $\mathbf{0}$ is a local minimizer of f. Does it follow that $\frac{\partial^2 f}{\partial x^2}(\mathbf{0})\ge 0$ and $\frac{\partial^2 f}{\partial y^2}(\mathbf{0})\ge 0$? State if this is true or false, and prove your claim or give a counterexample.

23) Let $f:\mathbb{R}^2\to\mathbb{R}$ with continuous second order derivatives. Assume

$$\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)=1$$

a) Is it possible for f to have a strict local minimizer (x_0,y_0)? Explain why not, or give an example where this is possible.
b) Is it possible for f to have a strict local maximizer (x_0,y_0)? Explain why not, or give an example where this is possible.

24). Let $f:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}$ be C^1 and have the property that $f(tx,ty)=tf(x,y)$ for all $(x,y)\ne(0,0)$ and all $t>0$. Prove that there exists C such that $|f(x,y)|\le C\|(x,y)\|$ for all $(x,y)\ne(0,0)$ and thus $\lim_{(x,y)\to(0,0)} f(x,y)=0$.

25) Assume $f:\mathbb{R}^2\to\mathbb{R}$ of class C^1 satisfies
$\frac{\partial f}{\partial x}(x,y)+2\frac{\partial f}{\partial y}(x,y)=0$ for all (x,y). Let $\phi(x,y)=f(x+y,x-y)$ What equation does ϕ satisfy?

26) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 and $\mathbf{x} \in \mathbb{R}^n$, $\nabla f(\mathbf{x}) = 0$ and $\nabla^2 f(\mathbf{x})$ is positive definite, then there exist $c > 0$, $\delta > 0$ such that $f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) \geq c\|\mathbf{h}\|$ for all $\|\mathbf{h}| < \delta$.

27) Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$. Assume $\|\mathbf{F}(\mathbf{h})\| \geq \|\mathbf{h}\|$ for all $\|\mathbf{h}\| \leq 1$. a) Prove that there exists $\delta > 0$ such that $\|\mathbf{DF}(\mathbf{0})\mathbf{h}\| \geq \frac{1}{2}\|\mathbf{h}\|$ for all $\|\mathbf{h}\| < \delta$.

b) Is it necessarily true that $\|\mathbf{DF}(\mathbf{0})\mathbf{h}\| \geq \frac{1}{2}\|\mathbf{h}\|$ for all $\mathbf{h} \in \mathbb{R}^n$?

c) Is it necessarily true that $\|\mathbf{DF}(\mathbf{0})\mathbf{h}\| \geq \|\mathbf{h}\|$ for all $\mathbf{h} \in \mathbb{R}^n$?

28) Let $F : \mathbf{R}^3 \to \mathbf{R}^2$ be C^1. Assume $F(0,0,0) = (0,0)$ and the derivative matrix equals

$$DF(0,0,0) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

a,) Circle the correct statement (no proof required).

There exists $r > 0$ and $g : (-r,r) \to \mathbf{R}$, $h : (-r,r) \to \mathbf{R}$, continuously differentiable, such that $g(0) = 0$, $h(0) = 0$ and

$F(x, g(x), h(x)) = (0,0)$ for all $|x| < r$

$F(g(y), y, h(y)) = (0,0)$ for all $|y| < r$

$F(g(z), h(z), z) = (0,0)$ for all $|z| < r$

(b) Are the other two statements, which aren't true for sure, possible with g, h continuously differentiable and $DF(0,0,0)$ as above)? An example or proof they are impossible is required.

30) Let $f_1,\ f_2 : \mathbf{R}^2 \to \mathbf{R}$ of class C^2. Consider the zero sets Z_1 of f_1, Z_2 of f_2. Thus $Z_1 = \{(x,y) | f_1(x,y) = 0\}$, and the same for Z_2. Assume $\nabla f_i(x,y) \neq 0$ if $(x,y) \in Z_i$, and assume there exists a function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f_1(x,y) = \lambda(x,y) \nabla f_2(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

Assume the intersection of Z_1 and Z_2 is non-empty.

Prove that $Z_1 \cap Z_2$ contains infinitely many points.

31) Let $X = (2, \infty)$ with the the usual metric $d(x,y) = |x - y|$.

(a) Define what it means for $T : X \to X$ to be a contraction.

(b) Prove or disprove: If $T : X \to X$ is a contraction, then there is at most one $x \in X$ such that $T(x) = x$. (X as above).

(c) Prove or disprove: If $T : X \to X$ is a contraction (X as above), then there exists at least one $x \in X$ such that $T(x) = x$.

32) Prove or disprove: There exists $F : \mathbf{R}^2 \to R^2$ of class C^1, with $DF(\mathbf{x})$ invertible for every $\mathbf{x} \in R^n$, such that the image set $F(\mathbf{R}^2)$ is sequentially compact.

33. You are given a C^1 function $g : \mathbb{R}^2 \to \mathbb{R}$, and two differentiable functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$. Assume that

$$f_i'(x) = g(x, f_i(x))$$

for $i = 1, i = 2$ and all x.

a) Prove that the set
$\{x \in \mathbb{R} | f_1(x) = f_2(x)\}$ is closed.

b) Prove that the set
$\{x \in \mathbb{R} | f_1(x) = f_2(x)\}$ is open.

34. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x, y) = (\sin(\frac{x+y}{2}), \cos(\frac{x-y}{3}))$. Prove there exists a unique solution to $F(x, y) = (x, y)$.

35. Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ C^1 and $\mathbf{DF}(\mathbf{x})$ is invertible for all $\mathbf{x} \in \mathbb{R}^n$. Prove that if V is open in $\mathbb{R}^n$, then $\mathbf{F}(V)$ is open.

36. The sphere $\{x^2 + y^2 + z^2 = 3\}$ and the cone $\{2x^2 + 3y^2 = 5z^2\}$ intersect in two disjoint oval-shaped curves C_1 and C_2, and assume C_1 is the one that goes through $(1, 1, 1)$. Write down the direction of the tangent vector to C_1 at $(1, 1, 1)$.

37. Find the maximum value of z on the curve (circle) given by the intersection of $x^2 + y^2 + z^2 = 1$ and $x + y + z = 0$.

38. Let $X = \{f \in C([0, 1], \mathbb{R}) | 0 \leq f(x) \leq 2 \, \forall x \in [0, 1]\}$, and let
$T : X \to C([0, 1], \mathbb{R})$ be defined by $(T(f))(x) = \int_0^x t f^2(t) dt$.

Decide if T maps X into X. If yes, prove it. If no, give an example of $f \in X$ such that $T(f)$ is not in X.

Also, decide if T is a contraction from X to X,

39. Let $X = \{f \in C([0, \pi], \mathbb{R}) | 1 - \pi \leq f(x) \leq 1 + \pi \, \forall x \in [0, \pi]\}$, and let
$T : X \to C([0, \pi], \mathbb{R})$ be defined by $(T(f))(x) = 1 + \int_0^x \cos(f(t)) dt$.

a) Carefully prove T maps X into X.

b) If f is a fixed "point" of T, what differential equation and initial condition does f satisfy? (You do not have to solve that equation.)

c) Decide if T is a contraction.

40) Let $X = \{f \in C([0, 1], \mathbb{R}) | 0 \leq f(x) \leq 1 \, \forall x \in [0, 1]\}$, and let
$T : X \to C([0, 1], \mathbb{R})$ be defined by $(T(f))(x) = \int_0^x f(t)^{2/3} dt$.

a) (10 pts) Prove that T maps X into X.
b) (10 pts) Decide of T is a contraction.

41) Let $\mathbf{F} : \mathbb{R}^{\mathbf{2}} \to \mathbb{R}^{\mathbf{2}}$ of class C^1 and assume that the derivative matrix $\mathbf{DF}(\mathbf{x})$ is symmetric and positive definite for every $\mathbf{x} \in \mathbb{R}^{\mathbf{2}}$. Prove $\mathbf{F}$ is globally one-to-one.

42) Let $h : [0,1] \to [0,1]$, continuous. Assume $h(1) = 0$. Also, define the sequence of functions $f_k : [0,1] \to \mathbb{R}$ by

$$f_k(x) = x^k h(x)$$

Prove f_k converges to 0 *uniformly.*

43) Carefully compute the following limits (if they exist), and justify your answer

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x) + \cos(2y) - 2}{\sqrt{x^2 + y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x) + \cos(2y) - 2}{x^2 + y^2}$$

44) Let $f : \mathbb{R}^2 \to \mathbb{R}$ with continuous second order partial derivatives. Assume $f(0,0) = 0$, $\nabla f(0,0) = 0$.

Find a necessary and sufficient condition for the Hessian matrix $\nabla^2 f(0,0)$ such that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2 + y^2} \text{ exists}$$

Justify your answer.

45) Let $f : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable. Assume $\frac{\partial f}{\partial x}(0,0) = 1$ and $\frac{\partial f}{\partial y}(0,0) = 2$. Let $g(x,y) = f(x+y, 3x+2y)$. Compute $\frac{\partial g}{\partial x}(0,0)$.

46) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 \text{ if } y = x^2 \\ 0 \text{ otherwise} \end{cases}$$

a) Fix $(x,y) \neq (0,0)$ and find, if possible, $\lim_{t\to 0} f(tx, ty)$
b) Find, if possible, $\lim_{(x,y)\to(0,0)} f(x,y)$.