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Global Weyl Modules for Twisted and Untwisted Loop Algebras

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ABSTRACT OF THE DISSERTATION

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A family of modules called global Weyl modules were defined in [7] over algebras of the form $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a simple finite–dimensional complex Lie algebra and A is a commutative associative algebra with unity. Part I of this dissertation contains a characterization the homomorphisms between these global Weyl modules, under certain restrictions on $\mathfrak{g}$ and A. The crucial tool in this section is the reconstruction of the fundamental global Weyl module from a local one. In Part II, global Weyl modules are defined for the first time for loop algebras which have been twisted by a graph automorphism of the Dynkin diagram. We analyze their relationship with the twisted local Weyl module, which was defined in [8], and with the untwisted global Weyl module.

Contents

Introduction

This manuscript is concerned with Lie algebras of the form $\mathfrak{g}\otimes A$, where $\mathfrak{g}$ is a simple, complex finite–dimensional Lie algebra and A is a commutative associative algebra with unity over $\mathbf{C}$. The main content of Part I first appeared as [3]. In it, we study the category $\mathcal{I}_A$ of $\mathfrak{g}\otimes A$–modules which are integrable (locally finite–dimensional) as $\mathfrak{g}$–modules. This category fails to be semi–simple, and it was proved in [11] that irreducible representations of the quantum affine algebra specialize at $q = 1$ to reducible, indecomposable representations of the loop algebra. This phenomenon is analogous to the one observed in modular representation theory: an irreducible finite–dimensional representation in characteristic zero becomes reducible in passing to characteristic p, and the resulting object is called a Weyl module.

This analogy motivated the definition of Weyl modules (global and local) for loop algebras in [11]. Their study was pursued for more general rings A in [7] and [12]. Thus, given any dominant integral weight of the simple Lie algebra $\mathfrak{g}$, one can define an infinite–dimensional object $W_A(\lambda)$ of $\mathcal{I}_A$, called the *global Weyl module*, via generators and relations. It was shown (see [7] for the most general case) that if A is finitely generated, then $W_A(\lambda)$ is a right module for a certain commutative finitely generated associative algebra $\mathbf{A}_\lambda$, which is canonically associated with A and λ. The local Weyl modules are obtained by taking the tensor product of $W_A(\lambda)$, over $\mathbf{A}_\lambda$, with simple $\mathbf{A}_\lambda$-modules. This construction is known as the Weyl functor $\mathbf{W}_A^\lambda$; equivalently, local Weyl modules can be given via generators and relations. The calculation of their dimension and character has led to a series of papers ([10], [9], [15], [23], [1]).

The local Weyl modules have been useful in understanding the blocks of the category $\mathcal{F}_A$ of finite–dimensional representations of $\mathfrak{g} \otimes A$, which gives homological information about this category. One motivation for Part I of this dissertation is to explore the use of the global Weyl modules to further understand the homological properties of the categories $\mathcal{F}_A$ and $\mathcal{I}_A$. The global Weyl modules have nice universal properties, and in fact they play a role similar to that of the Verma modules $M(\lambda)$ in the study of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for $\mathfrak{g}$ (for a more precise treatment of this topic, see [4]). A basic result about Verma modules is that $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu))$ is of dimension at most one and also that any non–zero map is injective. In Part I of the dissertation we prove an analogue of this result (Theorem 3) for global Weyl modules.

In Part II, we consider the *twisted loop (sub)algebras* of $\mathfrak{g} \otimes \mathbf{C}[t^{\pm 1}]$, which we denote by $L^\Gamma(\mathfrak{g})$. These are the algebras of fixed points in $\mathfrak{g} \otimes \mathbf{C}[t^{\pm 1}]$ under a group action of $\Gamma \cong \mathbf{Z}/m\mathbf{Z}$ obtained from an order m Dynkin diagram automorphism of $\mathfrak{g}$. Here Γ acts upon $\mathbf{C}^*$ by multiplication with an m^{th} primitive root of unity ξ; using the induced action on the coordinate ring $\mathbf{C}[t^{\pm 1}]$ of this variety (that is, $t \mapsto \xi^{-1} t$) we have a diagonal action on $\mathfrak{g} \otimes \mathbf{C}[t^{\pm 1}]$.

In [8], local Weyl modules were defined and studied for these algebras. The main result of that paper was that any local Weyl module of $L^\Gamma(\mathfrak{g})$ can be obtained by restricting a local Weyl module for $L(\mathfrak{g})$. As an application, one of the authors of [8] obtained in his thesis a parametrization of the blocks of the category of finite–dimensional representations of the twisted loop algebras. The notion of a global Weyl module for $L^\Gamma(\mathfrak{g})$, however, has not appeared in the literature thus far, and it is the main aim of Part II to fill this gap.

The fixed–point subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ under the action of Γ is a simple Lie algebra, called the *underlying* Lie algebra. Its weight theory is crucial in our development. In Part II, using the same tools as [7], we define, for any dominant integral weight λ of $\mathfrak{g}_0$, a global Weyl module $W^\Gamma(\lambda)$. We also describe its highest weight space $\mathbf{A}^\Gamma_\lambda$, by giving it a natural algebra structure, and define a Weyl functor $\mathbf{W}^\Gamma_\lambda$ from the category of left $\mathbf{A}^\Gamma_\lambda$-modules to the category of integrable $L^\Gamma(\mathfrak{g})$-modules. As in [7], we obtain, by using the Weyl functor, a homological characterization of Weyl modules. We also prove that there is a canonical embedding of $\mathbf{A}^\Gamma_\lambda$ into $\mathbf{A}_\mu$ for any $\mu \in P^+$ satisfying the condition that $\mu\,|_{\mathfrak{h}_0} = \lambda$, where $\mathfrak{h}_0$ is the fixed–point subalgebra of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

The global Weyl module $W(\mu)$ ($\mu \in P^+$) for the loop algebra $L(\mathfrak{g})$ is, via restriction, also a module for $L^\Gamma(\mathfrak{g})$. Furthermore, if λ is a dominant integral weight of $\mathfrak{g}_0$, then $\bigoplus_\mu W(\mu)$ (where the sum is taken over all μ such that $\mu\,|_{\mathfrak{h}_0} = \lambda$) is a $L^\Gamma(\mathfrak{g})$-module and the main theorem of Part II (Theorem 8) relates this module with $W^\Gamma(\lambda)$. Some motivation for this work also comes from the finite-dimensional representation theory of the quantum affine algebra, where relationships are known ([17, Theorem 4.15]) between Kirillov-Reshetikhin modules for the twisted and untwisted algebras.

Part I

Homomorphisms between global Weyl modules

Chapter 1

In this chapter we establish some notation and then recall the definition and some elementary properties of the global Weyl modules. Throughout, we assume that the reader is familiar with the material from an introductory graduate course in Lie theory.

1.1 Preliminary notation

Let $\mathbf{C}$ be the field of complex numbers and let $\mathbf{Z}$ (respectively $\mathbf{Z}_+$) be the set of integers (respectively non–negative integers). Given two complex vector spaces V, W let $V \otimes W$ (respectively, $\mathrm{Hom}(V, W)$) denote their tensor product over $\mathbf{C}$ (respectively the space of $\mathbf{C}$–linear maps from V to W).

Given a commutative and associative algebra A over $\mathbf{C}$, let $\mathrm{Max}\, A$ be the maximal spectrum of A and $\mod A$ the category of left A–modules. Given a right A-module M and an element $m \in M$, the annihilating (right) ideal of m is

$$\mathrm{Ann}_A\, m = \{a \in A \,:\, m.a = 0\}.$$

1.1.1

The assignment $x \mapsto x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{a}$ extends to a homomorphism of algebras

$$\Delta : \mathbf{U}(\mathfrak{a}) \to \mathbf{U}(\mathfrak{a}) \otimes \mathbf{U}(\mathfrak{a}),$$

and therefore defines a bialgebra structure on $\mathbf{U}(\mathfrak{a})$. In particular, if V, W are two $\mathfrak{a}$–modules then $V \otimes W$ and $\mathrm{Hom}_{\mathbf{C}}(V, W)$ are naturally $\mathbf{U}(\mathfrak{a})$-modules and $W \otimes V \cong V \otimes W$ as $\mathbf{U}(\mathfrak{a})$-modules. One can also define the trivial $\mathfrak{a}$–module structure on $\mathbf{C}$ and we have

$$V^{\mathfrak{a}} = \{v \in V : \mathfrak{a}v = 0\} \cong \mathrm{Hom}_{\mathfrak{a}}(\mathbf{C}, V).$$

Suppose that A is an associative commutative algebra over $\mathbf{C}$ with unity. Then $\mathfrak{a} \otimes A$ is canonically a Lie algebra, with the Lie bracket given by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \qquad x, y \in \mathfrak{a}, \qquad a, b \in A.$$

We shall identify $\mathfrak{a}$ with the Lie subalgebra $\mathfrak{a} \otimes 1$ of $\mathfrak{a} \otimes A$. Note that for any algebra homomorphism $\varphi : A \to A'$ the canonical map $1 \otimes \varphi : \mathfrak{g} \otimes A \to \mathfrak{g} \otimes A'$ is a homomorphism of Lie algebras and hence induces an algebra homomorphism $\mathbf{U}(\mathfrak{g} \otimes A) \to \mathbf{U}(\mathfrak{g} \otimes A')$.

1.2 Simple Lie algebras

1.2.1

Let $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$. Let Φ be the corresponding root system and fix a set $\{\alpha_i : i \in I\} \subset \mathfrak{h}^*$ (where $I = \{1, \ldots, \dim \mathfrak{h}\}$) of simple roots for Φ. The root lattice Q is the $\mathbf{Z}$–span of the simple roots while Q^+ is the $\mathbf{Z}_+$–span of the simple roots, and $\Phi^+ = \Phi \cap Q^+$ denotes the set of positive roots in Φ. Let $\mathrm{ht} : Q^+ \to \mathbf{Z}_+$ be the homomorphism of free semi-groups defined by setting $\mathrm{ht}(\alpha_i) = 1$, $i \in I$.

The restriction of the Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}$ to $\mathfrak{h} \times \mathfrak{h}$ induces a non–degenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$, and we let $\{\omega_i \,:\, i \in I\} \subset \mathfrak{h}^*$ be the fundamental weights defined by $2(\omega_j, \alpha_i) = \delta_{i,j}(\alpha_i, \alpha_i)$, $i, j \in I$. Let P (respectively P^+) be the $\mathbf{Z}$ (respectively $\mathbf{Z}_+$) span of the $\{\omega_i \,:\, i \in I\}$ and note that $Q \subseteq P$. Given $\lambda, \mu \in P$ we say that $\mu \le \lambda$ if and only if $\lambda - \mu \in Q^+$. Clearly $\le$ is a partial order on P. The set Φ^+ has a unique maximal element with respect to this order which is denoted by θ and is called the highest root of Φ^+. From now on, we normalize the bilinear form on $\mathfrak{h}^*$ so that $(\theta, \theta) = 2$.

1.2.2

Given $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ denote the root space

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \;:\; [h, x] = \alpha(h)x, \quad h \in \mathfrak{h}\},$$

and define subalgebras $\mathfrak{n}^\pm$ of $\mathfrak{g}$ by

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\pm\alpha}.$$

We have isomorphisms of vector spaces

$$\mathfrak{g} \cong \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathbf{U}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{n}^+). \tag{1.2.1}$$

For $\alpha \in \Phi^+$, fix elements $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$ and $h_\alpha \in \mathfrak{h}$ spanning a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$, i.e., we have

$$[h_\alpha, x_\alpha^\pm] = \pm 2x_\alpha^\pm, \qquad [x_\alpha^+, x_\alpha^-] = h_\alpha,$$

and more generally, assume that the set $\{x_\alpha^\pm : \alpha \in \Phi^+\} \cup \{h_i := h_{\alpha_i} \,:\, i \in I\}$ is a Chevalley basis for $\mathfrak{g}$.

1.3 Representation theory of simple Lie algebras

Given an $\mathfrak{h}$–module V, we say that V is a weight module if

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v,\ h \in \mathfrak{h}\},$$

and elements of the set wt $V = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$ are called weights of V. If $\dim V_\mu < \infty$ for alll $\mu \in \mathfrak{h}^*$, let ch V be the character of V, namely the element of the group ring $\mathbf{Z}[\mathfrak{h}^*]$ given by,

$$\text{ch } V = \sum_{\mu \in \mathfrak{h}^*} \dim V_\mu e(\mu),$$

where $e(\mu) \in \mathbf{Z}[\mathfrak{h}^*]$ is the element corresponding to $\mu \in \mathfrak{h}^*$. Observe that for two such modules V_1 and V_2, we have

$$\text{ch}(V_1 \oplus V_2) = \text{ch } V_1 + \text{ch } V_2, \qquad \text{ch}(V_1 \otimes V_2) = \text{ch } V_1 \text{ch } V_2.$$

1.3.1

Definition 1. For $\lambda \in P^+$, let $M(\lambda)$ be the left $\mathbf{U}(\mathfrak{g})$–module generated by an element m_λ with defining relations:

$$hm_\lambda = \lambda(h)m_\lambda, \qquad x^+_{\alpha_i} m_\lambda = 0, \qquad h \in \mathfrak{h}, \quad i \in I.$$

In other words, $M(\lambda)$ is the quotient of $\mathbf{U}(\mathfrak{g})$ by the left ideal $J(\lambda)$ generated by the vectors $\left\{x^+_{\alpha_i},\ h - \lambda(h)\ :\ h \in \mathfrak{h},\ i \in I\right\}$, and the vector m_λ is the image of $1 \in \mathbf{U}(\mathfrak{g})$ modulo this ideal.

As a direct consequence of the definition, we have the following lemma, which we isolate here for later comparison.

Lemma 1. Let $\lambda, \mu \in \mathfrak{h}^*$. Then,

(i) $M(\lambda)$ is not isomorphic to $M(\mu)$ if $\lambda \neq \mu$.

(ii) $\mathrm{Hom}(M(\lambda), M(\mu)) = 0$ unless $\lambda \leq \mu$. $\square$

The modules $M(\lambda)$, for $\lambda \in \mathfrak{h}^*$, are called Verma modules. They play an important role in the representation theory of semisimple complex Lie algebras $\mathfrak{g}$, and we have introduced them here for purposes of analogy with the global Weyl module. They are also useful for us as a way of producing a family of simple modules for $\mathbf{U}(\mathfrak{g})$. The following is an amalgamation of standard results on the simple quotients of Verma modules (for instance, see [5]).

Theorem 1. For any $\lambda \in \mathfrak{h}^*$, the Verma module $M(\lambda)$ has a unique irreducible quotient, denoted by $V(\lambda)$.

(i) The module $V(\lambda)$ is finite–dimensional if and only if $\lambda \in P^+$. Moreover, if V is a finite-dimensional irreducible $\mathfrak{g}$–module then there exists a unique $\lambda \in P^+$ such that V is isomorphic to $V(\lambda)$.

(ii) For $\lambda \in P^+$, $V(\lambda)$ is the left $\mathfrak{g}$–module generated by an element v_λ with defining relations:

$$hv_\lambda = \lambda(h)v_\lambda, \qquad x^+_{\alpha_i}v_\lambda = 0, \qquad (x^-_{\alpha_i})^{\lambda(h_{\alpha_i})+1}v_\lambda = 0, \qquad h \in \mathfrak{h},\ i \in I.$$

(iii) $V(\lambda)$ is a weight module, with

$$\text{wt } V(\lambda) \subset \lambda - Q^+ \quad \text{and} \quad \dim V(\lambda)_\lambda = 1.$$

$\square$

We shall say that V is a locally finite–dimensional $\mathfrak{g}$–module if

$$\dim \mathbf{U}(\mathfrak{g})v < \infty, \quad v \in V.$$

It is well–known that a locally finite–dimensional $\mathfrak{g}$–module is isomorphic to a direct sum of irreducible finite–dimensional modules, and moreover it is easy to see that as vector spaces,

$$V_\mu \cap V^{\mathfrak{n}^+} \cong \operatorname{Hom}_{\mathfrak{g}}(V(\mu), V), \quad \mu \in P^+.$$

1.4 Global Weyl modules

Assume from now on that A is an associative commutative algebra over $\mathbf{C}$ with unity. We recall the definition of the global Weyl modules. These were first introduced and studied in the case when $A = \mathbf{C}[t^{\pm 1}]$ in [11] and then later in [12] in the general case. We shall, however, follow the approach developed in [7]. We observe the similarity of this definition with that of the Verma module (Section 1.3.1).

Definition 2. For $\lambda \in P^+$, the *global Weyl module* $W_A(\lambda)$ is the left $\mathbf{U}(\mathfrak{g} \otimes A)$–module generated by an element w_λ with defining relations,

$$(\mathfrak{n}^+ \otimes A)w_\lambda = 0, \qquad hw_\lambda = \lambda(h)w_\lambda, \qquad (x^-_{\alpha_i})^{\lambda(h_{\alpha_i})+1}w_\lambda = 0, \tag{1.4.1}$$

where $h \in \mathfrak{h}$ and $i \in I$. In other words, $W_A(\lambda)$ is the quotient of $\mathbf{U}(\mathfrak{g} \otimes A)$ by the defining ideal $I(\lambda)$, where $I(\lambda)$ is generated by the set

$$\left\{x \otimes a, \ h \otimes 1 - \lambda(h), \ (x^-_{\alpha_i})^{\lambda(h_i)+1} \ : \ x \in \mathfrak{n}^+, \ \ i \in I, \ \ h \in \mathfrak{h}\right\}.$$

The following immediate consequence of the definition establishes a crucial property of the global Weyl module.

Lemma 2. Let V be any $\mathfrak{g} \otimes A$–module and let $\lambda \in P^+$. If $v \in V$ satisfies the relations in Equation 1.4.1, then the assignment $w_\lambda \mapsto v$ induces a $(\mathfrak{g} \otimes A)$–module homomorphism

$$W_A(\lambda) \to V.$$

$\square$

Suppose that φ is an algebra automorphism of A. Then $\varphi \otimes 1$ induces an automorphism of $\mathbf{U}(\mathfrak{g} \otimes A)$ as in Section 1.1.1, which clearly preserves each generator of the defining ideal ideal $I(\lambda)$ of $W_A(\lambda)$. Thus $I(\lambda)$ is also preserved, and we have an isomorphism of $\mathfrak{g} \otimes A$–modules,

$$W_A(\lambda) \cong (1 \otimes \varphi)^* W_A(\lambda). \tag{1.4.2}$$

1.4.1

The following construction shows immediately that $W_A(\lambda)$ is non–zero. Given any ideal $\mathfrak{I}$ of A, define an action of $\mathfrak{g} \otimes A$ on $V(\lambda) \otimes A/\mathfrak{I}$ by

$$(x \otimes a)(v \otimes b) = xv \otimes \bar{a}b, \qquad x \in \mathfrak{g},\, a \in A,\, b \in A/\mathfrak{I},$$

where $\bar{a}$ is the canonical image of a in $A/\mathfrak{I}$. In particular, if $a \notin \mathfrak{I}$ and $h \in \mathfrak{h}$ is such that $\lambda(h) \neq 0$ we have

$$(h \otimes a)(v_\lambda \otimes 1) = \lambda(h) v_\lambda \otimes \bar{a} \neq 0. \tag{1.4.3}$$

Clearly if $\mathfrak{I} \in \operatorname{Max} A$, then

$$V(\lambda) \otimes A/\mathfrak{I} \cong V(\lambda)$$

as $\mathfrak{g}$–modules and hence $v_\lambda \otimes 1$ generates $V(\lambda) \otimes A/\mathfrak{I}$ as a $\mathfrak{g}$–module (and so also as a $\mathfrak{g} \otimes A$–module). Since $v_\lambda \otimes 1$ satisfies the defining relations of $W_A(\lambda)$, we see from Lemma 2 that $V(\lambda) \otimes A/\mathfrak{I}$ is a non–zero quotient of $W_A(\lambda)$.

1.4.2

Given a weight module V of $\mathfrak{g} \otimes A$, and a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g} \otimes A$, set

$$V_\mu^{\mathfrak{a}} = V_\mu \cap V^{\mathfrak{a}}, \qquad \mu \in \mathfrak{h}^*.$$

The following lemma is proved by observing that $W_A(\lambda)$ is a highest weight module for $\mathfrak{g}$ and that the adjoint action of $\mathfrak{g}$ on $\mathbf{U}(\mathfrak{g} \otimes A)$ is locally finite.

Lemma 3. For $\lambda \in P^+$ the module $W_A(\lambda)$ is a locally finite–dimensional $\mathfrak{g}$–module and we have

$$W_A(\lambda) = \bigoplus_{\eta \in Q^+} W_A(\lambda)_{\lambda - \eta}$$

which in particular means that wt $W_A(\lambda) \subset \lambda - Q^+$. If V is a $\mathfrak{g} \otimes A$–module which is locally finite–dimensional as a $\mathfrak{g}$–module then we have an isomorphism of vector spaces,

$$\mathrm{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), V) \cong V_\lambda^{\mathfrak{n}^+ \otimes A}. \qquad \square$$

The following remark extends the analogy with Verma modules (cf. Lemma 1).

Remark. Fix $\lambda, \mu \in P^+$. Then

(i) $W_A(\lambda)$ is not isomorphic to $W_A(\mu)$ if $\lambda \neq \mu$.

(ii) $\mathrm{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu)) = 0$ unless $\lambda \leq \mu$.

1.4.3

The weight spaces $W_A(\lambda)_{\lambda-\eta}$ are not necessarily finite–dimensional, and to understand them, we proceed as follows. We first observe that one can regard $W_A(\lambda)$ as a right module for $\mathbf{U}(\mathfrak{h}\otimes A)$ by setting

$$(uw_\lambda)(h\otimes a) = u(h\otimes a)w_\lambda, \qquad u\in\mathbf{U}(\mathfrak{g}\otimes A),\quad h\in\mathfrak{h},\quad a\in A.$$

To see that this is well–defined, it suffices to show that for every generator x of $I(\lambda)$, we have $x(h\otimes a)w_\lambda = 0$. In other words, we must see that

$$(\mathfrak{n}^+\otimes A)(h\otimes a)w_\lambda = 0 = (h-\lambda(h))(h\otimes a)w_\lambda,$$
$$(x_{\alpha_i}^-)^{\lambda(h_i)+1}(h\otimes a)w_\lambda = 0, \quad h\in\mathfrak{h},\quad i\in I.$$

The first two conditions follow promptly from the defining relations of the module $W_A(\lambda)$. To see the last condition, recall that $W_A(\lambda)$ is a locally finite–dimensional module for $\mathfrak{g}$, and therefore the vector $(h\otimes a)w_\lambda$ generates a finite-dimensional module for the $\mathfrak{sl}_2$ triple $\langle x_{\alpha_i}^\pm, h_i\rangle$. In particular, since

$$x_{\alpha_i}^+(h\otimes a)w_\lambda = 0$$

for each $i\in I$, we see that the third condition is satisfied. $\square$

Since $\mathbf{U}(\mathfrak{h}\otimes A)$ is commutative, the algebra $\mathbf{A}_\lambda$ defined by

$$\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}\otimes A)/\operatorname{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)} w_\lambda$$

is a commutative associative algebra. It follows that $W_A(\lambda)$ is a $(\mathfrak{g}\otimes A,\mathbf{A}_\lambda)$-bimodule. Moreover, for all $\eta\in Q^+$, the weight space $W_A(\lambda)_{\lambda-\eta}$ is a right $\mathbf{A}_\lambda$–module: given $u\in$

$\mathbf{U}(\mathfrak{g}\otimes A)$ with $uw_\lambda \in W_A(\lambda)_{\lambda-\eta}$, let $\mathbf{a}\in \mathbf{A}_\lambda$ with preimage $a \in \mathbf{U}(\mathfrak{h}\otimes A)$. Then,

$$(h\otimes 1)(uw_\lambda.\mathbf{a}) = (h\otimes 1)(uaw_\lambda) = (u(h\otimes 1)aw_\lambda) + [h\otimes 1, u]\, aw_\lambda$$
$$= \lambda(h)uaw_\lambda - \eta(h)uaw_\lambda = (\lambda-\eta)(h)uw_\lambda.\mathbf{a}.$$

Clearly the assignment $w_\lambda \mapsto 1 + \mathrm{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)}\, w_\lambda$ induces an isomorphism

$$W_A(\lambda)_\lambda \cong_{\mathbf{A}_\lambda} \mathbf{A}_\lambda \tag{1.4.4}$$

of $\mathbf{A}_\lambda$–modules, where we regard $\mathbf{A}_\lambda$ as a right $\mathbf{A}_\lambda$-module through right multiplication. The following is immediate.

Lemma 4. For $\lambda \in P^+$, $\eta \in Q^+$ the subspaces $W_A(\lambda)^{\mathfrak{n}^+}_{\lambda-\eta}$ and $W_A(\lambda)^{\mathfrak{n}^+\otimes A}_{\lambda-\eta}$ are $\mathbf{A}_\lambda$–submodules of $W_A(\lambda)$ and we have

$$W_A(\lambda)^{\mathfrak{n}^+}_\lambda = W_A(\lambda)^{\mathfrak{n}^+\otimes A}_\lambda = W_A(\lambda)_\lambda.$$

1.4.4

For $\lambda, \mu \in P^+$, the space $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\mu), W_A(\lambda))$ has the natural structure of a right $\mathbf{A}_\lambda$–module: given a homomorphism $f : W_A(\mu) \to W_A(\lambda)$ and $\mathbf{a} \in \mathbf{A}_\lambda$, we define $f.\mathbf{a} : W_A(\mu) \to W_A(\lambda)$ by extending the assignment $w_\mu \mapsto w_\mu.\mathbf{a}$ to a homomorphism of $\mathfrak{g}\otimes A$–modules. To see that this is well–defined, by Lemma 2 it is enough to observe that $w_\mu.\mathbf{a}$ satisfies the defining relations of $W_A(\mu)$.

The following lemma establishes some basic properties of this module structure.

Lemma 5. Given $\lambda, \mu \in P^+$, we have

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\mu), W_A(\lambda)) \cong_{\mathbf{A}_\lambda} W_A(\lambda)^{\mathfrak{n}^+\otimes A}_\mu.$$

In particular,

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\lambda)) \cong_{\mathbf{A}_\lambda} \mathbf{A}_\lambda \cong_{\mathbf{A}_\lambda} W_A(\lambda)_\lambda^{\mathfrak{n}^+\otimes A}.$$

Proof. Let $\psi : W_A(\mu) \to W_A(\lambda)$ be a nonzero $\mathfrak{g} \otimes A$–module map. The vector $\psi(w_\mu) \in W_A(\lambda)$ clearly has weight μ and is annihilated by $\mathfrak{n}^+ \otimes A$. Thus, we obtain a well-defined map $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\mu), W_A(\lambda)) \to W_A(\lambda)_\mu$ of right $\mathbf{A}_\lambda$–modules given by $\psi \mapsto \psi(w_\mu)$. It is straightforward to see that this map is injective, since $W_A(\mu)$ is a cyclic $\mathbf{U}(\mathfrak{g}\otimes A)$–module and thus ψ is determined by $\psi(w_\mu)$. On the other hand, given any vector $w \in W_A(\lambda)_\mu^{\mathfrak{n}^+\otimes A}$, the assignment $w_\mu \mapsto w$ induces a well–defined homomorphism $W_A(\mu) \to W_A(\lambda)$ by Lemma 2, which establishes the first part of the lemma.The second part follows immediately from the first and from Equation 1.4.4, and the lemma is proved. □

Chapter 2

In this chapter, we state the main results of Part I and make some comments about the restrictions in their hypotheses.

2.1 Statement of results

First, let us establish some additional notation.

2.1.1

For $\mathbf{s} = (s_i)_{i\in I} \in \mathbf{Z}_+^I$, set

$$\mathbf{A}_{\mathbf{s}} = \bigotimes_{i\in I} \mathbf{A}_{\omega_i}^{\otimes s_i}, \qquad W_A(\mathbf{s}) = \bigotimes_{i\in I} W_A(\omega_i)^{\otimes s_i}, \qquad w_{\mathbf{s}} = \bigotimes_{i\in I} w_{\omega_i}^{\otimes s_i}, \tag{2.1.1}$$

where all tensor products are taken in the same (fixed) order. Given $k, \ell \in \mathbf{Z}_+$ let $\mathcal{R}_{k,\ell}$ be the algebra of polynomials $\mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}, u_1, \dots, u_\ell]$ with the convention that if $k = 0$ (respectively $\ell = 0$), $\mathcal{R}_{0,\ell}$ (respectively $\mathcal{R}_{k,0}$) is just the ring of polynomials (respectively Laurent polynomials) in ℓ (respectively k) variables.

2.1.2

The main result of Part I is the following.

Theorem 2. Assume that $A = \mathcal{R}_{k,\ell}$ for some $k, \ell \in \mathbf{Z}_+$. For all $\mathbf{s} \in \mathbf{Z}_+^n$ and $\mu \in P^+$, we have

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\mu), W_A(\mathbf{s})) \cong_{\mathbf{A_s}} W_A(\mathbf{s})_\mu^{\mathfrak{n}^+\otimes A} = \Big(\bigotimes_{i\in I} (W_A(\omega_i)^{\mathfrak{n}^+\otimes A})^{\otimes s_i} \Big)_\mu. \tag{2.1.2}$$

In the case when $\mathfrak{g}$ is a classical simple Lie algebra, we can make (2.1.2) more precise. Let I_0 be the set of $i \in I$ such that α_i occurs in θ with the coefficient $2/(\alpha_i, \alpha_i)$. In particular, $I_0 = I$ for $\mathfrak{g}$ of type A or C. Given $\mathbf{s} = (s_i)_{i\in I} \in \mathbf{Z}_+^I$ and $\lambda \in P^+$, let $\mathbf{c_s}(\lambda) \in \mathbf{Z}_+$ be the coefficient of $e(\lambda)$ in

$$\prod_{i\in I_0} e(\omega_i)^{s_i} \prod_{i\notin I_0} \left(\sum_{0\le j\le i/2} \binom{j+k-1}{j} e(\omega_{i-2j}) \right)^{s_i},$$

where $\omega_0 = 0$.

Corollary 6. Let $\lambda \in P^+$ and $\mathbf{s} \in \mathbf{Z}_+^I$. Assume either that $\mathfrak{g}$ is not an exceptional Lie algebra or that $s_i = 0$ if $i \notin I_0$. We have

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mathbf{s})) \cong_{\mathbf{A_s}} \mathbf{A_s}^{\oplus \mathbf{c_s}(\lambda)},$$

where we use the convention that $\mathbf{A_s}^{\oplus \mathbf{c_s}(\lambda)} = 0$ if $\mathbf{c_s}(\lambda) = 0$.

2.1.3

Our next result is the following. Recall from Lemma 5 that for all $\lambda \in P^+$ we have $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\lambda)) \cong_{\mathbf{A}_\lambda} \mathbf{A}_\lambda$.

Theorem 3. Let A be the ring $\mathcal{R}_{0,1}$ or $\mathcal{R}_{1,0}$. For all $\mu = \sum_{i\in I} s_i\omega_i \in P^+$ with $s_i = 0$ if $i \notin I_0$, and all $\lambda \in P^+$, we have

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mu)) = 0, \qquad \text{if } \lambda \neq \mu.$$

Further, any non–zero element of $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\mu), W_A(\mu))$ is injective. An analogous result holds when $A = \mathcal{R}_{k,\ell}$, $k, \ell \in \mathbf{Z}_+$, $\mathfrak{g} \cong \mathfrak{sl}_{n+1}$ and $\mu = s\omega_1$.

2.2 Remarks on the main results

We now make some comments on the various restrictions in the main results. The proof of Theorem 2 relies on an explicit construction of the fundamental global Weyl modules in terms of certain finite–dimensional modules called the fundamental local Weyl modules. A crucial ingredient of this construction is a natural bialgebra structure of $\mathcal{R}_{k,\ell}$. The proof of Corollary 6 depends on a deeper understanding of the $\mathfrak{g}$–module structure of the local fundamental Weyl modules. These results are unavailable for the exceptional Lie algebras when $k + \ell > 1$. In the case when $k + \ell = 1$, the structure of these modules for the exceptional algebras is known as a consequence of the work of many authors on the Kirillov–Reshetikhin conjecture (see [6] for extensive references on the subject). Hence, a precise statement of Corollary 6 could be made when $k + \ell = 1$ in a case by case and in a not very compact fashion. The interested reader is referred to [16] and [20].

2.2.1

Before discussing Theorem 3, we make the following conjecture.

Conjecture 7. Let $A = \mathcal{R}_{k,\ell}$ for some $k, \ell \in \mathbf{Z}_+$. Then for all $\lambda \in P^+$ and $\mathbf{s} \in \mathbf{Z}_+^n$, any non–zero element of $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mathbf{s}))$ is injective.

The proof of Theorem 3 will rely on the fact that this conjecture is true (see Section 4.3.2) when $k+\ell = 1$ and $s_i = 0$ if $i \notin I_0$ *as well as* on the fact that the fundamental local Weyl module is irreducible as a $\mathfrak{g}$–module if $i \in I_0$. We shall prove in Section 4.3.2 using Corollary 6 and the work of [12] that the conjecture is also true when $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and $\lambda = s\omega_1$. Remark 22 of this paper shows that $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mu))$ can be non–zero if we remove the restriction on μ.

2.2.2

Finally, we make some remarks on quantum analogs of this result. In the case of the quantum loop algebra, one also has analogous notions of global and local Weyl modules which were defined in [11], and one can construct the global fundamental Weyl module from the local Weyl module in a way analogous to the one given in this paper for $A = \mathbf{C}[t^{\pm 1}]$. It was shown in [11] for the quantum loop algebra of $\mathfrak{sl}_2$ that the canonical map from the global Weyl module into the tensor product of fundamental global Weyl modules is injective. For the general quantum loop algebra, this was established by Beck and Nakajima ([1]) using crystal and global bases. They also describe the space of extremal weight vectors in the tensor product of quantum fundamental global Weyl modules.

Chapter 3

In this chapter we recall some necessary results from [7] and also the definition and elementary properties of local Weyl modules.

3.1 The highest weight space

3.1.1

For $r \in \mathbf{Z}_+$, the symmetric group S_r acts naturally on $A^{\otimes r}$ and on $(\operatorname{Max} A)^{\times r}$ by permuting the factors. Let $(A^{\otimes r})^{S_r}$ be the corresponding ring of invariants and $(\operatorname{Max} A)^{\times r}/S_r$ the set of orbits, respectively, of these actions. If $r = r_1 + \ldots + r_n$, then we regard $S_{r_1} \times \cdots \times S_{r_n}$ as a subgroup of S_r in the canonical way: S_{r_1} permutes the first r_1 letters, S_{r_2} the next r_2 letters, and so on. Given $\lambda = \sum_{i \in I} r_i \omega_i \in P^+$, set

$$r_\lambda = \sum_{i \in I} r_i, \quad S_\lambda = S_{r_1} \times \cdots \times S_{r_n}, \qquad \mathbb{A}_\lambda = (A^{\otimes r_\lambda})^{S_\lambda}, \tag{3.1.1}$$

$$\operatorname{Max} \mathbb{A}_\lambda = (\operatorname{Max} A)^{r_\lambda}/S_\lambda. \tag{3.1.2}$$

The algebra $\mathbb{A}_\lambda$ is generated by elements of the form

$$\operatorname{sym}_\lambda^i(a) = 1^{\otimes(r_1+\cdots+r_{i-1})} \otimes \Big(\sum_{k=0}^{r_i-1} 1^{\otimes k} \otimes a \otimes 1^{\otimes(r_i-k-1)}\Big) \otimes 1^{\otimes(r_{i+1}+\cdots+r_n)}, \qquad a \in A,\, i \in I. \tag{3.1.3}$$

The following was proved in [7, Theorem 4].

Proposition 8. Let A be a finitely generated commutative associative algebra over $\mathbf{C}$ with trivial Jacobson radical. Then the homomorphism of associative algebras $\mathbf{U}(\mathfrak{h} \otimes A) \to \mathbb{A}_\lambda$ defined by

$$h_i \otimes a \mapsto \operatorname{sym}_\lambda^i(a), \quad i \in I, \quad a \in A$$

induces an isomorphism of algebras $\operatorname{sym}_\lambda : \mathbf{A}_\lambda \xrightarrow{\sim} \mathbb{A}_\lambda$. In particular, if A is a finitely generated integral domain then $\mathbf{A}_\lambda$ is isomorphic to an integral subdomain of $\mathbf{A}_{\mathbf{r}}$. $\square$

3.1.2

For $\lambda, \mu \in P^+$, it is clear that the tensor product $W_A(\lambda) \otimes W_A(\mu)$ has the natural structure of a $(\mathfrak{g} \otimes A, \mathbf{A}_\lambda \otimes \mathbf{A}_\mu)$-module. We recall from [7] that, in fact, there exists a $(\mathfrak{g} \otimes A, \mathbf{A}_{\lambda+\mu})$-bimodule structure on $W_A(\lambda) \otimes W_A(\mu)$.

It is clear from Definition 1.4 that the assignment $w_{\lambda+\mu} \mapsto w_\lambda \otimes w_\mu$ defines a homomorphism $\tau_{\lambda,\mu} : W_A(\lambda + \mu) \to W_A(\lambda) \otimes W_A(\mu)$ of $\mathfrak{g} \otimes A$-modules. The restriction of this map to $W_A(\lambda + \mu)_{\lambda+\mu}$ induces a homomorphism of algebras $\mathbf{A}_{\lambda+\mu} \to \mathbf{A}_\lambda \otimes \mathbf{A}_\mu$ as follows. Consider the restriction of the comultiplication Δ of $\mathbf{U}(\mathfrak{g} \otimes A)$ to $\mathbf{U}(\mathfrak{h} \otimes A)$. It is not hard to see that

$$\operatorname{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)\otimes\mathbf{U}(\mathfrak{h}\otimes A)}(w_\lambda \otimes w_\mu) \subset \operatorname{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)} w_\lambda \otimes \mathbf{U}(\mathfrak{h} \otimes A) + \mathbf{U}(\mathfrak{h} \otimes A) \otimes \operatorname{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)} w_\mu,$$

and hence we have

$$\Delta(\mathrm{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)}(w_{\lambda+\mu})) \subset \mathrm{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)}\, w_\lambda \otimes \mathbf{U}(\mathfrak{h}\otimes A) + \mathbf{U}(\mathfrak{h}\otimes A) \otimes \mathrm{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)}\, w_\mu.$$

It is now immediate that the comultiplication $\Delta : \mathbf{U}(\mathfrak{h}\otimes A) \to \mathbf{U}(\mathfrak{h}\otimes A)\otimes \mathbf{U}(\mathfrak{h}\otimes A)$ induces a homomorphism of algebras $\Delta_{\lambda,\mu} : \mathbf{A}_{\lambda+\mu} \to \mathbf{A}_\lambda \otimes \mathbf{A}_\mu$. This endows any right $\mathbf{A}_\lambda \otimes \mathbf{A}_\mu$-module (hence, in particular, $W_A(\lambda)\otimes W_A(\mu)$) with the structure of a right $\mathbf{A}_{\lambda+\mu}$-module. It was shown in [7] that $\tau_{\lambda,\mu}$ is then a homomorphism of $(\mathfrak{g}\otimes A, \mathbf{A}_{\lambda+\mu})$-bimodules. Summarizing, we have

Lemma 9. Let $\lambda_s \in P^+$, $1 \le s \le k$ and let $\lambda = \sum_{s=1}^k \lambda_s$. The natural map $W_A(\lambda) \to W_A(\lambda_1)\otimes\cdots\otimes W_A(\lambda_k)$ given by $w_\lambda \mapsto w_{\lambda_1}\otimes\cdots\otimes w_{\lambda_k}$ is a homomorphism of $(\mathfrak{g}\otimes A, \mathbf{A}_\lambda)$-bimodules. □

3.2 Local Weyl modules

For $\lambda \in P^+$, let $\mathbf{W}_A^\lambda$ be the right exact functor from $\mathbf{A}_\lambda - \mathrm{Mod}$ to the category of $\mathfrak{g}\otimes A$-modules given on objects by

$$\mathbf{W}_A^\lambda M = W_A(\lambda)\otimes_{\mathbf{A}_\lambda} M, \qquad M \in \mathbf{A}_\lambda - \mathrm{Mod}\,.$$

This is known as the Weyl functor; it plays a crucial role in all that follows. Clearly $\mathbf{W}_A^\lambda M$ is a weight module for $\mathfrak{g}$ and we have isomorphisms of vector spaces

$$(\mathbf{W}_A^\lambda M)_{\lambda-\eta} \cong (W_A(\lambda))_{\lambda-\eta} \otimes_{\mathbf{A}_\lambda} M, \qquad \eta \in Q^+,$$

$$(\mathbf{W}_A^\lambda M)_\lambda \cong (W_A(\lambda)_\lambda) \otimes_{\mathbf{A}_\lambda} M \cong w_\lambda \otimes_{\mathbf{C}} M.$$

Moreover, $\mathbf{W}_A^\lambda M$ is generated as a $\mathfrak{g}\otimes A$-module by the space $w_\lambda \otimes_{\mathbf{C}} M$ and

$$\mathbf{W}_A^\lambda M \cong \mathbf{W}_A^\mu N \iff \lambda = \mu,\ M \cong_{\mathbf{A}_\lambda} N.$$

Because $\mathbf{A}_\lambda$ is a commutative associative algebra over the algebraically closed field $\mathbf{C}$, the isomorphism classes of simple objects of $\mathbf{A}_\lambda - \mathrm{Mod}$ are given by the maximal ideals of $\mathbf{A}_\lambda$. Given $\mathbf{I} \in \operatorname{Max} \mathbf{A}_\lambda$, the quotient $\mathbf{A}_\lambda/\mathbf{I}$ is a simple object of $\mathbf{A}_\lambda - \mathrm{Mod}$ and has dimension one. The $\mathfrak{g} \otimes A$-modules $\mathbf{W}_A^\lambda \mathbf{A}_\lambda/\mathbf{I}$ are called the local Weyl modules and when $\lambda = \omega_i$, $i \in I$ we call them the fundamental local Weyl modules. It follows that $(\mathbf{W}_A^\lambda \mathbf{A}_\lambda/\mathbf{I})_\lambda$ is also a one-dimensional vector space spanned by

$$w_{\lambda,\mathbf{A}_\lambda/\mathbf{I}} = w_\lambda \otimes 1.$$

We note the following corollary.

Corollary 10. Suppose that $M \in \mathbf{A}_\lambda - \mathrm{Mod}$ is finite-dimensional. Then $\mathbf{W}_A^\lambda M$ is finite–dimensional. In particular, the local Weyl modules are finite–dimensional and have a unique irreducible quotient $\mathbf{V}_A^\lambda M$.

3.2.1

We note the following consequence of Proposition 8.

Lemma 11. For $i \in I$, we have $\mathbf{A}_{\omega_i} \cong A$ and $W_A(\omega_i)$ is a finitely generated right A-module. The fundamental local Weyl modules are given by

$$\mathbf{W}_A^{\omega_i} A/\mathfrak{I} = W_A(\omega_i) \otimes_A A/\mathfrak{I}, \qquad \mathfrak{I} \in \operatorname{Max} A.$$

In particular, we have

$$(h \otimes a) w_{\omega_i, A/\mathfrak{I}} = 0, \quad (h \otimes b) w_{\omega_i, A/\mathfrak{I}} = \omega_i(h) w_{\omega_i} \otimes \bar{b}, \qquad h \in \mathfrak{h},\ a \in \mathfrak{I},\ b \in A. \qquad \square$$

3.2.2

The following lemma is a special case of a result proved in [7] and we include the proof in this case for the reader's convenience.

Lemma 12. Let A be a finitely generated, commutative associative algebra. For $\mathfrak{I} \in \operatorname{Max} A$ there exists $N \in \mathbf{Z}_+$ such that for all $i \in I$,

$$(\mathfrak{g} \otimes \mathfrak{I}^N)\mathbf{W}_A^{\omega_i} A/\mathfrak{I} = 0. \tag{3.2.1}$$

Proof. First, we claim it is enough to show that for $j \in I$ there exists $N_j \in \mathbf{Z}_+$, with

$$(x_{\alpha_j}^- \otimes \mathfrak{I}^{N_j})w_{\omega_i,A/\mathfrak{I}} = 0. \tag{3.2.2}$$

Indeed, write $x_\theta^- = [x_{\alpha_{i_1}}^-[\cdots[x_{\alpha_{i_{p-1}}}^-, x_{\alpha_{i_p}}^-]\cdots]]$ for some $i_1, \ldots, i_p \in I$ (where we recall that θ is the highest root of $\mathfrak{g}$) and take $N = \sum_{j=1}^p N_{i_j}$. Together with the fact that

$$\mathbf{W}_A^{\omega_i} A/\mathfrak{I} = \mathbf{U}(\mathfrak{n}^- \otimes A)w_{\omega_i,A/\mathfrak{I}}, \qquad [x_\theta^-, \mathfrak{n}^-] = 0,$$

Equation 3.2.2 implies that

$$(x_\theta^- \otimes \mathfrak{I}^N)w_{\omega_i,A/\mathfrak{I}} = 0,$$

and because $\mathfrak{g}$ is a simple Lie algebra, this will prove the lemma.

It therefore remains to establish 3.2.2. Observe first that for $j \neq i \in I$, $k \in I$ and for all $a \in A$

$$x_{\alpha_k}^+(x_{\alpha_j}^- \otimes a)w_{\omega_i,A/\mathfrak{I}} = \delta_{k,j}(h_j \otimes a)w_{\omega_i,\mathfrak{I}} = 0,$$

by Lemma 3.2.1 and the defining relations of $W_A(\omega_i)$. Thus, $(x_{\alpha_j}^- \otimes a)w_{\omega_i,A/\mathfrak{I}} \in (\mathbf{W}_A^{\omega_i} A/\mathfrak{I})^{\mathfrak{n}^+}$ and since $\omega_i - \alpha_j \notin P^+$ we conclude that

$$(x_{\alpha_j}^- \otimes A)w_{\omega_i,A/\mathfrak{I}} = 0, \qquad j \neq i.$$

If $j = i$, then

$$0 = (x^+_{\alpha_i} \otimes a)(x^-_{\alpha_i})^2 w_{\omega_i} = 2((x^-_{\alpha_i} \otimes 1)(h_i \otimes a) - (x^-_{\alpha_i} \otimes a))w_{\omega_i}.$$

By Lemma 3.2.1, we have $(h_i \otimes a)w_{\omega_i, A/\mathfrak{I}} = 0$ if $a \in \mathfrak{I}$, and so we get

$$(x^-_{\alpha_i} \otimes a)w_{\omega_i, A/\mathfrak{I}} = 0, \qquad a \in \mathfrak{I}$$

which completes the proof.

□

3.3 Fundamental global Weyl modules

In this section we establish the main tool for proving Theorem 2. It is not, in general, clear how (or even if it is possible) to reconstruct the global Weyl module from a local Weyl module. The main result of this section is that it is possible to do so when $\lambda = \omega_i$ and $A = \mathcal{R}_{k,\ell}$ for some $k, \ell \in \mathbf{Z}_+$.

3.3.1

We begin with a general construction. The Lie algebra $(\mathfrak{g} \otimes A) \otimes A$ acts naturally on $V \otimes A$ for any $\mathfrak{g} \otimes A$-module V. Suppose that A is a bialgebra with the comultiplication $\mathbf{h} : A \to A \otimes A$. (It is useful to recall that A is a commutative associative algebra with identity). Then the comultiplication map $\mathbf{h}$ induces a homomorphism of Lie algebras $1 \otimes \mathbf{h} : \mathfrak{g} \otimes A \to \mathfrak{g} \otimes A \otimes A$ (cf. 1.1.1) and thus a $\mathfrak{g} \otimes A$-module structure on $V \otimes A$. Explicitly, the $(\mathfrak{g} \otimes A, A)$-bimodule structure on $V \otimes A$ is given by the following formulas:

$$(x \otimes a)(v \otimes b) = \sum_s (x \otimes a'_s)v \otimes a''_s b, \qquad (v \otimes b)a = v \otimes ba,\ v \in V,\ a, b \in A,$$

where $\mathbf{h}(a) = \sum_s a'_s \otimes a''_s$. We denote this bimodule by $(V \otimes A)_{\mathbf{h}}$ and observe that it is a free right A-module of rank equal to $\dim_{\mathbf{C}} V$. It is trivial to see that $(V \otimes A)_{\mathbf{h}}$ is a weight module for $\mathfrak{g} \otimes A$ if V is a weight module for $\mathfrak{g} \otimes A$ and that

$$((V \otimes A)_{\mathbf{h}})_\mu = V_\mu \otimes A, \qquad V^{\mathfrak{n}^+ \otimes A} \otimes A \subset (V \otimes A)_{\mathbf{h}}^{\mathfrak{n}^+ \otimes A}.$$

Moreover, if V_1, V_2 are $\mathfrak{g} \otimes A$-modules, one has a natural inclusion

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(V_1, V_2) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}\otimes A}((V_1 \otimes A)_{\mathbf{h}}, (V_2 \otimes A)_{\mathbf{h}}), \quad \eta \mapsto \eta \otimes 1. \tag{3.3.1}$$

In particular, if V is reducible, then $(V \otimes A)_{\mathbf{h}}$ is also a reducible $(\mathfrak{g} \otimes A)$-module.

3.3.2

Let $\mathbf{h}_{k,\ell} : \mathcal{R}_{k,\ell} \to \mathcal{R}_{k,\ell} \otimes \mathcal{R}_{k,\ell}$ be the comultiplication given by,

$$\mathbf{h}_{k,\ell}(t_s^{\pm 1}) = t_s^{\pm 1} \otimes t_s^{\pm 1}, \qquad \mathbf{h}_{k,\ell}(u_r) = u_r \otimes 1 + 1 \otimes u_r,$$

where $1 \le s \le k$ and $1 \le r \le \ell$. Any monomial $\mathbf{m} \in \mathcal{R}_{k,\ell}$ can be written uniquely as a product of monomials

$$\mathbf{m} = \mathbf{m}_u \mathbf{m}_t, \qquad \mathbf{m}_t \in \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}],\ \mathbf{m}_u \in \mathbf{C}[u_1, \dots, u_\ell].$$

Set $\deg t_s^{\pm 1} = \pm 1$ and $\deg u_r = 1$ for $1 \le s \le k$, $1 \le r \le \ell$ and let $\deg_t \mathbf{m}$ (respectively, $\deg_u \mathbf{m}$) be the total degree of $\mathbf{m}_t$ (respectively, $\mathbf{m}_u$) and for for any $f \in A$ define $\deg_t f$ and $\deg_u f$ in the obvious way. The next lemma is elementary.

Lemma 13. Let $\mathbf{m} = \mathbf{m}_t \mathbf{m}_u$ be a monomial in $\mathcal{R}_{k,\ell}$. Then $\mathbf{m}_t \in \mathcal{R}_{k,\ell}^\times$, $\mathbf{h}_{k,\ell}(\mathbf{m}_t) = \mathbf{m}_t \otimes \mathbf{m}_t$ and

$$\mathbf{h}_{k,\ell}(\mathbf{m}) = \mathbf{m} \otimes \mathbf{m}_t + \sum_q \mathbf{m}'_{u,q}\mathbf{m}_t \otimes \mathbf{m}''_{u,q}\mathbf{m}_t = \mathbf{m}_t \otimes \mathbf{m} + \sum_q \mathbf{m}''_{u,q}\mathbf{m}_t \otimes \mathbf{m}'_{u,q}\mathbf{m}_t, \tag{3.3.2}$$

where $\mathbf{m}'_{u,q}, \mathbf{m}''_{u,q}$ are (scalar multiples of) monomials in the u_r, $1 \leq r \leq \ell$, such that if $\mathbf{m}'_{u,q} \neq 0$, $\mathbf{m}''_{u,q} \neq 0$, then $\deg_u \mathbf{m}'_{u,q} < \deg_u \mathbf{m}$ and $\deg_u \mathbf{m}'_{u,q} + \deg_u \mathbf{m}''_{u,q} = \deg_u \mathbf{m}$. $\square$

3.3.3

For the rest of the section A denotes the algebra $\mathcal{R}_{k,\ell}$ for some $k, \ell \in \mathbf{Z}_+$ and $\mathfrak{I}$ the ideal of A generated by the elements $\{t_1 - 1, \ldots, t_k - 1, u_1, \ldots, u_\ell\}$.

Suppose that $\mathfrak{J} \in \operatorname{Max} \mathcal{R}_{k,\ell}$. It is clear that there exists an algebra automorphism $\varphi : \mathcal{R}_{k,\ell} \to \mathcal{R}_{k,\ell}$ such that $\varphi(\mathfrak{I}) = \mathfrak{J}$. As a consequence, we have an induced isomorphism of $\mathfrak{g} \otimes A$-modules,

$$\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \cong (1 \otimes \varphi)^* \mathbf{W}_A^{\omega_i} A/\mathfrak{I}. \tag{3.3.3}$$

Moreover, if we set

$$\mathbf{h}_{k,\ell}^{\varphi} = (\varphi \otimes \varphi) \circ \mathbf{h}_{k,\ell} \circ \varphi^{-1} : A \to A \otimes A,$$

then $\mathbf{h}_{k,\ell}^{\varphi}$ also defines a bialgebra structure on A and we have an isomorphism of $\mathfrak{g} \otimes A$-modules

$$(1 \otimes \varphi)^* (\mathbf{W}_A^{\omega_i} A/\mathfrak{I} \otimes A)_{\mathbf{h}_{k,\ell}} \cong (\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}^{\varphi}}. \tag{3.3.4}$$

This becomes an isomorphism of $(\mathfrak{g} \otimes A, A)$-bimodules if we twist the right A-module structure of $(\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}^{\varphi}}$ by φ.

3.4 Construction of fundamental global Weyl modules

We now reconstruct the global fundamental Weyl module from a local one.

Proposition 14. For all $i \in I$, the assignment $w_{\omega_i} \mapsto w_{\omega_i, A/\mathfrak{J}} \otimes 1$ defines an isomorphism of $(\mathfrak{g} \otimes A, A)$-bimodules

$$W_A(\omega_i) \xrightarrow{\cong} (\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}}.$$

Remark 15. It is clear from (3.3.4) and Section 1.4 that one can work with an arbitrary ideal $\mathfrak{J}$ provided that $\mathbf{h}_{k,\ell}$ is replaced by $\mathbf{h}_{k,\ell}^{\varphi}$, where φ is the unique automorphism of A such that $\varphi(\mathfrak{I}) = \mathfrak{J}$.

Proof. The element $w_{\omega_i, A/\mathfrak{J}} \otimes 1 \in (\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}}$ satisfies the relations in Definition 1.4 and hence the assignment $w_{\omega_i} \mapsto w_{\omega_i, A/\mathfrak{J}} \otimes 1$ defines a homomorphism of $\mathfrak{g} \otimes A$-modules

$$\mathbf{p} : W_A(\omega_i) \to (\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}}.$$

We begin by proving that $\mathbf{p}$ is a homomorphism of right A-modules. Using Lemma 3.2.1 and the definition of the right module structure on $W_A(\omega_i)$ we see that

$$\mathbf{p}((uw_{\omega_i})a) = \mathbf{p}(u(h_i \otimes a)w_{\omega_i}) = u(h_i \otimes a)(w_{\omega_i, A/\mathfrak{J}} \otimes 1), \qquad u \in \mathbf{U}(\mathfrak{g} \otimes A),\ a \in A.$$

This shows that it is enough to prove that for any monomial $\mathbf{m}$ in A, we have

$$(h_i \otimes \mathbf{m})(w_{\omega_i, A/\mathfrak{J}} \otimes 1) = w_{\omega_i, A/\mathfrak{J}} \otimes \mathbf{m}. \tag{3.4.1}$$

Write $\mathbf{m} = \mathbf{m}_t \mathbf{m}_u$ and observe that $\mathbf{m}_t - 1 \in \mathfrak{J}$ while

$$\deg_u \mathbf{m} > 0 \implies \mathbf{m} \in \mathfrak{J}.$$

Using (3.3.2) we get

$$\mathbf{h}_{k,\ell}(\mathbf{m}) - 1 \otimes \mathbf{m} \in \mathfrak{J} \otimes A$$

and since $(h_i \otimes \mathfrak{J} \otimes A)(w_{\omega_i, A/\mathfrak{J}} \otimes 1) = 0$ by Lemma 3.2.1, we have established (3.4.1).

To prove that $\mathbf{p}$ is surjective we must show that

$$\mathbf{U}(\mathfrak{g} \otimes A)(w_{\omega_i, A/\mathfrak{I}} \otimes 1) = (\mathbf{W}_A^{\omega_i} A/\mathfrak{I} \otimes A)_{\mathbf{h}_{k,\ell}},$$

and the remarks in Section 3.3.1 show that it is enough to prove

$$(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})_{\omega_i - \eta} \otimes A \subset \mathbf{U}(\mathfrak{g} \otimes A)(w_{\omega_i, A/\mathfrak{I}} \otimes 1), \qquad \eta \in Q^+. \tag{3.4.2}$$

The argument is by induction on $\operatorname{ht} \eta$, the induction base with $\operatorname{ht} \eta = 0$ being immediate from (3.4.1). For the inductive step assume that we have proved the result for all $\eta \in Q^+$ with $\operatorname{ht} \eta < k$. To prove the result for $\operatorname{ht} \eta = k$, it suffices to prove that for all $j \in I$ and all monomials $\mathbf{m}$ in A we have

$$((x_{\alpha_j}^- \otimes \mathbf{m})w) \otimes g \in \mathbf{U}(\mathfrak{g} \otimes A)(w_{\omega_i, A/\mathfrak{I}} \otimes 1),$$

where $w \in (\mathbf{W}_A^{\omega_i} A/\mathfrak{I})_{\omega_i - \eta'}$ with $\operatorname{ht} \eta' = k-1$ and $g \in A$. For this, we argue by a further induction on $\deg_u \mathbf{m}$. If $\deg_u \mathbf{m} = 0$ then $\mathbf{m} = \mathbf{m}_t$ and we have

$$((x_{\alpha_j}^- \otimes \mathbf{m})(w) \otimes g = (x_{\alpha_j}^- \otimes \mathbf{m})(w \otimes \mathbf{m}^{-1} g) \in \mathbf{U}(\mathfrak{g} \otimes A)(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})_{\omega_i - \eta'}.$$

This proves that the induction on $\deg_u \mathbf{m}$ starts. If $\deg_u \mathbf{m} > 0$ we use (3.3.2) to get

$$((x_{\alpha_j}^- \otimes \mathbf{m})w) \otimes g = (x_{\alpha_j}^- \otimes \mathbf{m})(w \otimes \mathbf{m}_t^{-1} g) - \sum_q ((x_{\alpha_j}^- \otimes \mathbf{m}'_{u,q} \mathbf{m}_t)w) \otimes \mathbf{m}''_{u,q} g.$$

Both terms on the right hand side are in $\mathbf{U}(\mathfrak{g} \otimes A)(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})_{\omega_i - \eta'}$: the first by the induction hypothesis on $\operatorname{ht} \eta'$ and the second by the induction hypothesis on $\deg_u \mathbf{m}$. This completes the proof of the surjectivity of $\mathbf{p}$.

To prove that $\mathbf{p}$ is injective, recall from Section 3.3.1 that $(\mathbf{W}_A^{\omega_i} A/\mathfrak{I} \otimes A)_{\mathbf{h}_{k,\ell}}$ is a free right A-module of rank equal to the dimension of $\mathbf{W}_A^{\omega_i} A/\mathfrak{I}$. Hence if K is the kernel of $\mathbf{p}$ we have an isomorphism of right A-modules,

$$W_A(\omega_i) \cong K \oplus (\mathbf{W}_A^{\omega_i} A/\mathfrak{I} \otimes A)_{\mathbf{h}_{k,\ell}}.$$

Using (3.3.3) we see that for any maximal ideal $\mathfrak{J}$ in A,

$$\dim(W_A(\omega_i) \otimes_A A/\mathfrak{J}) = \dim \mathbf{W}_A^{\omega_i} A/\mathfrak{J} = \dim \mathbf{W}_A^{\omega_i} A/\mathfrak{I} = \dim((\mathbf{W}_A^{\omega_i} A/\mathfrak{I} \otimes A)_{\mathbf{h}_{k,\ell}} \otimes_A A/\mathfrak{J}).$$

Therefore, $K \otimes_A A/\mathfrak{J} = 0$. Since K is finitely generated over the (Noetherian) ring A, Nakayama's Lemma implies that there is $a \in A$ with $a - 1 \in \mathfrak{J}$, so that $Ka = 0$. In particular, a becomes invertible in the localization $A_{\mathfrak{J}}$, from which it now follows that $K_{\mathfrak{J}} = 0$ for all $\mathfrak{J} \in \operatorname{Max} A$. Thus, $K = 0$. $\square$

Chapter 4

This chapter contains the proofs of the main theorems of Part I.

4.1 Proof of Theorem 2 and Corollary 6

We continue to assume that $A = \mathcal{R}_{k,\ell}$, $k, \ell \in \mathbf{Z}_+$ and that $\mathfrak{I}$ is the maximal ideal of A generated by $\{t_1 - 1, \dots, t_k - 1, u_1, \dots, u_\ell\}$. We also use the comultiplication $\mathbf{h}_{k,\ell}$ and denote it by just $\mathbf{h}$. Let $\mathfrak{M}_A \subset A$ be the set of monomials in the generators u_r, $t_s^{\pm 1}$, $1 \leq r \leq \ell$, $1 \leq s \leq k$.

4.1.1

The following proposition, together with Lemma 12 and Proposition 14, completes the proof of Theorem 2.

Proposition 16. Suppose that V_s, $1 \leq s \leq M$ are $\mathfrak{g} \otimes A$-modules such that there exists $N \in \mathbf{Z}_+$ with

$$(\mathfrak{n}^+ \otimes \mathfrak{I}^N) V_s = 0, \qquad 1 \leq s \leq M. \tag{4.1.1}$$

Then

$$((V_1 \otimes A)_{\mathbf{h}} \otimes \cdots \otimes (V_M \otimes A)_{\mathbf{h}})^{\mathfrak{n}^+ \otimes A} = (V_1^{\mathfrak{n}^+ \otimes A} \otimes A)_{\mathbf{h}} \otimes \cdots \otimes (V_M^{\mathfrak{n}^+ \otimes A} \otimes A)_{\mathbf{h}}.$$

4.1.2

The first step in the proof of Proposition 16 is the following. We need some notation. Let V be a $\mathfrak{g} \otimes A$-module and let $K \in \mathbf{Z}_+$. Define

$$V_{\geq K} = \{v \in V \,:\, (\mathfrak{n}^+ \otimes \mathbf{m})v = 0,\, \mathbf{m} \in \mathfrak{M}_A,\, |\deg_t \mathbf{m}| \geq K\}.$$

Note that $V_{\geq 0} = V^{\mathfrak{n}^+ \otimes A}$.

Lemma 17. Let V be a $\mathfrak{g} \otimes A$-module and $K \in \mathbf{Z}_+$. Then

$$((V \otimes A)_{\mathbf{h}})_{\geq K} = V_{\geq K} \otimes A. \tag{4.1.2}$$

In particular,

$$(V \otimes A)_{\mathbf{h}}^{\mathfrak{n}^+ \otimes A} = V^{\mathfrak{n}^+ \otimes A} \otimes A. \tag{4.1.3}$$

Proof. Let $v_{\mathbf{h}} \in (V \otimes A)_{\mathbf{h}}$ and write, $v_{\mathbf{h}} = \sum_p v_p \otimes g_p$, where $\{g_p\}_p$ is a linearly independent subset of A. By (3.3.2) we have

$$(x \otimes \mathbf{m})v_{\mathbf{h}} = \sum_p (x \otimes \mathbf{m})v_p \otimes \mathbf{m}_t g_p + \sum_{p,q} (x \otimes \mathbf{m}'_{u,q}\mathbf{m}_t)v_p \otimes \mathbf{m}''_{u,q}\mathbf{m}_t g_p.$$

with $\deg_u \mathbf{m}'_{u,q} < \deg_u \mathbf{m}$. Since $\deg_t \mathbf{m}'_{u,q}\mathbf{m}_t = \deg_t \mathbf{m}$, it follows that

$$V_{\geq K} \otimes A \subset ((V \otimes A)_{\mathbf{h}})_{\geq K}.$$

We prove the reverse inclusion by induction on $\deg_u \mathbf{m}$. Let $v_{\mathbf{h}} \in ((V \otimes A)_{\mathbf{h}})_{\geq K}$ and let $\deg_t \mathbf{m} \geq K$. If $\deg_u \mathbf{m} = 0$, then

$$0 = (x \otimes \mathbf{m})v_{\mathbf{h}} = \sum_p (x \otimes \mathbf{m})v_p \otimes g_p \mathbf{m}.$$

Since the set $\{g_p \mathbf{m}\}_p$ is also linearly independent, we see that $(x \otimes \mathbf{m})v_p = 0$ for all p. If $\deg_u \mathbf{m} > 0$, we use (3.3.2) to get

$$0 = (x \otimes \mathbf{m})v_{\mathbf{h}} = \sum_p (x \otimes \mathbf{m})v_p \otimes \mathbf{m}_t g_p + \sum_{p,q} (x \otimes \mathbf{m}'_{u,q}\mathbf{m}_t)v_p \otimes \mathbf{m}''_{u,q}\mathbf{m}_t g_p.$$

Since $\deg_u \mathbf{m}'_{u,q} < \deg_u \mathbf{m}$ all terms in the second sum are zero by the induction hypothesis, and the linear independence of the set $\{\mathbf{m}_t g_p\}_p$ gives $(x \otimes \mathbf{m})v_p = 0$ for all p. □

4.1.3

Proposition 18. Let U, V be $\mathfrak{g} \otimes A$-modules and suppose that for some $N \in \mathbf{Z}_+$

$$(\mathfrak{n}^+ \otimes \mathfrak{I}^N)V = 0.$$

Then

$$(U \otimes (V \otimes A)_{\mathbf{h}})^{\mathfrak{n}^+ \otimes A} = U^{\mathfrak{n}^+ \otimes A} \otimes (V^{\mathfrak{n}^+ \otimes A} \otimes A). \tag{4.1.4}$$

Before proving this proposition, we establish Proposition 16. The argument is by induction on M, with (4.1.3) showing that induction begins at $M = 1$. For $M > 1$, take

$$U = (V_1 \otimes A)_{\mathbf{h}} \otimes \cdots \otimes (V_{M-1} \otimes A)_{\mathbf{h}}, \qquad V = V_M.$$

The induction hypothesis applies to U and together with Proposition 18 completes the inductive step.

4.1.4

Lemma 19. Let $A = \mathcal{R}_{k,\ell}$ with $k > 0$. Let V be a $\mathfrak{g} \otimes A$-module and suppose that $(\mathfrak{n}^+ \otimes \mathfrak{I}^N)V = 0$ for some $N \in \mathbf{Z}_+$. Then for all $K \in \mathbf{Z}_+$ we have

$$V^{\mathfrak{n}^+ \otimes A} = V_{\geq K}.$$

Proof. It suffices to prove that $V_{\geq K} \subset V_{\geq K-1}$ for all $K \geq 1$. Since $(1-t_1^{\pm 1})^N \in \mathfrak{I}^N$ we have

$$0 = (x \otimes \mathbf{m}(1-t_1^{\pm 1})^N)v = (x \otimes \mathbf{m})v + \sum_{s=1}^{N} (-1)^s \binom{N}{s} (x \otimes \mathbf{m}t_1^{\pm s})v, \tag{4.1.5}$$

for all $x \in \mathfrak{n}^+$, $\mathbf{m} \in \mathfrak{M}_A$ and $v \in V$ Suppose that $v \in V_{\geq K}$ and take $\mathbf{m} \in \mathfrak{M}_A$ with $|\deg_t \mathbf{m}| = K-1$. If $\deg_t \mathbf{m} \geq 0$ (respectively, $\deg_t \mathbf{m} < 0$) then $|\deg_t \mathbf{m}t_1^s| \geq K$ (respectively, $|\deg_t \mathbf{m}t_1^{-s}| \geq K$) for all $s > 0$. Thus we conclude that all terms in the sum in (4.1.5) with the appropriate sign choice equal zero hence $(x \otimes \mathbf{m})v = 0$ and so $v \in V_{\geq K-1}$. □

4.1.5

Lemma 20. Let $A = \mathcal{R}_{0,\ell}$. Let V be a $\mathfrak{g} \otimes A$-module and suppose that $(\mathfrak{n}^+ \otimes \mathfrak{I}^N)V = 0$ for some $N \in \mathbf{Z}_+$. Let $K \geq N \in \mathbf{Z}_+$. Then

$$V^{\mathfrak{n}^+ \otimes A} \otimes A = \{v_{\mathbf{h}} \in (V \otimes A)_{\mathbf{h}} \,:\, (\mathfrak{n}^+ \otimes \mathbf{m})v_{\mathbf{h}} = 0,\, \mathbf{m} \in \mathfrak{M}_A,\, \deg_u \mathbf{m} \geq K\} \tag{4.1.6}$$

Proof. Since $(V \otimes A)_{\mathbf{h}}$ is a $(\mathfrak{g} \otimes A, A)$-bimodule the sets on both sides of (4.1.6) are right A-modules. Hence if $v_{\mathbf{h}}$ is an element of the set on the right hand side of (4.1.6) then $v_{\mathbf{h}} u_j^s$ is also in the right hand side of (4.1.6) for all $s \in \mathbf{Z}_+$. Write $v_{\mathbf{h}} = \sum_p v_p \otimes g_p$, where $\{g_p\}_p$ is a linearly independent subset of A. Since the u_j, $1 \leq j \leq \ell$ are primitive and $u_j^s \in \mathfrak{I}^N$ if $s \geq N$, we have for all $0 \leq r \leq N$

$$\begin{aligned} 0 = (x \otimes u_j^{(K+N-r)})(v_{\mathbf{h}})u_j^{(r)} &= \sum_{s=0}^{N} \Big(\sum_p ((x \otimes u_j^{(s)})v_p) \otimes u_j^{(K+N-r-s)} g_p u_j^{(r)} \Big) \\ &= \sum_{s=0}^{N} \binom{K+N-s}{r} \Big(\sum_p ((x \otimes u_j^{(s)})v_p) \otimes u_j^{(K+N-s)} g_p \Big). \end{aligned}$$

We claim that the matrix $C(N,K) = (\binom{K+N-s}{r})_{0 \leq s,r \leq N}$ is invertible. Assuming the claim, we get

$$\sum_p ((x \otimes u_j^{(s)})v_p) \otimes u_j^{(K+N-s)} g_p = 0, \qquad 0 \leq s \leq N,$$

and since the g_p are linearly independent this implies that

$$(x \otimes u_j^{(s)})v_p = 0, \quad 0 \le s \le N$$

and so $(x \otimes u_j^s)v_{\mathbf{h}} = 0$ for all $x \in \mathfrak{n}^+$, $s \in \mathbf{Z}_+$.

Now, let $\mathbf{m} \in \mathfrak{M}_A$ and let $\alpha \in \Phi^+$. Then $(h_\alpha \otimes \mathbf{m})v_{\mathbf{h}}$ is also an element of the right hand side of (4.1.6) and hence by the preceding argument, we get

$$\begin{aligned} 0 &= (x_\alpha \otimes 1)(h_\alpha \otimes \mathbf{m})v_{\mathbf{h}} \\ &= (h_\alpha \otimes \mathbf{m})(x_\alpha \otimes 1)v_{\mathbf{h}} - 2(x_\alpha \otimes \mathbf{m})v_{\mathbf{h}} = -(2x_\alpha \otimes \mathbf{m})v_{\mathbf{h}}, \end{aligned}$$

thus proving that $v_{\mathbf{h}} \in (V \otimes A)_{\mathbf{h}}^{\mathfrak{n}^+ \otimes A} = V^{\mathfrak{n}^+ \otimes A} \otimes A$ by (4.1.3).

To prove the claim, let u be an indeterminate and let $\{p_r \in \mathbf{C}[u] : 0 \le r \le N\}$ be a collection of polynomials such that $\deg p_r = r$ (in particular, we assume that p_0 is a non-zero constant polynomial). Then for any tuple $(a_0, \dots, a_N) \in \mathbf{C}^{N+1}$, we have $\det(p_r(a_s))_{0 \le r,s \le N} = c \det(a_s^r)_{0 \le r,s \le N} = c \prod_{0 \le r < s \le N}(a_s - a_r)$, where c is the product of highest coeffcients of the p_r, $0 \le r \le N$. Since $\binom{u}{r}$ is a polynomial in u of degree r with highest coefficient $1/r!$, we obtain with $a_s = N + K - s$,

$$\det C(N,K) = \Big(\prod_{r=1}^{N} r!\Big)^{-1} \prod_{0 \le r < s \le N} (r - s) = (-1)^{N(N+1)/2}. \qquad \square$$

4.2 Proof of Proposition 18

Now we have all the necessary ingredients to prove Proposition 18.

Proof of Proposition 18. Let $v_{\mathbf{h}} \in (U \otimes (V \otimes A)_{\mathbf{h}})^{\mathfrak{n}^+ \otimes A}$ and write $v_{\mathbf{h}} = \sum_{p,s} w_s \otimes v_{s,p} \otimes g_p$, where $\{w_s\}_s$ and $\{g_p\}_p$ are linearly independent subsets of U and A respectively. We have

$$0 = (x \otimes \mathbf{m})v_{\mathbf{h}} = \sum_{s,p} \Big(((x \otimes \mathbf{m})w_s) \otimes v_{s,p} \otimes g_p + w_s \otimes (x \otimes \mathbf{m})(v_{s,p} \otimes g_p) \Big) \tag{4.2.1}$$

$$= \sum_{s,p} ((x \otimes \mathbf{m})w_s) \otimes v_{s,p} \otimes g_p$$

$$+ \sum_{s,p} w_s \otimes \Big((x \otimes \mathbf{m})v_{s,p} \otimes g_p \mathbf{m}_t + \sum_q (x \otimes \mathbf{m}'_{u,q}\mathbf{m}_t)v_{s,p} \otimes \mathbf{m}''_{u,q}\mathbf{m}_t g_p \Big), \tag{4.2.2}$$

Suppose first that $A = \mathcal{R}_{k,\ell}$ with $k > 0$ and let $K = \max_p |\deg_t g_p| + 1$. If $\mathbf{m}$ is such that $|\deg_t \mathbf{m}| \geq K$, then the set $\{g_p\}_p$ is linearly independent from the set $\{\mathbf{m}''_{u,q}\mathbf{m}_t g_p\}_{p,q}$ and hence we must have that

$$\sum_{s,p} ((x \otimes \mathbf{m})w_s) \otimes v_{s,p} \otimes g_p = 0, \qquad \sum_{s,p} w_s \otimes (x \otimes \mathbf{m})(v_{s,p} \otimes g_p) = 0 \tag{4.2.3}$$

and using the linear independence of the elements $\{w_s\}_s$ we conclude that for all s

$$\sum_p (v_{s,p} \otimes g_p) \in ((V \otimes A)_{\mathbf{h}})_{\geq K} = V_{\geq K} \otimes A = V^{\mathfrak{n}^+ \otimes A} \otimes A,$$

using (4.1.2) and Lemma 19. This proves Proposition 18 in the case when $k > 0$.

Suppose now that $A = \mathcal{R}_{0,\ell}$ and let $N \in \mathbf{Z}_+$ be such that $(\mathfrak{n}^+ \otimes \mathfrak{I}^N)V = 0$. Let $K = N + 1 + \max_p \{\deg_u g_p\}$ and let $x \in \mathfrak{n}^+$. If $\deg_u \mathbf{m} \geq K$ then $\mathbf{m} \in \mathfrak{I}^N$ and so $(x \otimes \mathbf{m})v_{s,p} = 0$. Furthermore, $(x \otimes \mathbf{m}'_{u,q})v_{s,p} \neq 0$ implies that $\deg_u \mathbf{m}'_{u,q} < N$. By Lemma 13 it follows that $\deg_u \mathbf{m}''_{u,q} > \max_p \{\deg g_p\}$. Therefore, the non-zero terms, if any, in the second sum in (4.2.2) are linearly independent from those in the first sum and we obtain (4.2.3). Furthermore, we have

$$0 = \sum_p \sum_{\{q \,:\, \deg_u \mathbf{m}'_{u,q} < N\}} (x \otimes \mathbf{m}'_{u,q})v_{s,p} \otimes \mathbf{m}''_{u,q} g_p$$

and as before we conclude that $(x \otimes \mathbf{m}'_{u,q})v_{s,p} = 0$ when $\deg_u \mathbf{m}'_{u,q} < N$. Thus, $(x \otimes \mathbf{m})\big(\sum_p v_{s,p} \otimes g_p\big) = 0$ for all s and it remains to apply Lemma 20. $\square$

4.2.1

We conclude this section with a proof of Corollary 6. The following is a special case of Theorem 7.1 and Proposition 7.7 of [7].

Theorem 4. Let $\mathfrak{I} \in \operatorname{Max} A$, where $A = \mathcal{R}_{k,\ell}$. Then

$$\dim(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})^{\mathfrak{n}^+\otimes A} = \dim(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})_{\omega_i} = 1, \qquad i \in I_0,$$

If $\mathfrak{g}$ is of type B_n or D_n and $i \notin I_0$ then

$$\dim(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})_{\mu}^{\mathfrak{n}^+\otimes A} = \begin{cases} 0, & \mu \neq \omega_{i-2j}, \\ \binom{j+k-1}{j}, & \mu = \omega_{i-2j},\ i-2j \geq 0 \end{cases}$$

where $\omega_0 = 0$. □

Alternately, one may reformulate this result in the following way. The subspace $(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})^{\mathfrak{n}^+\otimes A}$ is an $\mathfrak{h}$-module with character given by

$$\operatorname{ch}(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})^{\mathfrak{n}^+\otimes A} = \sum_{j: i-2j\geq 0} \binom{j+k-1}{j} e(\omega_{i-2j}).$$

Hence, using Theorem 2, we get

$$\operatorname{ch}\mathbf{W}_A(\mathbf{s})^{\mathfrak{n}^+\otimes A} = \prod_{i\in I} (\operatorname{ch}(\mathbf{W}_A^{\omega_i} A/\mathfrak{I})^{\mathfrak{n}^+\otimes A})^{s_i},$$

and Corollary 6 follows.

4.3 Proof of Theorem 3

Given $\mathbf{s} \in \mathbf{Z}_+^I$, define $\mu_{\mathbf{s}} \in P^+$ by $\mu_{\mathbf{s}} = \sum_{i\in I} s_i\omega_i$ and let $\tau_{\mathbf{s}} : W_A(\mu_{\mathbf{s}}) \to W_A(\mathbf{s})$ be the natural map of $(\mathfrak{g} \otimes A, A)$-bimodules defined in the above lemma and satisfying $\tau_{\mathbf{s}}(w_{\mu_{\mathbf{s}}}) = w_{\mathbf{s}}$. Since $W_A(\mathbf{s})_{\mu_{\mathbf{s}}} = w_{\mathbf{s}} \otimes \mathbf{A}_{\mathbf{s}}$, we see that any non-zero element τ of

$\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\mu_{\mathbf{s}}), W_A(\mathbf{s}))$ is given by composing $\tau_{\mathbf{s}}$ with right multiplication by an element of $\mathbf{A}_{\mathbf{s}}$, i.e. $\tau = \tau_{\mathbf{s}}\mathbf{a}$ with $\mathbf{a} \in \mathbf{A}_{\mathbf{s}}$.

4.3.1

Lemma 21. Assume that $\lambda \in P^+$ and $\mathbf{s} \in \mathbf{Z}_+^I$ satisfy

(i) any non-zero element of $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mathbf{s}))$ is injective,

(ii) the map $\tau_{\mathbf{s}} : W_A(\mu_{\mathbf{s}}) \to W_A(\mathbf{s})$ is injective.

Then any non-zero element of $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mu_{\mathbf{s}}))$ is injective. Moreover, if $s_i = 0$ for all $i \notin I_0$, then

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mu_{\mathbf{s}})) = 0, \qquad \lambda \neq \mu_{\mathbf{s}}.$$

Proof. Let $\eta \in \mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mu_{\mathbf{s}}))$. If $\eta \neq 0$, then $\tau_{\mathbf{s}} \cdot \eta \in \mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mathbf{s}))$ is non-zero since $\tau_{\mathbf{s}}$ is injective. Hence $\tau_{\mathbf{s}} \cdot \eta$ is injective which forces η to be injective. If we now assume that $\lambda \neq \mu_{\mathbf{s}}$ and that $s_i = 0$ if $i \notin I_0$, then it follows from Corollary 6 that $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mathbf{s})) = 0$ and hence it follows that $\eta = 0$ in this case. $\square$

Remark 22. Using Theorem 4, we see that if $\mathfrak{g}$ is of type B_n or D_n with $n \geq 6$ and $i = 4$, then $(\mathbf{W}_A^{\omega_4} A/\mathfrak{I})_{\omega_2}^{\mathfrak{n}^+\otimes A} \neq 0$ or equivalently

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(\mathbf{W}_A^{\omega_2} A/\mathfrak{I}, \mathbf{W}_A^{\omega_4} A/\mathfrak{I}) \neq 0.$$

Using Proposition 14 and (3.3.1) we get

$$\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\omega_2), W_A(\omega_4)) \neq 0,$$

which in particular proves that the last assertion of the above Lemma and hence Theorem 3 fail in this case.

4.3.2

From now on, we shall assume that $\mathbf{s} \in \mathbf{Z}_+^I$ is such that $s_i = 0$ if $i \notin I_0$. By Corollary 6, we see that $\mathrm{Hom}_{\mathfrak{g}\otimes A}(W_A(\lambda), W_A(\mu_{\mathbf{s}})) = 0$ if $\lambda \neq \mu_{\mathbf{s}}$ and the first condition of Lemma 21 is trivially satisfied. Hence Theorem 3 will follow if we show that $\mu_{\mathbf{s}}$ satisfies both conditions in Lemma 21. By the discussion at the start of Section 5, we see that proving that $\mu_{\mathbf{s}}$ satisfies the first condition is equivalent to proving that $\tau_{\mathbf{s}}\mathbf{a}$ is injective for all $\mathbf{a} \in \mathbf{A}_{\mathbf{s}}$. In other words, Theorem 3 follows if we establish the following.

Proposition 23. Let $\mathbf{s} \in \mathbf{Z}_+^I$ be such that $s_i = 0$ if $i \notin I_0$. For all $\mathbf{a} \in \mathbf{A}_{\mathbf{s}}$, the canonical map $\tau_\mu \mathbf{a} : W_A(\mu_{\mathbf{s}}) \to W_A(\mathbf{s})$ given by extending $w_{\mu_{\mathbf{s}}} \to w_{\mathbf{s}}\mathbf{a}$ is injective, in the following cases:

(i) $A = \mathcal{R}_{0,1}$ or $\mathcal{R}_{1,0}$,

(ii) $A = \mathcal{R}_{k,\ell}$, $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\mathbf{s} = (s, 0, \ldots, 0) \in \mathbf{Z}_+^I$, $s > 0$.

The rest of the section is devoted to proving the proposition.

4.3.3

We begin by proving the following Lemma.

Lemma 24. Let A be a finitely generated integral domain. Let $\mathbf{s} \in \mathbf{Z}_+^I$ be such that $s_i = 0$ if $i \notin I_0$. Then $\tau_{\mu_{\mathbf{s}}}\mathbf{a} : W_A(\mu_{\mathbf{s}}) \to W_A(\mathbf{s})$ is injective for $\mathbf{a} \in \mathbf{A}_{\mathbf{s}} \setminus \{0\}$ if and only if $\tau_{\mu_{\mathbf{s}}}$ is injective.

Proof. Consider the map $\rho_{\mathbf{a}} : W_A(\mathbf{s}) \to W_A(\mathbf{s})$ given by

$$\rho_{\mathbf{a}}(w) = w\mathbf{a}, \qquad w \in W_A(\mathbf{s}).$$

This is clearly a map of $(\mathfrak{g}\otimes A, \mathbf{A}_{\mathbf{s}})$-bimodules. Since

$$W_A(\mathbf{s})_{\mu_{\mathbf{s}}} = w_{\mathbf{s}} \otimes \mathbf{A}_{\mathbf{s}},$$

and $\mathbf{A}_{\mathbf{s}}$ is an integral domain, it follows that that the restriction of $\rho_{\mathbf{a}}$ to $W_A(\mathbf{s})_{\mu_{\mathbf{s}}}$ is injective, and so

$$\ker \rho_{\mathbf{a}} \cap W_A(\mathbf{s})_{\mu_{\mathbf{s}}} = \{0\}.$$

Since wt $W_A(\mathbf{s}) \subset \mu_{\mathbf{s}} - Q^+$, it follows that if $\ker \rho_{\mathbf{a}}$ is non-zero, there must exist $w' \in \ker \rho_{\mathbf{a}}$ with

$$(\mathfrak{n}^+ \otimes A)w' = 0.$$

But this is impossible by Corollary 6 and hence $\ker \rho_{\mathbf{a}} = 0$. Since $\tau_{\mu_{\mathbf{s}}}\mathbf{a} = \rho_{\mathbf{a}}\tau_{\mu_{\mathbf{s}}}$, the Lemma follows. $\square$

4.3.4

We now prove that $\tau_{\mu_{\mathbf{s}}}$ is injective. This was proved in [11] for $\mathfrak{g} = \mathfrak{sl}_2$ and $A = \mathcal{R}_{1,0}$ and in [12] for $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathbf{s} = (s, 0, \ldots, 0) \in \mathbf{Z}_+^n$, $s > 0$ and for any finitely generated integral domain A.

Since

$$\tau_{\mathbf{s}} W_A(\mu_{\mathbf{s}})_{\mu_{\mathbf{s}}} \cong_{\mathbf{A}_{\mu_{\mathbf{s}}}} (\mathbf{U}(\mathfrak{g}\otimes A)w_{\mathbf{s}})_{\mu_{\mathbf{s}}} \cong_{\mathbf{A}_{\mu_{\mathbf{s}}}} \mathbf{A}_{\mu_{\mathbf{s}}},$$

the following proposition completes the proof of Proposition 23.

Proposition 25. Let $\mu \in P^+$ and let $\boldsymbol{\pi} : W_A(\mu) \to W$ be a surjective map of $(\mathfrak{g}\otimes A, \mathbf{A}_\mu)$-bi-modules such that the restriction of $\boldsymbol{\pi}$ to $W_A(\mu)_\mu$ is an isomorphism of right $\mathbf{A}_\mu$-modules. If $A = \mathcal{R}_{0,1}$ or $\mathcal{R}_{1,0}$ and $\mu = \sum_{i\in I_0} s_i\omega_i$, then $\boldsymbol{\pi}$ is an isomorphism.

4.3.5

Assume from now on that A is either $\mathcal{R}_{0,1}$ or $\mathcal{R}_{1,0}$. The following is well-known.

Proposition 26. For all $r \in \mathbf{Z}_+$, the ring $(\mathcal{R}_{0,1}^{\otimes r})^{S_r}$ is isomorphic to $\mathcal{R}_{0,r}$ and $(\mathcal{R}_{1,0}^{\otimes r})^{S_r}$ is isomorphic to $\mathbf{C}[t_1, t_2, \dots, t_r, t_r^{-1}]$.

The proposition implies that $\operatorname{Max} \mathbf{A}_\lambda$ is an irreducible variety. Given $\lambda \in P^+$, define $\mathcal{D}_\lambda \subset \operatorname{Max} \mathbf{A}_\lambda$ by: $\mathbf{I} \in \mathcal{D}_\lambda$ if and only if the S_{r_λ}-orbit of $\operatorname{sym}_\lambda \mathbf{I}$ is of maximal size, i.e, $\operatorname{sym}_\lambda \mathbf{I}$ is the S_{r_λ}-orbit of $((t-a_{1,1}), \dots, (t-a_{1,r_1}), \dots, (t-a_{n,1}), \dots, (t-a_{n,r_n})) \in (\operatorname{Max} A)^{\times r_\lambda}$ for some $a_{i,r} \in \mathbf{C}$ (respectively $a_{i,r} \in \mathbf{C}^\times$) with $a_{i,r} \neq a_{j,s}$ if $(i,r) \neq (j,s)$. The set of such orbits is clearly Zariski open in $\operatorname{Max} \mathbb{A}_\lambda$. Since $\operatorname{sym}_\lambda$ induces an isomorphism of algebraic varieties $\operatorname{Max} \mathbb{A}_\lambda \to \operatorname{Max} \mathbf{A}_\lambda$, we conclude that $\mathcal{D}_\lambda$ is Zariski open, hence is dense in $\operatorname{Max} \mathbf{A}_\lambda$. Therefore, given any non-zero $\mathbf{a} \in \mathbf{A}_\lambda$ there exists $\mathbf{I} \in \mathcal{D}_\lambda$ with $\mathbf{a} \notin \mathbf{I}$.

4.3.6

We shall need the following theorem.

Theorem 5. Let $A = \mathcal{R}_{0,1}$ or $\mathcal{R}_{1,0}$ and let $\lambda = \sum_{i \in I} r_i \omega_i \in P^+$.

(i) The right $\mathbf{A}_\lambda$-module $W_A(\lambda)$ is free of rank d_λ, where

$$d_\lambda = \prod_{i \in I} (\dim \mathbf{W}_A^{\omega_i}(A/\mathfrak{I}))^{r_i},$$

for any $\mathfrak{I} \in \operatorname{Max} A$.

(ii) Let $\mathbf{I} \in \mathcal{D}_\lambda$. Then

$$\mathbf{W}_A^\lambda(\mathbf{A}_\lambda/\mathbf{I}) \cong \bigotimes_{i \in I} \bigotimes_{r=1}^{r_i} \mathbf{W}_A^{\omega_i}(A/\mathfrak{I}_{i,r}),$$

where $\mathfrak{I}_{i,r} \in \operatorname{Max} A$ is the ideal generated by $(t - a_{i,r})$. If, in addition, we have $r_i = 0$ for $i \notin I_0$, then $\mathbf{W}_A^\lambda(\mathbf{A}_\lambda/\mathbf{I})$ is an irreducible $\mathfrak{g} \otimes A$- module.

Part (i) of the Theorem was proved in [11] for $\mathfrak{sl}_2$, in [9] for $\mathfrak{sl}_{r+1}$ and in [15] for algebras of type A, D, E. The general case can be deduced from the quantum case, using results of [1, 19, 22]. Part (ii) of the Theorem was proved in [11] in a different language and in [7] in the language of this paper.

4.4 Proof of Proposition 25

Proof of Proposition 25. Let $\{w_s\}_{1 \le s \le d_\mu}$ be an $\mathbf{A}_\mu$-basis of $W_A(\mu)$ (cf. Theorem 5(i)). Then for all $\mathbf{I} \in \max \mathbf{A}_\mu$, $\{w_s \otimes 1\}_{1 \le s \le d_\mu}$ is a $\mathbf{C}$-basis of $\mathbf{W}_A^\mu \mathbf{A}_\mu/\mathbf{I}$. Suppose that $w \in \ker \boldsymbol{\pi}$ and write

$$w = \sum_{s=1}^{d_\mu} w_s \mathbf{a}_s, \qquad \mathbf{a}_s \in \mathbf{A}_\mu.$$

If $w \neq 0$, let $\mathbf{a}$ be the product of the non-zero elements of the set $\{\mathbf{a}_s : 1 \le s \le d_\mu\}$. Since $\mathbf{A}_\mu$ is an integral domain we see that $\mathbf{a} \neq 0$. By the discussion in Section 4.3.5 we can choose $\mathbf{I} \in \mathcal{D}_\mu$ with $\mathbf{a} \notin \mathbf{I}$. Then $\mathbf{a}_s \neq 0$ implies that $\mathbf{a}_s \notin \mathbf{I}$ and hence $\bar{w} := w \otimes 1 = \sum_{s=1}^{d_\mu} w_s \otimes \bar{\mathbf{a}}_s \neq 0$, where $\bar{\mathbf{a}}_s$ is the canonical image of $\mathbf{a}_s$ in $\mathbf{A}_\mu/\mathbf{I}$. Notice that Theorem 5(ii) implies that $\mathbf{W}_A^\mu(\mathbf{A}_\mu/\mathbf{I})$ is a simple $\mathfrak{g} \otimes A$-module.

Since $\boldsymbol{\pi}$ is surjective, W is generated by $\boldsymbol{\pi}(w_\mu)$. Setting $W' = \boldsymbol{\pi}(W_A(\mu)\mathbf{I})$, we see that

$$W'_\mu = \boldsymbol{\pi}((W_A(\mu)\mathbf{I})_\mu) = \boldsymbol{\pi}(w_\mu)\mathbf{I}.$$

In particular, this proves that $\boldsymbol{\pi}(w_\mu) \notin W'$, hence W' is a proper submodule of W and

$$(W/W')\mathbf{I} = 0.$$

This implies that $\boldsymbol{\pi}$ induces a well-defined non-zero surjective homomorphism of $\mathfrak{g} \otimes A$-modules $\bar{\boldsymbol{\pi}} : \mathbf{W}^{\mu}_{A}(\mathbf{A}_{\mu}/\mathbf{I}) \to W/W' \to 0$. In fact since $\mathbf{W}^{\mu}_{A}(\mathbf{A}_{\mu}/\mathbf{I})$ is simple, we see that $\bar{\boldsymbol{\pi}}$ is an isomorphism. But now we have

$$0 = \boldsymbol{\pi}(w) = \bar{\boldsymbol{\pi}}(\bar{w}),$$

forcing $\bar{w} = 0$ which is a contradiction caused by our assumption that $w \neq 0$.

The proof of Proposition 25 is complete. $\square$

Part II

Global Weyl modules for the twisted loop algebra

Chapter 5

For Part II of the manuscript, we alter our notation to fit the context of the twisted loop algebras. From now on there will be two simple Lie algebras under consideration (one being the set of fixed points of the other under an automorphism). As such, we need to introduce some alternate notation, as the context will not always make it clear. This chapter is devoted to some explication and reminders of preliminary results, in addition to these notational changes.

5.1 Preliminaries

5.1.1

As in Part I, Let $\mathfrak{g}$ be a finite–dimensional simple Lie algebra of rank n with Cartan matrix $(a_{ij})_{i,j\in I}$, where $I = \{1, \cdots, n\}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let R denote the corresponding set of roots. Let $\{\alpha_i\}_{i\in I}$ (resp. $\{\omega_i\}_{i\in I}$) be a set of simple roots (resp. fundamental weights) and Q (resp. Q^+), P (resp. P^+) be the integer span (resp. $\mathbf{Z}_+$–span) of the simple roots and fundamental weights respectively. Denote by $\leq$ the usual partial order on P,

$$\lambda, \mu \in P, \quad \lambda \leq \mu \iff \mu - \lambda \in Q^+.$$

Set $R^+ = R \cap Q^+$ and let θ be the unique maximal element in R^+ with respect to the partial order.

Fix a Chevalley basis, which we now denote with capital letters by $X_\alpha^\pm$, H_i, $\alpha \in R^+$, $i \in I$ of $\mathfrak{g}$ and set $X_i^\pm = X_{\alpha_i}^\pm$, $H_\alpha = [X_\alpha^+, X_\alpha^-]$ and note that $H_i = H_{\alpha_i}$. For each $\alpha \in R^+$, the subalgebra of $\mathfrak{g}$ spanned by $\{X_\alpha^\pm, H_\alpha\}$ is isomorphic to $\mathfrak{sl}_2$. Define subalgebras $\mathfrak{n}^\pm$ of $\mathfrak{g}$ by

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathbf{C} X_\alpha^\pm,$$

and note that $\mathfrak{g}$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Let σ be a permutation of I satisfying $a_{\sigma(i)\sigma(j)} = a_{ij}$ for each $i, j \in I$. Then the assignment

$$X_i^\pm \mapsto X_{\sigma(i)}^\pm, \quad H_i \mapsto H_{\sigma(i)}, \quad i \in I,$$

extends to an automorphism of $\mathfrak{g}$ called a *diagram automorphism* and also denoted by σ. Fix such an automorphism, say of order m, and take $\Gamma = \langle \sigma \rangle = \mathbf{Z}/m\mathbf{Z}$. The character group G of Γ is defined as the set of group homomorphisms $\Gamma \to \mathbf{C}^*$; fixing ζ a primitive m^{th} root of 1, we obtain a G-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{s=0}^{m-1} \mathfrak{g}_s$$

where

$$\mathfrak{g}_s = \{x \in g \ : \ \sigma(x) = \zeta^s x\}.$$

Given any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is preserved by Γ, set $\mathfrak{a}_s = \mathfrak{g}_s \cap \mathfrak{a}$. The following is well known (see for example [5] or [18, Chapter 8]). The fixed–point subalgebra $\mathfrak{g}_0$ is a simple Lie algebra and $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, and we denote by R_0 the corresponding set of roots. We fix a set of simple roots $\{\alpha_i\}_{i\in I_0}$, and let Q_0 (resp. Q_0^+), P_0 (resp. P_0^+) be the integer span (resp. $\mathbf{Z}_+$–span) of the simple roots $\{\alpha_i\}_{i\in I_0}$ and the weights $\{\omega_i\}_{i\in I_0}$ (resp. in the case where $\mathfrak{g}$ is of type A_{2n} the span of $\{\omega_i\}_{i\in I_0\setminus\{\mathrm{rk}\ \mathfrak{g}_0\}} \cup \{2\omega_{\mathrm{rk}\ \mathfrak{g}_0}\}$), respectively. This conflict in notation between the roots of $\mathfrak{g}_0$ and the roots of $\mathfrak{g}$ will not cause a problem in practice, where it will always be clear from context which we mean. The rank of $\mathfrak{g}_0$ is equal to the number of orbits of I under the induced action of Γ. We identify this set of orbits with an index set for the simple roots of $\mathfrak{g}_0$, and further identify this with a subset $I_0 = \{1, \ldots, \mathrm{rk}\ \mathfrak{g}_0\} \subset I$ by adopting the standard labeling as the first rk $\mathfrak{g}_0$ nodes of I. Moreover, $\mathfrak{g}_s$ is an irreducible representation of $\mathfrak{g}_0$ for all s, and

$$\mathfrak{n}^{\pm} \cap \mathfrak{g}_0 = \mathfrak{n}_0^{\pm} = \bigoplus_{\alpha\in R_0^+} (\mathfrak{g})_{\pm\alpha}.$$

We set $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$, so we have $\mathfrak{g} = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ is a triangular decomposition of $\mathfrak{g}_0$.

As $\mathfrak{h}_0 \subseteq \mathfrak{h}$, we have a natural map $\mathfrak{h}^* \twoheadrightarrow \mathfrak{h}_0^*$ given by restriction; furthermore our choice of Chevalley basis elements $\{h_i\}$ is such that $P \twoheadrightarrow P_0$. If $\lambda \in P$, we denote its image under this projection by $\bar{\lambda} \in P_0$. We will frequently suppress the bar if it is clear from context whether a functional lies in P or in P_0.

As diagram automorphisms, the group Γ acts upon the nodes $I = \{1, \ldots, n\}$ of the Dynkin diagram of $\mathfrak{g}$, and for a node i of this diagram we denote by Γ_i the stabilizer of i in Γ. More generally, Γ acts on R and we denote by Γ_α the stabilizer of α. For $\alpha \in R_0$, we often choose a preimage lying in R, and when this will not cause confusion, we also label it α. For $0 \le k < m$ and $\alpha \in R_0$, we define the following elements $h_\alpha(k) \in \mathfrak{h} \cap \mathfrak{g}_k$, $x_\alpha^\pm(k) \in \mathfrak{n}^\pm \cap \mathfrak{g}_k$:

$$h_\alpha(k) = \frac{1}{|\Gamma_\alpha|}\sum_{j=0}^{m-1}(\zeta^k)^j H_{\sigma^j(\alpha)}, \quad x_\alpha^\pm(k) = \frac{1}{|\Gamma_\alpha|}\sum_{j=0}^{m-1}(\zeta^k)^j X_{\sigma^j(\alpha)}^\pm.$$

In the case that $\mathfrak{g}$ is of type A_{2n} and $\alpha \in R_0$ is a short root, we also use the formula above to define additional elements $x_{2\alpha}^\pm(1)$, satisfying

$$\mathbf{C}x_{2\alpha}^\pm(1) = \mathbf{C}[X_\alpha^\pm, X_{\sigma(\alpha)}^\pm].$$

Observe that if $\Gamma_\alpha = \Gamma$, then $h_\alpha(\epsilon) = 0$ for all $\epsilon \neq 0$. We set $h_i(k) := h_{\alpha_i}(k)$, $x_i^\pm(k) := x_{\alpha_i}^\pm(k)$ and write $h_i = h_i(0)$, $x_i^\pm = x_i^\pm(0)$.

Then for all $x_\alpha^\pm(0) \in (\mathfrak{g}_0)_\alpha$, $h_\alpha(0) \in \mathfrak{h}$, the vectors $\{x_\alpha^\pm(0), h_\alpha(0)\}$ generate a Lie algebra isomorphic to $\mathfrak{sl}_2$, and $\{x_\alpha^\pm(0), h_i\}_{i \in I_0, \alpha \in R_0^+}$ is a Chevalley basis of $\mathfrak{g}_0$; see [14] for details. In the case when Γ is trivial, we recover the untwisted case in the sense that $\mathfrak{g}_0 = \mathfrak{g}$, $x_\alpha(0) = X_\alpha$, $h_\alpha(0) = H_\alpha$, $P_0^\pm = P^\pm$ and $Q_0^\pm = Q^\pm$.

5.1.2

Let $A = \mathbf{C}\left[t^{\pm 1}\right]$ and let A_+ be a fixed vector space complement to the subspace $\mathbf{C}$ of A. Given a Lie algebra $\mathfrak{a}$, define a Lie algebra structure on $\mathfrak{a} \otimes A$, by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \quad x, y \in \mathfrak{g}, \quad a, b \in A.$$

Definition 3. The Lie algebra $\mathfrak{a} \otimes A$ is called the *loop algebra* of $\mathfrak{a}$ and is denoted by $L(\mathfrak{a})$.

We will denote by $\Gamma : A \to A$ the group action of Γ on A given by extending the map $\sigma : t \mapsto \zeta t$ to an algebra homomorphism (recall that ζ is a primitive m^{th} root of unity).

Then A decomposes as

$$A = \bigoplus_{s=0}^{m-1} A_s,$$

where $A_s = \{a \in A : \sigma(a) = \zeta^s a\}$. We then have $A_0 = \mathbf{C}\left[t^{\pm m}\right]$ and $A_s = t^s A_0$. The linear extension of the map $\sigma : x \otimes t^k \mapsto \sigma(x) \otimes \sigma(t^k)$ for all $x \in \mathfrak{g}$, $k \in \mathbf{Z}$ is a Lie algebra automorphism of $\mathfrak{g} \otimes A$, and the set of fixed points

$$(\mathfrak{g} \otimes A)^\Gamma = \bigoplus_{s=0}^{m-1} \mathfrak{g}_s \otimes A_{-s}$$

is a Lie subalgebra of $\mathfrak{g} \otimes A$.

Definition 4. The Lie algebra $(\mathfrak{g} \otimes A)^\Gamma$ is called the *twisted loop algebra* of $\mathfrak{g}$ with respect to Γ; we will denote this algebra by $L^\Gamma(\mathfrak{g})$.

These loop algebras occur as a main ingredient in a realization of the affine Kac–Moody algebras and also of the extended affine Lie algebras; see for example [5, Chapter 18] or [18] for details. For any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is invariant under the action of Γ, we set $L^\Gamma(\mathfrak{a}) = (\mathfrak{a} \otimes A)^\Gamma$.

As Γ is generated by a diagram automorphism of $\mathfrak{g}$, the subalgebras $\mathfrak{n}^\pm, \mathfrak{h}$ of $\mathfrak{g}$ are each preserved by Γ and hence $L^\Gamma(\mathfrak{g})$ inherits the triangular decomposition of $\mathfrak{g}$:

$$L^\Gamma(\mathfrak{g}) = L^\Gamma(\mathfrak{n}^-) \oplus L^\Gamma(\mathfrak{h}) \oplus L^\Gamma(\mathfrak{n}^+).$$

We briefly mention a more geometric realization of these loop algebras. The ring $\mathbf{C}\left[t^{\pm 1}\right]$ is the coordinate ring of the affine variety $\mathbf{C}^*$. The Lie algebra $\mathfrak{g}$ can be viewed as an affine variety, and if we denote by $M(\mathbf{C}^*, \mathfrak{g})$ the Lie algebra of regular maps from $\mathbf{C}^*$ to $\mathfrak{g}$ (where the bracket is defined pointwise), the group action of Γ on $\mathfrak{g}$ and on A (hence on $\mathbf{C}^*$) extends to an action $\Gamma : M(\mathbf{C}^*, \mathfrak{g}) \to M(\mathbf{C}^*, \mathfrak{g})$. Then it is easy to see that

$M(\mathbf{C}^*, \mathfrak{g})^\Gamma \cong L^\Gamma(\mathfrak{g})$; we call such a realization of $L^\Gamma(\mathfrak{g})$ an *equivariant map algebra* (see [24] for more details).

We identify $\mathfrak{a}$ with the Lie subalgebra $\mathfrak{a} \otimes \mathbf{C}$ of $\mathfrak{a} \otimes A$. Similarly, if $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{a}$, then $\mathfrak{b} \otimes A$ is naturally isomorphic to a subalgebra of $\mathfrak{a} \otimes A$. Finally we denote by $\mathbf{U}(\mathfrak{g} \otimes A_+)$ the subspace of $\mathbf{U}(\mathfrak{g} \otimes A)$ spanned by monomials in the elements $x \otimes a$ where $x \in \mathfrak{g}$, $a \in A_+$.

If J_0 is any ideal in A_0, then $\bigoplus_{s=0}^{m-1} \mathfrak{g}_s \otimes t^{-s} J_0$ is clearly an ideal of $L^\Gamma(\mathfrak{g})$; conversely, the following can be deduced from [8] or [21].

Lemma 27. Let J be an ideal of $L^\Gamma(\mathfrak{g})$. Then there exists an ideal $J_0 \subseteq A_0$ such that

$$J = \bigoplus_{s \in \mathbf{Z}} \mathfrak{g}_s \otimes t^{-s} J_0.$$

5.1.3

A very important tool for understanding and analyzing modules for loop algebras has been the use of results for $L(\mathfrak{sl}_2)$. In the twisted loop setting, we will once again use results for the smallest available twisted loop algebra. Namely, let $\mathfrak{g} = \mathfrak{sl}_3$ and Γ be induced by the non-trivial Dynkin diagram automorphism of $\mathfrak{g}$. Then in our notation, the fixed–point algebra is denoted by $L^\Gamma(\mathfrak{sl}_3)$. In this case $\mathfrak{g}_0 \cong \mathfrak{sl}_2$ and $\mathfrak{g}_1 \cong V(4\omega)$, the five–dimensional irreducible $\mathfrak{sl}_2$-module.

In contrast with the loop case, the twisted loop algebras are in some sense built from copies of $L(\mathfrak{sl}_2)$ and $L^\Gamma(\mathfrak{sl}_3)$. The following lemma, proved in [14], makes this idea precise.

Lemma 28.

(i) If $\mathfrak{g}$ is of type A_{2n}, then we have canonical isomorphisms

$$L(\mathfrak{sl}_2) \cong \mathrm{sp}\left\{x_\alpha^\pm(j) \otimes t^{ms-j},\ h_\alpha(j) \otimes t^{ms-j} \mid s \in \mathbb{Z}\ ,\ 0 \le j \le m-1\right\},$$

if α is a long root, and

$$L^\Gamma(\mathfrak{sl}_3) \cong \mathrm{sp}\left\{x_\alpha^\pm(j) \otimes t^{ms-j},\ x_{2\alpha}^\pm(j+1) \otimes t^{ms-j},\ h_\alpha(j) \otimes t^{ms-j} \mid s \in \mathbb{Z},\ 0 \le j \le m-1\right\},$$

if α is a short root.

(ii) If $\mathfrak{g}$ is not of type A_{2n}, then we have canonical isomorphisms

$$L(\mathfrak{sl}_2) \cong \mathrm{sp}\left\{x_\alpha^\pm(0) \otimes t^{ms},\ h_\alpha(0) \otimes t^{ms} \mid s \in \mathbb{Z}\right\},$$

if α is a long root, and

$$L(\mathfrak{sl}_2) \cong \mathrm{sp}\left\{x_\alpha^\pm(j) \otimes t^{ms-j},\ h_\alpha(j) \otimes t^{ms-j} \mid s \in \mathbb{Z}\ ,\ 0 \le j \le m-1\right\},$$

if α is a short root.

$\square$

5.1.4

In this section, we will recall some crucial results on the classification of simple finite–dimensional modules for $L(\mathfrak{g})$. We begin with the definition of an evaluation module. Given $\lambda \in P^+$ and $a \in \mathbf{C}^*$, the the $\mathfrak{g}$-module $V(\lambda)$ described in Theorem 1 has an $L(\mathfrak{g})$–module structure given by

$$(x \otimes t^n).v = a^n x.v \quad \text{for all} \ \ x \in \mathfrak{g}\ ,\ v \in V(\lambda).$$

We denote this module by $V_a(\lambda)$. Clearly, since $V(\lambda)$ is a simple $\mathfrak{g}$-module, $V_a(\lambda)$ is a simple $L(\mathfrak{g})$-module. This result has a generalization for tensor products of simple modules. To state this result, we will first introduce some useful terminology, due to [24]:

Let Ξ be the monoid of finitely supported functions from $\mathbf{C}^*$ to P^+. Thus, for $\xi \in \Xi$,

$$\mathrm{supp}(\xi) := \{a \in \mathbf{C}^* \mid \xi(a) \neq 0\} \subset \mathbf{C}^*$$

is a finite set. We define the *weight* of $\xi \in \Xi$ by the formula $\mathrm{wt}(\xi) := \sum_{a \in \mathrm{supp}(\xi)} \xi(a) \in P^+$. Consequently we have

$$\Xi = \bigcup_{\lambda \in P^+} \Xi_\lambda,$$

where $\Xi_\lambda = \{\xi \in \Xi \mid \mathrm{wt}(\xi) = \lambda\}$. We associate to each $\xi \in \Xi$ an $L(\mathfrak{g})$-module

$$V_\xi := \bigotimes_{a \in \mathrm{supp}(\xi)} V_a(\xi(a)).$$

The following characterization of simple finite–dimensional $L(\mathfrak{g})$-modules was proved in [10].

Theorem 6. *V_ξ is a simple finite–dimensional $L(\mathfrak{g})$–module. Moreover, if V is a simple finite–dimensional $L(\mathfrak{g})$-module, then there exists $\xi \in \Xi$, such that $V \cong V_\xi$.*

5.1.5

Before recalling the results on simple finite–dimensional modules for $L^\Gamma(\mathfrak{g})$, we will introduce the necessary notion of admissible sets.

Definition 5. A finite subset $X \subset \mathbf{C}^*$ is called admissible, if for all $a \neq b \in X$ we have

$$\Gamma.a \cap \Gamma.b = \emptyset.$$

We say a finitely supported function $\xi \in \Xi$ is *admissible* if its support $\mathrm{supp}(\xi)$ is an admissible set. Furthermore, for every finite subset $X \subset \mathbf{C}^*$, we denote by X_{adm} a maximal admissible subset (clearly this set is not unique, but for our purposes the uniqueness will not be necessary).

Now, observe that any Dynkin diagram automorphism σ induces an automorphism of P^+ given by the formula $\sigma(\omega_i) = \omega_{\sigma(i)}$. As stated in Section 5.1.2, the group of automorphisms Γ acts also on $\mathbf{C}^*$ by multiplication with ζ, a primitive m^{th} root of unity. We say $\xi \in \Xi$ is *equivariant* with respect to Γ, if

$$\xi(\sigma(a)) = \sigma(\xi(a)) \text{ for all } a \in \mathbf{C}^* \text{ and } \sigma \in \Gamma.$$

We denote by Ξ^Γ the set of equivariant functions in Ξ. The following was proved in [24].

Theorem 7. Ξ^Γ parametrizes the simple finite–dimensional $L^\Gamma(\mathfrak{g})$-modules.

For the reader's convenience, we recall here the assigment of a simple module to an equivariant function. In order to do so, we introduce the symmetrizer map $\Sigma : \Xi \longrightarrow \Xi^\Gamma$, given by

$$\xi \mapsto \sum_{\sigma \in \Gamma} \sigma \circ \xi \circ \sigma^{-1}.$$

Clearly, this function is well-defined, since the right hand side is by construction equivariant. Given $\chi \in \Xi^\Gamma$, a function $\xi \in \Xi$ is called χ-*admissible* if $\Sigma(\xi) = \chi$ and $\text{supp}(\xi)$ is an admissible set. Before continuing, we observe that for each $\chi \in \Xi^\Gamma$, there exists at least one χ-admissible function, constructed as follows.

Let $\chi \in \Xi^\Gamma$ and choose a maximal admissible subset $X_{\text{adm}} \subset \text{supp}(\chi)$. For $a \in \mathbf{C}^*$, define $\xi \in \Xi$ by

$$\xi(a) := \begin{cases} \chi(a) & \text{if } a \in X_{\text{adm}} \\ 0 & a \notin X_{\text{adm}} \end{cases}.$$

Then ξ is finitely supported, and $\text{supp}(\xi)$ is admissible by construction, with $\Sigma(\chi) = \xi$.

The following was shown in ([21],[24]):

Lemma 29. Suppose $\xi \in \Xi$ is admissible. Then the $L(\mathfrak{g})$-module V_ξ is simple as an $L^\Gamma(\mathfrak{g})$-module. Moreover, every simple $L^\Gamma(\mathfrak{g})$-module is obtained in this way.

The parametrization of Theorem 7 is completed by observing that for two admissible functions $\xi_1, \xi_2 \in \Xi$ with $\Sigma(\xi_1) = \Sigma(\xi_2)$, we have

$$V_{\xi_1} \cong V_{\xi_2} \text{ as } L^\Gamma(\mathfrak{g})\text{-modules.}$$

We shall also define the *weight* of an equivariant function $\chi \in \Xi^\Gamma$. This was done before for elements from Ξ, but it is important to note that although $\Xi^\Gamma \subset \Xi$, the weight of an element in Ξ^Γ considered as an equivariant function is different from its weight considered as an element in Ξ. To define the weight of χ, let $\xi \in \Xi$ be χ-admissible and set

$$\mathrm{wt}_0(\chi) = \overline{\mathrm{wt}(\xi)} \in P_0^+.$$

We observe here that the weight is independent of the choice of ξ.

5.2 The category $\mathcal{I}^\Gamma$

In this section we will (by analogy with [7]) define the category of locally finite modules and the global Weyl modules. We keep the exposition as short as possible without sacrificing necessary detail.

5.2.1

Let $\mathcal{I}^\Gamma$ be the category whose objects are modules for $L^\Gamma(\mathfrak{g})$ which are locally finite–dimensional $\mathfrak{g}_0$–modules and whose morphisms are

$$\mathrm{Hom}_{\mathcal{I}^\Gamma}(V, V') = \mathrm{Hom}_{L^\Gamma(\mathfrak{g})}(V, V'), \quad V, V' \in \mathcal{I}^\Gamma.$$

Clearly $\mathcal{I}^\Gamma$ is an abelian category and is closed under tensor products. We shall use the following elementary result often without mention in the rest of the paper.

Lemma 30. Let $V \in \operatorname{Ob}\mathcal{I}^\Gamma$.

(i) If $V_\lambda \neq 0$ and $\operatorname{wt} V \subset \lambda - Q_0^+$, then $\lambda \in P_0^+$ and

$$L^\Gamma(\mathfrak{n}^+).V_\lambda = 0, \quad (x_i^-)^{\lambda(h_i)+1}.V_\lambda = 0, \quad i \in I_0.$$

If in addition, $V = \mathbf{U}(L^\Gamma(\mathfrak{g})).V_\lambda$ and $\dim V_\lambda = 1$, then V has a unique irreducible quotient.

(ii) If $V = \mathbf{U}(L^\Gamma(\mathfrak{g})).V_\lambda$ and $L^\Gamma(\mathfrak{n}^+).V_\lambda = 0$, then $\operatorname{wt} V \subset \lambda - Q_0^+$.

(iii) If $V \in \mathcal{I}^\Gamma$ is irreducible and finite–dimensional, then there exists $\lambda \in \operatorname{wt} V$ such that

$$\dim V_\lambda = 1, \quad \operatorname{wt} V \subset \lambda - Q_0^+.$$

$\square$

5.2.2

We recall here the definition of the global Weyl module for $L(\mathfrak{g})$ (due to [10]); it will play a crucial role in all that follows.

Definition 6. Let $\lambda \in P^+$. The global Weyl module $W(\lambda)$ is generated by a non-zero vector w_λ, subject to the defining relations:

$$L(\mathfrak{n}^+).w_\lambda = 0, \quad (H \otimes 1).w_\lambda = \lambda(H)w_\lambda, \quad (X_i^-)^{\lambda(H_i)+1}.w_\lambda = 0, \qquad i \in I, \quad H \in \mathfrak{h}.$$

The study of these modules initiated a series of papers ([10], [9], [15], [23], [1]), and we give here a brief summary of the results contained therein. $W(\lambda)$ is an integrable projective module in a certain category (see Section 5.2.3). Furthermore, $W(\lambda)$ is a free module of finite rank over the algebra

$$\mathbf{A}_\lambda \coloneqq \mathbf{U}(L(\mathfrak{h}))/\operatorname{Ann}_{\mathbf{U}(L(\mathfrak{h}))} w_\lambda,$$

which is isomorphic to a Laurent polynomial ring in finitely many variables.

5.2.3

Given an integrable left $\mathfrak{g}_0$-module V, it is a standard fact of relative homological algebra that

$$P^\Gamma(V) := \mathbf{U}(L^\Gamma(\mathfrak{g})) \otimes_{\mathfrak{g}_0} V$$

is a projective $L^\Gamma(\mathfrak{g})$-module. Moreover $P^\Gamma(V)$ lies in $\mathcal{I}^\Gamma$. Furthermore, if $\lambda \in P_0^+$, then $P^\Gamma(V(\lambda))$ is generated as an $L^\Gamma(\mathfrak{g})$-module by a non-zero element v with relations

$$\mathfrak{n}_0^+.v = 0 \quad h.v = \lambda(h)v, \quad (x_i^-)^{\lambda(h_i)+1}.v = 0, \qquad i \in I_0, \quad h \in \mathfrak{h}_0.$$

For $\nu \in P_0^+$ and $V \in \operatorname{Ob} \mathcal{I}^\Gamma$, let $V^\nu \in \operatorname{Ob} \mathcal{I}^\Gamma$ be defined by:

$$V^\nu \coloneqq V/\sum_{\mu \not\leq \nu} \mathbf{U}(L^\Gamma(\mathfrak{g}))V_\mu. \tag{5.2.1}$$

Equivalently, this is the unique maximal $L^\Gamma(\mathfrak{g})$ quotient W of V satisfying wt $W \subset \nu - Q_0^+$. A morphism $\pi : V \to V'$ of objects in $\mathcal{I}^\Gamma$ clearly induces a morphism $\pi^\nu : V^\nu \to (V')^\nu$. Let $\mathcal{I}_\nu^\Gamma$ be the full subcategory of objects $V \in \mathcal{I}^\Gamma$ such that $V = V^\nu$. If $V \in \operatorname{Ob} \mathcal{I}_\nu^\Gamma$, then its weights are bounded above by ν and, since it is integrable, it decomposes into a direct sum simple finite–dimensional $\mathfrak{g}_0$-modules. Hence,

$$V \in \operatorname{Ob} \mathcal{I}_\nu^\Gamma \implies \#\mathrm{wt}\ V < \infty. \tag{5.2.2}$$

Remark 31. If $\lambda, \nu \in P_0^+$, then $P^\Gamma(V(\lambda))^\nu \in \operatorname{Ob}\mathcal{I}_\nu^\Gamma$.

We are now able to define the main object of study for this paper:

Definition 7. The *global Weyl module* of weight $\lambda \in P_0^+$ for $L^\Gamma(\mathfrak{g})$ is

$$W^\Gamma(\lambda) := P^\Gamma(V(\lambda))^\lambda.$$

The following proposition is proved analogously to [7, Proposition 4].

Proposition 32. $W^\Gamma(\lambda)$ is generated by a nonzero element w_λ with relations

$$L^\Gamma(\mathfrak{n}^+).w_\lambda = 0, \quad h.w_\lambda = \lambda(h)w_\lambda, \quad (x_i^-)^{\lambda(h_i(0))+1}.w_\lambda = 0, \quad i \in I_0, \ \ h \in \mathfrak{h}_0. \tag{5.2.3}$$

5.2.4

For $\mu \in P^+$, $W(\mu)$ may be viewed as a module for $L^\Gamma(\mathfrak{g})$ by restriction, in which case the highest weight vector will be of weight $\bar{\mu}$. It follows that there is a natural map $W^\Gamma(\bar{\mu}) \longrightarrow W(\mu)$. The immediate questions, whether this map is injective or surjective, must be answered in the negative in general. Nevertheless, we shall find an $L(\mathfrak{g})$–module into which a global Weyl module for $L^\Gamma(\mathfrak{g})$ embeds: let $\lambda \in P_0^+$ and consider

$$W := \bigoplus_{\mu \in P^+ \,:\, \bar{\mu} = \lambda} W(\mu).$$

This is clearly a module for $L(\mathfrak{g})$ and hence, by restriction, for $L^\Gamma(\mathfrak{g})$. The main result of our paper is that $W^\Gamma(\lambda)$ appears as a submodule of W.

Theorem 8. Let $\lambda \in P_0^+$. There is an injective homomorphism of $L^\Gamma(\mathfrak{g})$-modules

$$W^\Gamma(\lambda) \hookrightarrow \bigoplus_{\bar{\mu}=\lambda} W(\mu),$$

induced by the assignment

$$w_\lambda \mapsto w := \sum_{\overline{\mu}=\lambda} w_\mu.$$

By Proposition 32, it is clear that this assignment gives a homomorphism of $L^\Gamma(\mathfrak{g})$–modules. The remainder of this manuscript is devoted to the proof that the map is, in fact, injective.

Chapter 6

In this chapter, we continue to follow the methods established in [7] by analyzing the highest weight space of the twisted global Weyl module, and using it to define a twisted Weyl functor.

6.1 The Weyl functor and its properties

For $\lambda \in P_0^+$, we denote by $\operatorname{Ann}_{\mathbf{U}(L^\Gamma(\mathfrak{h}))} w_\lambda$ the annihilator of w_λ in $\mathbf{U}(L^\Gamma(\mathfrak{h}))$. This is an ideal in $\mathbf{U}(L^\Gamma(\mathfrak{h}))$, and we define $\mathbf{A}_\lambda^\Gamma$ as the quotient of $\mathbf{U}(L^\Gamma(\mathfrak{h}))$ by this ideal:

$$\mathbf{A}_\lambda^\Gamma := \mathbf{U}(L^\Gamma(\mathfrak{h}))/ \operatorname{Ann}_{\mathbf{U}(L^\Gamma(\mathfrak{h}))} w_\lambda.$$

Clearly, $\mathbf{A}_\lambda^\Gamma$ is a commutative associative algebra and we will see in Theorem 10 that it is finitely generated.

6.1.1

We define a right module action of $\mathbf{A}_\lambda^\Gamma$ on $W^\Gamma(\lambda)$ as follows: for $a \in \mathbf{A}_\lambda^\Gamma$ and $u \in \mathbf{U}(L^\Gamma(\mathfrak{g}))$,

$$uw_\lambda.a := ua.w_\lambda.$$

The verification that this map is well–defined is straightforward; see [7] for details. For all $\mu \in P_0^+$, the subspaces $W^\Gamma(\lambda)_\mu$ are $L^\Gamma(\mathfrak{h})$–submodules for both the left and right actions and

$$\begin{aligned}\operatorname{Ann}_{\mathbf{U}(L^\Gamma(\mathfrak{h}))} w_\lambda &= \left\{u \in \mathbf{U}(L^\Gamma(\mathfrak{h})) : w_\lambda.u = 0 = u.w_\lambda\right\} \\ &= \left\{u \in \mathbf{U}(L^\Gamma(\mathfrak{h})) : W^\Gamma(\lambda).u = 0\right\}.\end{aligned}$$

Therefore $W^\Gamma(\lambda)$ is an $\left(L^\Gamma(\mathfrak{g}), \mathbf{A}_\lambda^\Gamma\right)$–bimodule and each subspace $W^\Gamma(\lambda)_\mu$ is a right $\mathbf{A}_\lambda^\Gamma$–module. Moreover, $W^\Gamma(\lambda)_\lambda$ is an $\mathbf{A}_\lambda^\Gamma$–module and

$$W^\Gamma(\lambda)_\lambda \cong_{\mathbf{A}_\lambda^\Gamma} \mathbf{A}_\lambda^\Gamma.$$

Let $\operatorname{mod} \mathbf{A}_\lambda^\Gamma$ be the category of left $\mathbf{A}_\lambda^\Gamma$–modules.

Definition 8. Let $\mathbf{W}_\lambda^\Gamma : \operatorname{mod} \mathbf{A}_\lambda^\Gamma \to \mathcal{I}_\lambda^\Gamma$ be the right exact functor given by

$$\mathbf{W}_\lambda^\Gamma M = W^\Gamma(\lambda) \otimes_{\mathbf{A}_\lambda^\Gamma} M, \qquad \mathbf{W}_\lambda^\Gamma f = 1 \otimes f,$$

where $M \in \operatorname{Ob} \operatorname{mod} \mathbf{A}_\lambda^\Gamma$ and $f \in \operatorname{Hom}_{\mathbf{A}_\lambda^\Gamma}(M, M')$ for some $M' \in \operatorname{Ob} \operatorname{mod} \mathbf{A}_\lambda^\Gamma$. We call this functor the *(twisted) Weyl functor*.

The $\mathfrak{g}_0$–action on $\mathbf{W}_\lambda^\Gamma M$ is locally finite (since $W^\Gamma(\lambda) \in \operatorname{Ob} \mathcal{I}_\lambda^\Gamma$), so that $\mathbf{W}_\lambda^\Gamma M \in \operatorname{Ob} \mathcal{I}_\lambda^\Gamma$, and

$$\mathbf{W}_\lambda^\Gamma \mathbf{A}_\lambda^\Gamma \cong_{L^\Gamma(\mathfrak{g})} W^\Gamma(\lambda), \qquad (\mathbf{W}_\lambda^\Gamma M)_\mu \cong W^\Gamma(\lambda)_\mu \otimes_{\mathbf{A}_\lambda^\Gamma} M,$$

for $\mu \in P_0$, $M \in \operatorname{Ob} \operatorname{mod} \mathbf{A}_\lambda^\Gamma$.

6.1.2

In this section we adapt results from [7] and state them without proofs, since these proofs carry over almost verbatim from the case of untwisted loop algebras. The first ingredient we shall require is the restriction functor, which will be right adjoint to the Weyl functor. For this we need the following lemma, whose proof can be found in [7, Lemma 4].

Lemma 33. For all $\lambda \in P_0^+$ and $V \in \operatorname{Ob} \mathcal{I}_\lambda^\Gamma$ we have $\operatorname{Ann}_{L^\Gamma(\mathfrak{h})}(w_\lambda).V_\lambda = 0$.

As a consequence, we see that the left action of $\mathbf{U}(L^\Gamma(\mathfrak{h}))$ on V_λ induces a left action of $\mathbf{A}_\lambda^\Gamma$ on V_λ, and we denote the resulting $\mathbf{A}_\lambda^\Gamma$–module by $\mathbf{R}_\Gamma^\lambda V$. Given $\pi \in \operatorname{Hom}_{\mathcal{I}_\lambda^\Gamma}(V, V')$ the restriction $\pi_\lambda : V_\lambda \to V'_\lambda$ is a morphism of $\mathbf{A}_\lambda^\Gamma$–modules and

$$V \mapsto \mathbf{R}_\Gamma^\lambda V, \quad \pi \mapsto \mathbf{R}_\Gamma^\lambda \pi = \pi_\lambda$$

defines a functor

$$\mathbf{R}_\Gamma^\lambda : \mathcal{I}_\lambda^\Gamma \longrightarrow \operatorname{mod} \mathbf{A}_\lambda^\Gamma$$

which is exact since restriction of π to a weight space is exact. If $M \in \operatorname{Ob} \operatorname{mod} \mathbf{A}_\lambda^\Gamma$, we have an isomorphism of left $\mathbf{A}_\lambda^\Gamma$–modules,

$$\mathbf{R}_\Gamma^\lambda \mathbf{W}_\lambda^\Gamma M = (\mathbf{W}_\lambda^\Gamma M)_\lambda = W^\Gamma(\lambda)_\lambda \otimes_{\mathbf{A}_\lambda^\Gamma} M \cong w_\lambda \mathbf{A}_\lambda^\Gamma \otimes_{\mathbf{A}_\lambda^\Gamma} M \cong M,$$

and hence an isomorphism of functors $\operatorname{Id}_{\mathbf{A}_\lambda^\Gamma} \cong \mathbf{R}_\Gamma^\lambda \mathbf{W}_\lambda^\Gamma$.

We may apply the restriction functor to an object of $\mathcal{I}_\lambda^\Gamma$, then apply the Weyl functor and obtain once again an object of $\mathcal{I}_\lambda^\Gamma$. The following proposition shows the relationship between these two $L^\Gamma(\mathfrak{g})$–modules.

Proposition 34. Let $\lambda \in P_0^+$ and $V \in \operatorname{Ob} \mathcal{I}_\lambda^\Gamma$. There exists a canonical map of $L^\Gamma(\mathfrak{g})$–modules $\eta_V : \mathbf{W}_\lambda^\Gamma \mathbf{R}_\Gamma^\lambda V \to V$ such that $\eta : \mathbf{W}_\lambda^\Gamma \mathbf{R}_\Gamma^\lambda \Rightarrow \operatorname{Id}_{\mathcal{I}_\lambda^\Gamma}$ is a natural transformation of functors and $\mathbf{R}_\Gamma^\lambda$ is a right adjoint to $\mathbf{W}_\lambda^\Gamma$.

As an immediate consequence we obtain with standard homological methods:

Corollary 35. The functor $\mathbf{W}_\lambda^\Gamma$ maps projective objects to projective objects.

We now have all the ingredients necessary to state the main result (Theorem 1) of [7] in the case of twisted loop algebras. The proof carries over almost identically, hence will be omitted.

Theorem 9. Let $\lambda \in P_0^+$ and $V \in \operatorname{Ob}\mathcal{I}_\lambda^\Gamma$. Then $V \cong \mathbf{W}_\lambda^\Gamma \mathbf{R}_\Gamma^\lambda V$ iff for all $U \in \operatorname{Ob}\mathcal{I}_\lambda^\Gamma$ with $U_\lambda = 0$, we have

$$\operatorname{Hom}_{\mathcal{I}_\lambda^\Gamma}(V, U) = 0, \quad \operatorname{Ext}^1_{\mathcal{I}_\lambda^\Gamma}(V, U) = 0. \tag{6.1.1}$$

6.1.3

Another consequence of Proposition 34 is the one-to-one correspondence between maximal ideals of $\mathbf{A}_\lambda^\Gamma$ and simple modules of $L^\Gamma(\mathfrak{g})$ of highest weight λ.

Lemma 36. For $\lambda \in P_0^+$, there exists a natural correspondence between maximal ideals of $\mathbf{A}_\lambda^\Gamma$ and Ξ_λ^Γ.

Proof. Let $\mathbf{M} \in \operatorname{Max}\mathbf{A}_\lambda^\Gamma$. Then $\mathbf{W}_\lambda^\Gamma(\mathbf{A}_\lambda^\Gamma/\mathbf{M})$ has a unique simple quotient V of highest weight λ, so by Theorem 6, there exists $\xi_\mathbf{M} \in \Xi_\lambda^\Gamma$ so that $V \cong V_{\xi_\mathbf{M}}$. On the other hand, let $\xi \in \Xi_\lambda^\Gamma$ and V_ξ be the corresponding simple $L^\Gamma(\mathfrak{g})$-module (Theorem 7). Then $\mathbf{R}_\Gamma^\lambda V_\xi$ is a simple $\mathbf{A}_\lambda^\Gamma$-module, and so there exists $\mathbf{M}_\xi \in \operatorname{Max}\mathbf{A}_\lambda^\Gamma$ such that

$$\mathbf{R}_\Gamma^\lambda V_\xi \cong \mathbf{A}_\lambda^\Gamma/\mathbf{M}_\xi.$$

Since $\mathbf{R}_\Gamma^\lambda$ is right adjoint to $\mathbf{W}_\lambda^\Gamma$, we have $\xi_{\mathbf{M}_\xi} = \xi$ and $\mathbf{M} = \mathbf{M}_{\xi_\mathbf{M}}$. $\square$

6.1.4

In this section, we will prove that the global Weyl module is finitely generated as a right $\mathbf{A}_\lambda^\Gamma$–module. This result is analogous to [7, Theorem 2], but requires a new proof when Γ is nontrivial.

To clarify the importance of this result, first recall that the global Weyl module is infinite–dimensional, and even decomposes into infinitely many simple $\mathfrak{g}_0$-modules. By applying the Weyl functor on one–dimensional $\mathbf{A}_\lambda^\Gamma$-modules we obtain the so called *local Weyl modules* (see Section 7.1). Once we show that the global Weyl module is finitely generated, we can deduce that the local Weyl modules are finite–dimensional (see [8, Proposition 4.2]).

Theorem 10. $W^\Gamma(\lambda)$ is a finitely generated right $\mathbf{A}_\lambda^\Gamma$–module.

Let u be an indeterminate and for $\alpha \in R_0^+$, we define for all $\ell \geq 1$ the following power series in u with coefficients in $\mathbf{U}(L^\Gamma(\mathfrak{h}))$:

$$\mathbf{p}_{\alpha,\ell}(u) = \exp\left(-\sum_{k=1}^{\infty} \frac{h_\alpha \otimes t^{\ell k}}{k} u^k\right).$$

Let $\mathbf{p}_{\alpha,\ell}(u) = \sum_{j=0}^{\infty} p_{\alpha,\ell}^j u^j$. Note that $p_{\alpha,\ell}^0 = 1$, and that $p_{\alpha,\ell}^j$ is contained in the subalgebra generated by $\left\{h_\alpha \otimes t^{\ell k} : 0 \leq k \leq j\right\}$. For the proof, we need the following lemma, which is an immediate consequence of [8, Lemma 2.3] via the substitution $t \mapsto t^\ell$.

Lemma 37. Let $\alpha \in R_0^+$ and $r \in \mathbf{Z}_+$. Then if

$$\ell \in \begin{cases} \mathbb{Z}_{>0}\ , & \text{if } \alpha \in (R_0)_s \text{ and } \mathfrak{g} \text{ is not of type } A_{2n} \\ m\mathbb{Z}_{>0}\ , & \text{if } \alpha \in (R_0)_l \text{ and } \mathfrak{g} \text{ is not of type } A_{2n} \\ 2\mathbb{Z}_{>0}\ , & \text{if } \alpha \in (R_0)_s \text{ and } \mathfrak{g} \text{ is of type } A_{2n} \\ \mathbb{Z}_{>0}\ , & \text{if } \alpha \in (R_0)_l \text{ and } \mathfrak{g} \text{ is of type } A_{2n} \end{cases}$$

we have

$$(x_\alpha^+ \otimes t^\ell)^r \left(x_\alpha^- \otimes 1\right)^{r+1} + (-1)^{r+1} \left(\sum_{j=0}^{r} \left(x_\alpha^- \otimes t^{\ell j}\right) p_{\alpha,\ell}^{r-j} \right) \in \mathbf{U}(L^\Gamma(\mathfrak{g}))\mathbf{U}(L^\Gamma(\mathfrak{n}^+))_+.$$

□

Using this containment, we now prove the theorem.

Proof. Since $W^\Gamma(\lambda)$ is an object of $\mathcal{I}_\lambda^\Gamma$ (see Remark 31), we know that $\#\mathrm{wt}\ W^\Gamma(\lambda) < \infty$. It follows that for any monomial $u \in \mathbf{U}(L^\Gamma(\mathfrak{g}))$ with $\mathrm{wt}_{\mathfrak{g}_0}(u) \neq 0$, and any $v \in W^\Gamma(\lambda)$, there exists $N > 0$ such that $u^N.v = 0$. In particular, $W^\Gamma(\lambda)$ is locally finite–dimensional for any vector of the form $x \otimes t^k$, $x \in \mathfrak{n}^- \cap \mathfrak{g}_\epsilon$, $k \equiv -\epsilon \mod m$.

Let $\lambda \in P_0^+$ and $\alpha \in R_0^+$. We set $\ell = 1$ if $\mathfrak{g}$ is of type A_{2n} and α is a long root, or $\mathfrak{g}$ is not of type A_{2n} and α is a short root. In any other case, we set $\ell = m$. By setting $r = \lambda(h_\alpha)$, we see from the defining relations of $W^\Gamma(\lambda)$ and the lemma above that

$$x_\alpha^- \otimes t^{\ell r}.w_\lambda = (-1)^r \left(\sum_{j=0}^{r-1} x_\alpha^- \otimes t^{\ell j} p_{\alpha,\ell}^{r-j} \right).w_\lambda, \tag{6.1.2}$$

which, after an inductive argument, implies

$$\left(x_\alpha^- \otimes t^{\ell k}\right).w_\lambda \in \mathrm{sp}\left\{ \left(x_\alpha^- \otimes t^{\ell s}\right) w_\lambda \mathbf{A}_\lambda^\Gamma : 0 \le s < \lambda(h_\alpha) \right\}.$$

Additionally we must consider the elements $x_\nu^- \otimes t$ and $x_{2\nu}^- \otimes t$, when $\mathfrak{g}$ is of type A_{2n} and $\nu \in (R_0)_s$. We proceed with the latter case, the former being very similar. Set $\beta = 2\nu$ and let $\mathfrak{a}$ be the Lie algebra generated by the elements

$$x_\beta^+ \otimes t^{2q+1}, \qquad x_\beta^- \otimes t^{2q-1}, \qquad \frac{1}{2} h_\nu \otimes t^{2q}, \qquad q \in \mathbf{Z},$$

then we have a Lie algebra ismorphism to $L(\mathfrak{sl}_2)$ (by Lemma 28), given by

$$x_\beta^+ \otimes t^{2q+1} \mapsto x_\alpha^+ \otimes t^q \ , \ x_\beta^- \otimes t^{2q-1} \mapsto x_\alpha^- \otimes t^q \ , \ \frac{1}{2} h_\nu \otimes t^{2q} \mapsto h \otimes t^q.$$

Lemma 37 gives us

$$(x_\beta^+ \otimes t^3)^r \left(x_\beta^- \otimes t^{-1} \right)^{r+1} + (-1)^{r+1} \left(\sum_{j=0}^{r} \left(x_\beta^- \otimes t^{2j-1} \right) p_\beta^{r-j} \right) \in \mathbf{U}(L^\Gamma(\mathfrak{g}))\mathbf{U}(L^\Gamma(\mathfrak{n}^+)_+),$$

where we define $p_\beta^j \in \mathbf{A}_\lambda^\Gamma$ by

$$\sum_{j=0}^{\infty} p_\beta^j u^j = \exp \left(- \sum_{k=1}^{\infty} \frac{\frac{1}{2} h_\nu \otimes t^{2k}}{k} u^k \right).$$

Again, since $W^\Gamma(\lambda)$ is integrable, we have

$$x_\beta^- \otimes t^{2r-1}.w_\lambda = (-1)^r \left(\sum_{j=0}^{r-1} \left(x_\beta^- \otimes t^{2j-1} \right) p_\beta^{r-j} \right).w_\lambda$$

for $r \gg 0$, and the second case is proven.

To complete the proof of the theorem, let $\{\beta_1, \cdots, \beta_N\}$ be an enumeration of $R_0^- \cup R_1^-$ (resp. $R_0^- \cup R_1^- \cup R_2^-$). Using the PBW theorem, we can see that elements of the form

$$\left((\mathfrak{g}_{\epsilon_1})_{\beta_{i_1}} \otimes t^{r_1} \right) \left((\mathfrak{g}_{\epsilon_2})_{\beta_{i_2}} \otimes t^{r_2} \right) \cdots \left((\mathfrak{g}_{\epsilon_\ell})_{\beta_{i_\ell}} \otimes t^{r_\ell} \right).w_\lambda$$

for $0 \le \epsilon_j < m$, $\beta_{i_j} \in \{\beta_1, \ldots, \beta_N\}$, $1 \le i_1 \le i_2 \le \cdots \le i_\ell \le N$, $\ell \in \mathbf{Z}_+$, and $r_i \equiv \epsilon_i$ mod m, generate $W^\Gamma(\lambda)$ as a right module for $\mathbf{A}_\lambda^\Gamma$. Using this spanning set and the fact that $\#\mathrm{wt}\ W^\Gamma(\lambda) < \infty$, an inductive argument on the length ℓ of a monomial from this spanning set shows that $W^\Gamma(\lambda)$ is finitely generated as an $\mathbf{A}_\lambda^\Gamma$–module. $\square$

6.2 The algebra $\mathbf{A}_\lambda^\Gamma$

In this section we will give an explicit description of the algebra $\mathbf{A}_\lambda^\Gamma$ and deduce that it is finitely generated.

6.2.1

We continue to follow the model of [7]. That is, we identify $\mathbf{A}_\lambda^\Gamma$ with a ring of symmetric polynomials. To begin, recall that a basis for $L^\Gamma(\mathfrak{h})$ is given by the set

$$\{h_i(\epsilon) \otimes t^{mk-\epsilon} \ : \ i \in I_0, \ 1 \le \epsilon \le m-1, \ k \in \mathbf{Z}\} = \{h_i(\bar{k}) \otimes t^{-k} \ : \ k \in \mathbf{Z}, \ i \in I_0\},$$

where $\bar{k}$ denotes the least nonnegative residue of k modulo m.

Set $A\,(k) = \mathbf{C}\left[t^{\pm k}\right]$ and for $\lambda = \sum_{i \in I_0} r_i \omega_i \in P_0^+$, define

$$\mathbb{A}_\lambda^\Gamma = \bigotimes_{i \in I_0} \left(A(|\Gamma_i|)^{\otimes r_i}\right)^{S_{r_i}}.$$

Now we identify a natural generating set of $\mathbb{A}_\lambda^\Gamma$.

For $N \in \mathbf{Z}_+$ and B any associative algebra over $\mathbf{C}$, we define a homomorphism of algebras $\mathrm{sym}_N : B \to B^{\otimes N}$ by the assignment

$$b \mapsto \sum_{\ell=0}^{N-1} 1^{\otimes \ell} \otimes b \otimes 1^{\otimes N-\ell-1}.$$

Now set

$$\mathrm{sym}_\lambda^i(b) = 1^{\otimes \lambda(\sum_{j<i} h_j)} \otimes \mathrm{sym}_{\lambda(h_i)}(b) \otimes 1^{\otimes \lambda(\sum_{j>i} h_j)}$$

for $i \in I_0$. Taking $B = A(|\Gamma_i|)$ for $i \in I_0$, we clearly have

$$\{\mathrm{sym}_\lambda^i(t^k) \ : \ k \in |\Gamma_i|\mathbf{Z}\} \subset \mathbb{A}_\lambda^\Gamma.$$

The following lemma makes clear the role of these elements in generating $\mathbb{A}_\lambda^\Gamma$.

Lemma 38. The set

$$\{\mathrm{sym}_\lambda^i(t^k) \ : \ i \in I_0, \ k \in |\Gamma_i|\mathbf{Z}, \ |k| \le \lambda(h_i)\}$$

generates $\mathbb{A}_\lambda^\Gamma$ as an algebra over $\mathbf{C}$.

Proof. Fix $i \in I_0$ and set $N = \lambda(h_i)$. It is well known that the algebra $A((|\Gamma_i|)^{\otimes N})^{S_N}$ is isomorphic to the polynomial algebra $\mathbf{C}[f_1, f_2 \ldots, f_N, f_N^{-1}]$, where the f_ℓ are the elementary symmetric functions in the N variables $t_1^{|\Gamma_i|}, \ldots, t_N^{|\Gamma_i|}$. On the other hand, the element $\mathrm{sym}_\lambda^i(t^k)$ corresponds to the power sum g_k of degree $k|\Gamma_i|$ in the factor $A(|\Gamma_i|)^{\otimes N}$, so that we have

$$\mathbf{C}[g_1, \ldots, g_N] \cong \mathbf{C}[t_1^{|\Gamma_i|}, \ldots, t_N^{|\Gamma_i|}]^{S_N} \cong \mathbf{C}[f_1, \ldots, f_N].$$

In order to prove the lemma, it therefore suffices to check that f_N^{-1} lies in $\mathbf{C}[g_{-1}, \ldots, g_{-N}]$. But this is clear, since

$$f_N^{-1} \in \mathbf{C}[t_1^{-|\Gamma_i|}, \ldots, t_N^{-|\Gamma_i|}]^{S_N} \cong \mathbf{C}[g_{-1}, \ldots, g_{-N}].$$

□

As a consequence of Lemma 38, we see that the assignment

$$h_i(\bar{k}) \otimes t^{-k} \mapsto \mathrm{sym}_\lambda^i(t^{-k}), \quad i \in I_0, \quad k \in \mathbf{Z}$$

extends to a surjective homomorphism of algebras $\tilde{\tau}_\lambda : \mathbf{U}(L^\Gamma(\mathfrak{h})) \twoheadrightarrow \mathbb{A}_\lambda^\Gamma$. We shall show in the rest of this section that $\tilde{\tau}_\lambda$ descends to an isomorphism $\tau_\lambda : \mathbf{A}_\lambda^\Gamma \cong \mathbb{A}_\lambda^\Gamma$.

6.2.2

The first step is to provide a natural correspondence between Ξ_λ^Γ and the maximal spectrum of $\mathbb{A}_\lambda^\Gamma$. This description of $\mathrm{Max}(\mathbb{A}_\lambda^\Gamma)$ will be used in the sequel to show that $\mathbb{A}_\lambda^\Gamma$ and $\mathbf{A}_\lambda^\Gamma$ are isomorphic as algebras. To describe the correspondence, we introduce an alternate description of the maximal ideals in $\mathbb{A}_\lambda^\Gamma$ in terms of multisets on the maximal ideals of $A(|\Gamma_i|)$.

For any set S, let $\mathcal{M}(S)$ be the set of functions $f : S \to \mathbf{Z}_+$ satisfying the condition that $f(s) = 0$ for all but finitely many $s \in S$. Such a function is called a *finite multiset* on S. $\mathcal{M}(S)$ forms a commutative monoid under the usual addition of functions. The size of $f \in \mathcal{M}(s)$ is given by the formula

$$|f| = \sum_{s \in S} f(s).$$

We also note that any element of $\mathcal{M}(S)$ can be written uniquely as a $\mathbf{Z}_+$-linear combination of characteristic functions χ_s for $s \in S$, defined by $\chi_s(b) = \delta_{s,b}$ for $b \in S$.

6.2.3

We shall use this language to describe the maximal ideals of the tensor product $\mathbb{A}^\Gamma_\lambda$. It is clearly enough to classify the maximal spectrum of rings $(A(|\Gamma_i|)^{\otimes \ell})^{S_\ell}$ for $\ell \in \mathbf{Z}_+$. Such ideals are given precisely by unordered ℓ-tuples (with possible repetition) of maximal ideals of $A(|\Gamma_i|)$, $i \in I_0$, i.e. by elements $f \in \mathcal{M}\left(\mathrm{Max}(A(|\Gamma_i|))\right)$ with $|f| = \ell$.

Since the maximal ideals of $A(|\Gamma_i|)$ are principal ideals generated by polynomials $t^{|\Gamma_i|} - a^{|\Gamma_i|}$ for $a \in \mathbf{C}^*$, we may view elements of $\mathcal{M}\left(\mathrm{Max}(A(|\Gamma_i|))\right)$ as multisets consisting of orbits of $\mathbf{C}^*$ under the action of Γ_i.

Abbreviating $\mathcal{M}\left(\mathrm{Max}(A(|\Gamma_i|))\right)$ by $\mathcal{M}_i$, the product $\hat{\mathcal{M}} = \prod_{i \in I_0} \mathcal{M}_i$ is also a commutative monoid, and for $\hat{f} \in \hat{\mathcal{M}}$ we set $\mathrm{wt}(\hat{f}) = \sum_{i \in I_0} |f_i| \omega_i \in P_0^+$. Defining

$$\hat{\mathcal{M}}_\lambda = \left\{ \hat{f} \in \hat{\mathcal{M}} : \mathrm{wt}(\hat{f}) = \lambda \right\}, \quad \lambda \in P_0^+,$$

we see that $\mathrm{Max}\left(\mathbb{A}^\Gamma_\lambda\right)$ is in bijective correspondence with $\hat{\mathcal{M}}_\lambda$.

In this language, any $\xi \in \Xi$ can be written uniquely as a linear combination of fundamental weights ω_i, $i \in I$, with coefficients from $\mathcal{M}(\mathbf{C}^*)$:

$$\Xi^\Gamma = \left\{ \sum_{i \in I} f_i \omega_i : f_i \in \mathcal{M}(\mathbf{C}^*) \right\},$$

where for $\xi = \sum_{i \in I} f_i \omega_i$ and $c \in \mathrm{Max}(A)$, we have $\xi(c) = \sum_{i \in I} f_i(c) \omega_i \in P^+$.

6.2.4

Next, we exhibit an isomorphism of monoids between $\hat{\mathcal{M}}$ and Ξ^Γ. Observe that for each $i \in I_0$ there is a surjective morphism of monoids

$$\mathcal{M}(\mathbf{C}^*) \overset{\pi_i}{\twoheadrightarrow} \mathcal{M}_i,$$

defined by extending the assignment $\chi_a \mapsto \chi_{\bar{a}}$, where $\bar{a}$ is the orbit of a under the action of Γ_i. If Γ_i is trivial, then of course π_i is simply the identity map. We describe some of its further properties in the following lemma.

Lemma 39. Let $\xi = \sum_{i \in I} f_i \omega_i$ lie in Ξ^Γ. Then

(1) For all $\gamma \in \Gamma$ and $i \in I$ we have $\pi_i(f_i) = \pi_i(f_{\gamma(i)})$.

(2) For each $a \in \mathbf{C}^*$ and $i \in I$, we have

$$\pi_i(f_i)(\bar{a}) = |\Gamma_i| f_i(a) \in |\Gamma_i| \mathbf{Z}_+. \tag{6.2.1}$$

Proof. For the first part, observe that by the equivariance of ξ we have

$$f_i(\gamma(a)) = f_{\gamma(i)}(a) \tag{6.2.2}$$

for $i \in I$, $\gamma \in \Gamma$ and $a \in \mathrm{Max}(A)$. The result now follows by the definition of π_i.

For the second assertion, there is nothing to prove unless $\Gamma_i = \Gamma$, that is $\gamma(i) = i$ for all $\gamma \in \Gamma$, in which case the result follows from Equation (6.2.2) and the definition of π_i. $\square$

Now construct a morphism of monoids $\alpha : \Xi^\Gamma \to \hat{\mathcal{M}}$: given an equivariant function $\xi \in \Xi^\Gamma$, write $\xi = \sum_{i \in I} f_i \omega_i$ and define $\alpha(\xi) \in \hat{\mathcal{M}}$ by the formula

$$\alpha(\xi) := \left(\frac{1}{|\Gamma_i|} \pi_i(f_i) \right)_{i \in I_0}. \tag{6.2.3}$$

It follows from part (ii) of Lemma 39 that α is injective, so it remains to show that it is surjective. For this, fix

$$\hat{g} = (g_i)_{i \in I_0} \in \hat{\mathcal{M}}.$$

To find a preimage of $\hat{g}$ under α, define $f_i \in \mathcal{M}(\mathbf{C}^*)$ for $i \in I_0$ by

$$f_i(a) = \begin{cases} g_i(a) & if \quad \Gamma_i = 1, \\ g_i(\bar{a}) & if \quad \Gamma_i = \Gamma, \end{cases}$$

so that f_i is clearly constant on the orbits of $\mathbf{C}^*$ under Γ_i. It follows that f_i satisfies Equation 6.2.1, and hence $\pi_i(f_i) = |\Gamma_i| g_i$. Hence, α is an isomorphism.

Finally, we show that α induces a bijection $\mathrm{Max}(\mathbb{A}^\Gamma_\lambda) \leftrightarrow \Xi^\Gamma_\lambda$. For this, it suffices to show that for $\xi \in \Xi^\Gamma$, we have

$$\mathrm{wt}_0(\xi) = \mathrm{wt}(\alpha(\xi)), \tag{6.2.4}$$

which follows from the observation that

$$\mathrm{wt}_0(\xi) = \sum_{i \in I_0} \frac{1}{|\Gamma_i|} |f_i| \, \omega_i, \quad \text{for} \quad \xi = \sum_{i \in I} f_i \omega_i.$$

6.2.5

The next step is to show that $\tilde{\tau}_\lambda$ descends to a surjective homomorphism of algebras

$$\tau_\lambda : \mathbf{A}_\lambda^\Gamma \to \mathbb{A}_\lambda^\Gamma.$$

For $\xi \in \Xi_\lambda^\Gamma$, define $\mathrm{ev}_\xi : \mathbf{U}(L^\Gamma(\mathfrak{h})) \to \mathbf{C}$ by extending the assignment

$$h_i(\bar{k}) \otimes t^{-k} \mapsto \sum_{\bar{a} \subset \mathrm{supp}(\xi)} a^{-k} \xi(a)(h_i(\bar{k})), \quad i \in I_0, \quad k \in \mathbf{Z}, \tag{6.2.5}$$

so that

$$u.v = \mathrm{ev}_\xi(u)v, \quad v \in (V_\xi)_\lambda, \quad u \in \mathbf{U}(L^\Gamma(h)).$$

Since V_ξ is a quotient of $W^\Gamma(\lambda)$ for $\xi \in \Xi_\lambda^\Gamma$, it follows immediately that

$$\mathrm{Ann}_{\mathbf{U}(L^\Gamma(\mathfrak{h}))}\, w_\lambda \subseteq \bigcap_{\xi \in \Xi_\lambda^\Gamma} \ker(\mathrm{ev}_\xi). \tag{6.2.6}$$

For $\hat{f} \in \hat{\mathcal{M}}_\lambda$, write $\mathrm{ev}_{\hat{f}} : \mathbb{A}_\lambda^\Gamma \to \mathbf{C}$ for evaluation at the maximal ideal of $\mathbb{A}_\lambda^\Gamma$ corresponding to $\hat{f}$. Applying the relevant definitions, we have

$$\mathrm{ev}_{\alpha^{-1}(\hat{f})} = \mathrm{ev}_{\hat{f}} \circ \tilde{\tau}_\lambda. \tag{6.2.7}$$

We can now complete the proof that $\mathbb{A}_\lambda^\Gamma$ is a quotient of $\mathbf{A}_\lambda^\Gamma$.

Proposition 40.

$$\mathrm{Ann}_{\mathbf{U}(L^\Gamma(\mathfrak{h}))}\, w_\lambda^\Gamma \subseteq \ker(\tilde{\tau}_\lambda).$$

Proof. It follows immediately from Equation 6.2.7 that

$$\bigcap_{\hat{f} \in \hat{\mathcal{M}}_\lambda} \ker(\mathrm{ev}_{\hat{f}} \circ \tilde{\tau}_\lambda) = \bigcap_{\xi \in \Xi_\lambda^\Gamma} \ker(\mathrm{ev}_\xi). \tag{6.2.8}$$

On the other hand, since the Jacobson radical $J(A(s)) = 0$ for all $s \in \mathbf{Z}_+$, we see that $J(\mathbb{A}_\lambda^\Gamma) = 0$. In particular,

$$\ker(\tilde{\tau}_\lambda) = \bigcap_{\hat{f} \in \hat{\mathcal{M}}_\lambda} \ker(\mathrm{ev}_{\hat{f}} \circ \tilde{\tau}_\lambda). \tag{6.2.9}$$

The proposition now follows from Equations 6.2.6, 6.2.8 and 6.2.9. □

Corollary 41. The map $\tilde{\tau}_\lambda$ descends to a surjective homomorphism of algebras

$$\tau_\lambda : \mathbf{A}_\lambda^\Gamma \to \mathbb{A}_\lambda^\Gamma.$$

6.2.6

It remains to show that τ_λ is injective. For this, we adapt the argument of [7], by identifying a spanning set of $\mathbf{A}_\lambda^\Gamma$ which is mapped to a linearly independent subset of $\mathbb{A}_\lambda^\Gamma$.

Lemma 42. The images of elements

$$\left\{ \prod_{i \in I_0} \prod_{s=1}^{m_i} (h_i(\bar{k}_{i,s}) \otimes t^{-k_{i,s}}) \ : \ 0 \le m_i \le \lambda(h_i),\ k_{i,s} \in \mathbf{Z} \right\}$$

span $\mathbf{A}_\lambda^\Gamma$.

Proof. It is clearly enough to prove that for each $i \in I_0$ and $k_1, \dots, k_\ell \in \mathbf{Z}$ we have

$$\prod_{s=1}^{\ell} (h_i(\bar{k_s}) \otimes t^{-k_s}) w_\lambda \in \mathrm{sp} \left\{ \prod_{s=1}^{r} (h_i(\bar{\ell_s}) \otimes t^{-\ell_s}) w_\lambda \ : \ r \le \lambda(h_i) \right\}. \tag{6.2.10}$$

We shall prove this statement as Corollary 44 below. Assuming it, the lemma follows. □

In order to establish Equation 6.2.10, we shall first prove the following, more general, proposition. For any element a of an associative algebra and any $n \in \mathbf{Z}$, we denote by $a^{(n)}$ the divided power $a^n/n!$, with the convention that $a^{(n)} = 0$ for $n < 0$.

Proposition 43. Let $k, \ell \in \mathbf{Z}_+$ with $k \leq \ell$. Given $\epsilon_1, \ldots, \epsilon_k \in \mathbf{Z}$ and $i \in I_0$, we have

$$\prod_{s=1}^{k}(x_i^+(\bar{\epsilon}_s) \otimes t^{-\epsilon_s})(x_i^-)^{(\ell)} = \sum_{r=0}^{2k}(x_i^-)^{(\ell-r)}P_r(\epsilon_1, \ldots, \epsilon_k),$$

where $P_r(\epsilon_1, \ldots, \epsilon_k)$ is an element of $U(L^\Gamma(\mathfrak{g}))$ in the standard PBW order, having length at most k and consisting of homogeneous elements of weight $(k-r)\alpha_i$ as an $\mathfrak{sl}_2(i)$-module. Moreover, $P_k(\epsilon_1, \ldots, \epsilon_k)$ has, except for terms ending in $L^\Gamma(\mathfrak{n}^+)$, a unique term of length k, which is $\prod_{s=1}^{k}(h_i(\bar{\epsilon}_s) \otimes t^{-\epsilon_s})$.

Proof. The proof proceeds by induction on k. For the base case, a simple induction on ℓ shows that

$$\begin{aligned}(x_i^+(\bar{\epsilon}) \otimes t^{-\epsilon})(x_i^-)^{(\ell)} &= (x_i^-)^{(\ell)}(x_i^+(\bar{\epsilon}) \otimes t^{-\epsilon}) \\ &\quad + (x_i^-)^{(\ell-1)}(h_i(\bar{\epsilon}) \otimes t^{-\epsilon}) + (x_i^-)^{(\ell-2)}(-x_i^-(\bar{\epsilon}) \otimes t^{-\epsilon}).\end{aligned} \tag{6.2.11}$$

Now assuming the result for $k < \ell$, we prove it for $k+1$. By the induction hypothesis and repeated use of Equation 6.2.11, we have

$$\begin{aligned}\prod_{s=1}^{k+1}(x_i^+(\bar{\epsilon}_s) \otimes t^{-\epsilon_s})(x_i^-)^{(\ell)} &= \sum_{r=0}^{2k}(x_i^-)^{(\ell-r)}(x_i^+(\bar{\epsilon}_{k+1}) \otimes t^{-\epsilon_{k+1}})P_r(\epsilon_1, \ldots, \epsilon_k) \\ &\quad + \sum_{r=0}^{2k}(x_i^-)^{(\ell-r-1)}(h_i(\bar{\epsilon}_{k+1}) \otimes t^{-\epsilon_{k+1}})P_r(\epsilon_1, \ldots, \epsilon_k). \\ &\quad + \sum_{r=0}^{2k}(x_i^-)^{(\ell-r-2)}(x_i^-(\bar{\epsilon}_{k+1}) \otimes t^{-\epsilon_{k+1}})P_r(\epsilon_1, \ldots, \epsilon_k).\end{aligned}$$

Reindexing, this is

$$\sum_{r=0}^{2(k+1)}(x_i^-)^{(\ell-r)}P_r(\epsilon_1, \ldots, \epsilon_{k+1}),$$

where

$$\begin{aligned}P_r(\epsilon_1, \ldots, \epsilon_{k+1}) &= (x_i^+(\bar{\epsilon}_{k+1}) \otimes t^{-\epsilon_{k+1}})P_r(\epsilon_1, \ldots, \epsilon_k) \\ &\quad + (h_i(\bar{\epsilon}_{k+1}) \otimes t^{-\epsilon_{k+1}})P_{r-1}(\epsilon_1, \ldots, \epsilon_k) + (x_i^-(\bar{\epsilon}_{k+1}) \otimes t^{-\epsilon_{k+1}})P_{r-2}(\epsilon_1, \ldots, \epsilon_k).\end{aligned}$$

(Here $P_j(\epsilon_1, \dots, \epsilon_k) = 0$ if $j < 0$ or $j > 2k$.) This element, once it is commuted into PBW order, clearly has the correct weight and maximum length.

It only remains to analyze $P_{k+1}(\epsilon_1, \dots, \epsilon_{k+1})$. Any monomial from the third term of this element ends in a summand of $P_{k-1}(\epsilon_1, \dots, \epsilon_k)$, which has weight 2 and hence, being already in PBW order, must end in some term from $L^\Gamma(\mathfrak{n}^+)$. By the induction hypothesis, the second term contains the desired product

$$\prod_{s=1}^{k+1} (h_i(\bar{\epsilon}_s) \otimes t^{-\epsilon_s})$$

as its unique term of length $k+1$ (modulo $L^\Gamma(\mathfrak{n}^+)$).

To deal with the term

$$(x_i^+(\bar{\epsilon}_{k+1}) \otimes t^{-\epsilon_{k+1}}) P_r(\epsilon_1, \dots, \epsilon_k),$$

we observe that by weight considerations, any monomial not ending in $L^\Gamma(\mathfrak{n}^+)$ must be of the form

$$(x_i^-(\bar{\delta}) \otimes t^{-\delta}) \prod_{p=1}^{q} (h_i(\bar{\delta}_s) \otimes t^{-\delta_s}), \quad q \le k-1.$$

Now applying the element $(x_i^+(\bar{\epsilon}_{k+1}) \otimes t^{-\epsilon_{k+1}})$ and commuting to PBW order yields terms that end in $L^\Gamma(\mathfrak{n}^+)$, together with a term of length $q+1 \le k$. $\square$

Corollary 44. Fix $i \in I_0$ and $k_1, \dots, k_\ell \in \mathbf{Z}$. Then

$$\prod_{s=1}^{\ell} (h_i(\bar{k_s}) \otimes t^{-k_s}) w_\lambda \in \mathrm{sp} \left\{ \prod_{s=1}^{r} (h_i(\bar{\ell_s}) \otimes t^{-\ell_s}) w_\lambda \ : \ r \le \lambda(h_i) \right\}.$$

Proof. By setting $k = \ell$ in Proposition 43, we see that

$$0 = \prod_{s=1}^{\ell} (x_i^+(\bar{k_s}) \otimes t^{-k_s})(x_i^- \otimes 1)^\ell w_\lambda = \prod_{s=1}^{\ell} (h_i(\bar{k_s}) \otimes t^{-k_s}) w_\lambda + H.w_\lambda, \qquad \ell \ge \lambda(h_i) + 1,$$

where H lies in the span of elements of the form $\prod_{s=1}^{r} (h_i(\bar{\ell_s}) \otimes t^{-\ell_s})$ with $r < \ell$. The statement of the corollary now follows by induction on ℓ. $\square$

6.2.7

It still remains to show that the images under τ_λ of the elements from Lemma 42 form a linearly independent subset of $\mathbb{A}_\lambda^\Gamma$. Now, these are

$$\left\{ \bigotimes_{i\in I_0} \prod_{s=1}^{m_i} \mathrm{sym}_{\lambda(h_i)}(t^{-k_{i,s}}) \;:\; 0 \le m_i \le \lambda(h_i),\; k_{i,s} \in \mathbf{Z} \right\}.$$

Since the tensor product preserves linear independence, it therefore suffices to check that for each $i \in I_0$ the set of products

$$\left\{ \prod_{s=1}^{m_i} \mathrm{sym}_{\lambda(h_i)}(t^{-k_{i,s}}) \;:\; 0 \le m_i \le \lambda(h_i),\; k_{i,s} \in \mathbf{Z} \right\}$$

is linearly independent. A slightly more general statement can be found in [7]; we reproduce it and include a proof here for convenience. Recall from Section 6.2.1 that for any associative algebra B we have a map $\mathrm{sym}_N : B \to B^{\otimes N}$ mapping

$$b \mapsto \sum_{\ell=0}^{N-1} 1^{\otimes \ell} \otimes b \otimes 1^{\otimes N-\ell-1}.$$

Lemma 45. Let $b_0, b_1, \ldots \in B$ form a countable ordered basis, with $b_0 = 1$ and $b_r \in B_+$ for $r > 1$. Then the elements

$$\left\{ \prod_{s=1}^{\ell} \mathrm{sym}_N(b_{r_s}) \;:\; r_s \in \mathbf{Z}_+, \ell \le N \right\}$$

are linearly independent in $B^{\otimes N}$.

Proof. The projections onto the summand $B_+^{\otimes \ell} \otimes 1^{\otimes N-\ell}$ of the elements listed are

$$\sum_{\sigma \in S_{r_\ell}} \sigma.(b_{r_1} \otimes b_{r_2} \otimes \cdots \otimes b_{r_\ell}) \otimes 1^{\otimes N-\ell},$$

where S_ℓ acts in the obvious way on $B^{\otimes \ell}$. Since these are clearly linearly independent by the choice of b_r as basis elements, the proof is complete. $\square$

6.2.8

In this subsection, we compare the algebras $\mathbf{A}_\lambda$ and $\mathbf{A}_{\overline{\lambda}}^{\Gamma}$. For this purpose, let us examine again the symmetrizer map defined in Section 5.1.5:

$$\Sigma : \Xi \longrightarrow \Xi^{\Gamma} \quad , \quad \xi \mapsto \sum_{\sigma \in \Gamma} \sigma \circ \xi \circ \sigma^{-1}.$$

If we restrict this map to the functions of weight $\lambda \in P^{+}$, we obtain a map

$$\Sigma : \Xi_\lambda \longrightarrow \Xi_{\overline{\lambda}}^{\Gamma}.$$

Let V_ξ be the simple $L(\mathfrak{g})$-module associated to $\xi \in \Xi_\lambda$. It follows from the discussion in Section 5.1.5 that V_ξ is a simple $L^{\Gamma}(\mathfrak{g})$-module if and only if $\mathrm{supp}(\xi)$ is admissible. On the other hand, V_ξ has, viewed as a $L^{\Gamma}(\mathfrak{g})$-module, a unique simple quotient of highest weight $\overline{\lambda}$; in fact this quotient is isomorphic to $V_{\Sigma(\xi)}$, as we shall show in Proposition 46.

Recall the evaluation map defined for any $\xi \in \Xi$ in [7] or, for $\eta \in \Xi_{\overline{\lambda}}^{\Gamma}$, in Section 6.2.5:

$$\mathrm{ev}_\xi : \mathbf{U}(L(\mathfrak{h})) \longrightarrow \mathbf{C} \qquad \text{and} \qquad \mathrm{ev}_\eta : \mathbf{U}(L^{\Gamma}(\mathfrak{h})) \longrightarrow \mathbf{C}.$$

The following theorem gives a natural embedding between the algebras $\mathbf{A}_\lambda$ and $\mathbf{A}_{\overline{\lambda}}^{\Gamma}$, and also gives a necessary and sufficient condition on λ for this embedding to be surjective.

Theorem 11. Let $\lambda = \sum m_i \omega_i \in P^{+}$, then there exists a natural injective map

$$\iota : \mathbf{A}_{\overline{\lambda}}^{\Gamma} \hookrightarrow \mathbf{A}_\lambda.$$

Furthermore, ι is surjective iff for each $i \in I$, λ satisfies the following:

1. If $\Gamma.i = \{i\}$, then $m_i = 0$

2. If $m_i \neq 0$, then $m_{\sigma(i)} = 0$ for all $\sigma \in \Gamma \setminus \{1\}$.

Proof. We have seen that $\mathbf{A}_\lambda^\Gamma \cong \mathbb{A}_\lambda^\Gamma$ and from [7], we have

$$\mathbf{A}_\lambda \cong \mathbb{A}_\lambda = \bigotimes_{i \in I} (A(1)^{\otimes m_i})^{S_{m_i}}.$$

All of these isomorphisms are, by construction, compatible with the embedding of $\mathbf{U}(L^\Gamma(\mathfrak{h}))$ into $\mathbf{U}(L(\mathfrak{h}))$. So it remains to show that

$$\mathbb{A}_\lambda^\Gamma \hookrightarrow \mathbb{A}_\lambda.$$

It is sufficient to show this for each $i \in I_0$. Recall that we have identified I_0 with a subset of I. We proceed with two exhaustive cases:

First assume that $i \in I$ such that $\Gamma.i = \{i\}$, so that $\Gamma_i = \Gamma$. Then $A(|\Gamma_i|) \subsetneq A(1)$, so we have

$$((A(|\Gamma_i|))^{\otimes m_i})^{S_{m_i}} \subsetneq (A(1)^{\otimes m_i})^{S_{m_i}},$$

if $m_i > 0$.

In the other case, $\Gamma_i = \{1\}$, and we set $n_i = \sum_{\sigma \in \Gamma} m_{\sigma(i)}$. Then we have

$$(A(1)^{\otimes n_i})^{S_{n_i}} \subseteq \bigotimes_{\sigma \in \Gamma} (A(1)^{\otimes m_{\sigma(i)}})^{S_{m_{\sigma(i)}}},$$

with equality if and only if the right hand side consists of only one non-trivial tensor factor– i.e., $m_{\sigma(i)} = 0$ for $\sigma \neq 1$, which proves the theorem. ☐

6.2.9

Let $\lambda \in P^+$ amd $\xi \in \Xi_\lambda$. We have seen in Lemma 36 that we can associate to ξ a maximal ideal $\mathbf{M}_\xi \in \operatorname{Max} \mathbf{A}_\lambda$. The simple module $\mathbf{A}_\lambda / \mathbf{M}_\xi$ will be denoted by $\mathbf{C}_\xi$. Similarly, for $\chi \in \Xi_{\lambda}^{\Gamma}$ we denote the simple $\mathbf{A}_{\lambda}^{\Gamma}$-module by $\mathbf{C}_{\chi}$.

Using the embedding $\mathbf{A}_{\lambda}^{\Gamma} \hookrightarrow \mathbf{A}_{\lambda}$ of Theorem 11, we see that for every $\xi \in \Xi_{\lambda}$, $\mathbf{C}_{\xi}$ is a simple $\mathbf{A}_{\lambda}^{\Gamma}$-module. Then we have by construction of the symmetrizer the following:

Proposition 46. Let $\xi \in \Xi_{\lambda}$. Then $\mathbf{C}_{\xi} \cong \mathbf{C}_{\Sigma(\xi)}$ as $\mathbf{A}_{\lambda}^{\Gamma}$-modules.

Proof. In [21, Equation (5.18)], the symmetrizer map was given for multiloop algebras. It was shown that for admissible $\xi \in \Xi_{\lambda}$, $\mathbf{C}_{\xi} \cong \mathbf{C}_{\Sigma(\xi)}$ as $\mathbf{A}_{\lambda}^{\Gamma}$-modules. If ξ is not admissible, then V_{ξ} is not simple as a $L^{\Gamma}(\mathfrak{g})$-module, but has a unique simple quotient. Denote this simple quotient by $V_{\xi'}$; then $\xi' \in \Xi_{\lambda}$ is admissible, $\mathbf{C}_{\xi} \cong \mathbf{C}_{\xi'}$ as $\mathbf{A}_{\lambda}^{\Gamma}$-modules and $\Sigma(\xi') = \Sigma(\xi)$. $\square$

Chapter 7

In this final chapter, we complete the proof of Theorem 8. To do this, we first study the local Weyl modules by applying the twisted Weyl functor to the simple $\mathbf{A}_\lambda^\Gamma$–modules.

7.1 Local Weyl modules

We have seen in Lemma 36 that simple $\mathbf{A}_\lambda^\Gamma$–modules are parametrized by Ξ_λ^Γ.

Definition 9. The *(twisted) local Weyl module* associated to $\chi \in \Xi_\lambda^\Gamma$ is the $L^\Gamma(\mathfrak{g})$-module

$$\mathbf{W}_\lambda^\Gamma \mathbf{C}_\chi \coloneqq W^\Gamma(\lambda) \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\chi$$

One compelling reason to study the local Weyl modules is the fact that they admit the following universal property:

Proposition 47. Let $V \in \operatorname{Ob} \mathcal{I}_\lambda^\Gamma$ such that V is generated by a highest weight vector v of weight λ, and suppose $\dim V_\lambda = 1$. Then there exists $\chi \in \Xi_\lambda^\Gamma$ such that the assigment $w_\lambda \otimes 1 \mapsto v$ extends to a surjective map

$$\mathbf{W}_\lambda^\Gamma \mathbf{C}_\chi \twoheadrightarrow V.$$

Proof. By Lemma 33 and since V is generated by v, the assigment $w_\lambda \mapsto v$ extends to a surjective map

$$W^\Gamma(\lambda) \twoheadrightarrow V.$$

Furthermore, V_λ is an $\mathbf{A}_\lambda^\Gamma$-module and since $\dim V_\lambda = 1$, this module is simple. Hence by the discussion in Section 6.2.9, there exists $\chi \in \Xi_\lambda^\Gamma$, such that $V_\lambda \cong \mathbf{C}_\chi$ as $\mathbf{A}_\lambda^\Gamma$-modules. We can deduce that the map induced by $w_\lambda \mapsto v$ factors through the kernel of ev_χ and we have: $w_\lambda \otimes 1 \mapsto v$ extends to a surjective map

$$\mathbf{W}_\lambda^\Gamma \mathbf{C}_\chi \twoheadrightarrow V,$$

and the proposition is proven. $\square$

Local Weyl modules for twisted loop algebras have been defined before in [8], as well as in [13] with two different approaches. We will compare these definitions and show their equivalences; we begin by defining them for $L(\mathfrak{g})$.

7.1.1

Let $\lambda \in P^+$ and $\xi \in \Xi_\lambda$. The local Weyl module associated to ξ, as defined in [7], is

$$W(\xi) \coloneqq W(\lambda) \otimes_{\mathbf{A}_\lambda} \mathbf{C}_\xi.$$

Local Weyl modules had been defined previously in [10], but we will use the notation from [7]. It was shown in the aforementioned series of papers ([10], [9], [15], [23], [1]) that

$$\dim W(\xi) = \prod_{i \in I} (\dim W(\xi_i))^{m_i},$$

where $\lambda = \sum m_i\omega_i$ and ξ_i is any element of Ξ_{ω_i}. This implies that the dimension of $W(\xi)$ is independent of ξ, but depends only on λ. Furthermore, it has been shown (for instance in [7]), that $W(\xi)$ has V_ξ as its unique simple quotient.

7.1.2

In [13], local Weyl modules for $L^\Gamma(\mathfrak{g})$ were defined to be the restriction of the untwisted local Weyl module for $L(\mathfrak{g})$; they are parametrized by equivariant finitely supported functions. We should mention, that in [13] local Weyl modules were defined in a much more general context. Namely, a finite group Γ acting freely on an affine scheme X and $\mathfrak{g}$ by automorphisms, which clearly includes the case of twisted loop algebras.

Specifically, let $\chi \in \Xi^\Gamma$ and let ξ be a χ-admissible function as in Section 5.1.5. Then one defines by restriction the $L^\Gamma(\mathfrak{g})$-module

$$W^\Gamma(\chi) := W(\xi).$$

Now, since ξ is admissible, it follows that $W(\xi)$ is a cyclic $\mathbf{U}(L^\Gamma(\mathfrak{g}))$-module ([13, Theorem 4.5]), and it was established in [13, Proposition 3.5] that the definition of $W^\Gamma(\chi)$ is independent of the choice of such ξ. Moreover, the modules $W^\Gamma(\chi)$ satisfy a universal property ([13, Theorem 4.5]) similar to the universal property of local Weyl modules for loop algebras ([10]) and generalized current algebras ([7, Theorem 1]).

7.1.3

In [8], local Weyl modules for the twisted loop algebra were defined by a generator w and certain relations. They were parametrized by a set of n-tuples (where $n = |I_0|$) of polynomials $\pi = (\pi_i)_{i\in I_0}$ with constant term 1, and we will denote these modules by $W^\Gamma(\pi)$. Their universal property was proven in [8, Theorem 2]; we cite an abbreviated version here.

Theorem 12. Let $\lambda \in P_0^+$ and suppose that V is a finite-dimensional $L^\Gamma(\mathfrak{g})$-module generated by a one-dimensional highest weight space of weight λ. Choose a vector $v_\lambda \in V_\lambda$. Then there exists an n-tuple $(\pi_i)_{i\in I_0}$ of polynomials such that the assigment $w \mapsto v_\lambda$ extends to a surjective map of $L^\Gamma(\mathfrak{g})$-modules

$$W^\Gamma(\pi) \twoheadrightarrow V.$$

The following is immediate from the universal properties established above.

Corollary 48. For each $\chi \in \Xi_\lambda^\Gamma$, there exists a n-tuple of polynomials (π) such that

$$\mathbf{W}_\lambda^\Gamma \mathbf{C}_\chi \cong W^\Gamma(\chi) \cong W^\Gamma(\pi),$$

and vice versa.

7.1.4

In [8], the dimension and character of local Weyl modules have been computed. We recall this result ([8, Theorem 2]) here since it will be useful in the proof of Theorem 8.

Theorem 13. Let $\lambda \in P^+$, and $\chi \in \Xi_{\overline{\lambda}}^\Gamma$, then

$$\dim W^\Gamma(\overline{\lambda}) \otimes_{\mathbf{A}_{\overline{\lambda}}^\Gamma} \mathbf{C}_\chi = \operatorname{rank}_{\mathbf{A}_\lambda} W(\lambda) = \prod (\operatorname{rank}_{\mathbf{A}_{\omega_i}} W(\omega_i))^{m_i}.$$

In particular, the dimension is independent of χ and depends only on $\overline{\lambda}$. Moreover, the $\mathfrak{g}_0$ character is also independent of χ.

7.1.5

Using the fact that $\mathbf{A}_\lambda^\Gamma$ is a Laurent polynomial ring (Section 6.2) and the fact that

$$\dim W^\Gamma(\lambda) \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\chi$$

is independent of χ, we can conclude a result which was previously known for untwisted loop and current algebras:

Theorem 14. For $\lambda \in P_0^+$, $W^\Gamma(\lambda)$ is a free right $\mathbf{A}_\lambda^\Gamma$-module with

$$\operatorname{rank}_{\mathbf{A}_\lambda^\Gamma} W^\Gamma(\lambda) = \dim \mathbf{W}_\lambda^\Gamma \mathbf{C}_\chi$$

for some, and hence for any, $\chi \in \Xi_\lambda^\Gamma$.

7.2 Proof of Theorem 8

It remains to prove the main theorem.

Theorem 15. For $\lambda \in P_0^+$, we have

$$W^\Gamma(\lambda) \hookrightarrow \bigoplus_{\overline{\mu}=\lambda} W(\mu),$$

where the map is induced by

$$w_\lambda \mapsto w := \sum_{\overline{\mu}=\lambda} w_\mu.$$

By construction, we have a surjective map

$$W^\Gamma(\lambda) \twoheadrightarrow \mathbf{U}(L^\Gamma(\mathfrak{g})).w$$

The idea of the proof is to show that both sides are free $\mathbf{A}_\lambda^\Gamma$-modules of the same rank. Together with the surjectivity of the above map, this will complete the proof.

7.2.1

We have seen in Theorem 11 that $\mathbf{A}_\lambda^\Gamma \subset \mathbf{A}_\mu$ is a subalgebra, for any μ satisfying $\overline{\mu} = \lambda$. It follows that $W(\mu)$ is a right module for $\mathbf{A}_\lambda^\Gamma$ and hence $\bigoplus_{\overline{\mu}=\lambda} W(\mu)$ is a right module for $\mathbf{A}_\lambda^\Gamma$. Finally, the submodule $\mathbf{U}(L^\Gamma(\mathfrak{g})).w \subset \bigoplus_{\overline{\mu}=\lambda} W(\mu)$ is a right module for $\mathbf{A}_\lambda^\Gamma$, being a quotient of $W^\Gamma(\lambda)$.

We want to show that $\mathbf{U}(L^\Gamma(\mathfrak{g})).w$ is a free $\mathbf{A}_\lambda^\Gamma$-module of the same rank as $W^\Gamma(\lambda)$. Now because $\mathbf{A}_\lambda^\Gamma$ is a polynomial algebra, in order to prove the freeness of $\mathbf{U}(L^\Gamma(\mathfrak{g})).w$ it suffices to show that the dimension of

$$\mathbf{U}(L^\Gamma(\mathfrak{g})).w \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\chi$$

is independent of the maximal ideal $\chi \in \Xi_\lambda^\Gamma$.

In order to prove this, we will need the following lemma:

Lemma 49. For each $\chi \in \Xi_\lambda^\Gamma$, there exists $\tau \in P^+$ and $\xi \in \Xi_\tau$ such that ξ is χ-admissible and

$$\mathbf{U}(L^\Gamma(\mathfrak{g})).w \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\chi \cong_{L^\Gamma(\mathfrak{g})} W(\tau) \otimes_{\mathbf{A}_\tau} \mathbf{C}_\xi.$$

Assuming the lemma, we prove Theorem 8 as follows. Observe that the dimension of the right hand side in Lemma 49 is independent of ξ and depends only on τ: it is equal to the rank of $W(\tau)$ as a $\mathbf{A}_\tau$-module. By Theorem 13 we know that for $\tau = \sum m_i\omega_i$,

$$\operatorname{rank}_{\mathbf{A}_\tau} W(\tau) = \prod_{i\in I} (\operatorname{rank}_{\mathbf{A}_{\omega_i}} W(\omega_i))^{m_i}.$$

On the other hand

$$\operatorname{rank}_{\mathbf{A}_{\omega_i}} W(\omega_i) = \operatorname{rank}_{\mathbf{A}_{\omega_{\sigma(i)}}} W(\omega_{\sigma(i)}).$$

To see this, one may recall, that σ is an automorphism of $L(\mathfrak{g})$, and $W(\omega_{\sigma(i)})$ is isomorphic to the pullback of the module $W(\omega_i)$ by the automorphism σ^{-1}.

Using this and the rank formula for the global Weyl module, we obtain that for all $\tau_1, \tau_2 \in P^+$ with $\overline{\tau_1} = \overline{\tau_2}$,

$$\operatorname{rank}_{\mathbf{A}_{\tau_1}} W(\tau_1) = \operatorname{rank}_{\mathbf{A}_{\tau_2}} W(\tau_2).$$

It follows that the dimension of $\mathbf{U}(L^\Gamma(\mathfrak{g})).w \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\chi$ is independent of χ, and hence $\mathbf{U}(L^\Gamma(\mathfrak{g})).w \subset \bigoplus_{\overline{\mu}=\lambda} W(\mu)$ is a projective $\mathbf{A}_\lambda^\Gamma$-module. Since $\mathbf{A}_\lambda^\Gamma$ is a polynomial ring, it now follows from the famous result of Quillen that $\mathbf{U}(L^\Gamma(\mathfrak{g})).w$ is a *free* $\mathbf{A}_\lambda^\Gamma$-module.

Together with Theorem 14, this gives for $\overline{\tau} = \lambda$,

$$\operatorname{rank}_{\mathbf{A}_\tau} W(\tau) = \operatorname{rank}_{\mathbf{A}_\lambda^\Gamma} W^\Gamma(\lambda).$$

We therefore conclude that the rank of $\mathbf{U}(L^\Gamma(\mathfrak{g})).w \subset \bigoplus_{\overline{\mu}=\lambda} W(\mu)$ as a $\mathbf{A}_\lambda^\Gamma$-module is equal to the rank of $W^\Gamma(\lambda)$ as a $\mathbf{A}_\lambda^\Gamma$-module. Since we already have a surjective map

$$W^\Gamma(\lambda) \twoheadrightarrow \mathbf{U}(L^\Gamma(\mathfrak{g})).w$$

and both modules are free $\mathbf{A}_\lambda^\Gamma$-modules, the map is an isomorphism and the theorem is proven.

7.2.2

It remains to prove Lemma 49:

Proof. We start by defining projection maps π_τ, for $\overline{\tau} = \lambda$, onto the τ-th component of $\bigoplus_{\overline{\mu}=\lambda} W(\mu)$.

$$\pi_\tau : \bigoplus_{\overline{\mu}=\lambda} W(\mu) \twoheadrightarrow W(\tau),$$

and by restriction we obtain maps

$$\pi_\tau : \mathbf{U}(L^\Gamma(\mathfrak{g})).w \longrightarrow W(\tau),$$

where $w = \sum_{\overline{\mu}=\lambda} w_\mu$. By construction, we have

$$\pi_\tau(\mathbf{U}(L^\Gamma(\mathfrak{g})).w) = \mathbf{U}(L^\Gamma(\mathfrak{g})).w_\tau \subset W(\tau),$$

the $L^\Gamma(\mathfrak{g})$-submodule of $W(\tau)$ generated through the highest weight vector w_τ.

For $\chi \in \Xi_\lambda^\Gamma$, let $\xi \in \Xi$ be a χ-admissible function (whose existence is assured by the discussion in Section 5.1.5) and let $\tau = \mathrm{wt}(\xi)$. Consider the local $L(\mathfrak{g})$-Weyl module

$$W(\tau) \otimes_{\mathbf{A}_\tau} \mathbf{C}_\xi.$$

We see, since the support of ξ is admissible, that this is a cyclic $L^\Gamma(\mathfrak{g})$-module, generated by $w_\tau \otimes \mathbf{C}_\xi$. In fact, $W(\tau) \otimes_{\mathbf{A}_\tau} \mathbf{C}_\xi$ is by restriction a local Weyl module for $L^\Gamma(\mathfrak{g})$, but by construction $\mathbf{C}_\chi \cong \mathbf{C}_\xi$ as $\mathbf{A}_\lambda^\Gamma$-modules. Therefore, we have

$$W(\tau) \otimes_{\mathbf{A}_\tau} \mathbf{C}_\xi \cong_{L^\Gamma(\mathfrak{g})} W^\Gamma(\lambda) \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\chi. \tag{7.2.1}$$

Since $\mathbf{C}_\chi \cong \mathbf{C}_\xi$, we have the trivial isomorphism

$$\mathbf{U}(L^\Gamma(\mathfrak{g})).w \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\chi \cong_{L^\Gamma(\mathfrak{g})} \mathbf{U}(L^\Gamma(\mathfrak{g})).w \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\xi.$$

We use the projection map π_τ to obtain

$$\mathbf{U}(L^\Gamma(\mathfrak{g})).w \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\xi. \twoheadrightarrow \big(\mathbf{U}(L^\Gamma(\mathfrak{g})).w_\tau\big) \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\xi.$$

Combining this projection map with the fact that $\mathbf{A}_\lambda^\Gamma \subset \mathbf{A}_\tau$ is a subalgebra, we have as $L^\Gamma(\mathfrak{g})$-modules

$$\mathbf{U}(L^\Gamma(\mathfrak{g})).w \otimes_{\mathbf{A}_\lambda^\Gamma} \mathbf{C}_\chi \twoheadrightarrow \big(\mathbf{U}(L^\Gamma(\mathfrak{g})).w_\tau\big) \otimes_{\mathbf{A}_\tau} \mathbf{C}_\xi. \tag{7.2.2}$$

With the considerations above (we use that the support of ξ is admissible), we obtain, that

$$W(\tau) \otimes_{\mathbf{A}_\tau} \mathbf{C}_\xi = (\mathbf{U}(L(\mathfrak{g})).w_\tau) \otimes_{\mathbf{A}_\tau} \mathbf{C}_\xi = \big(\mathbf{U}(L^\Gamma(\mathfrak{g})).w_\tau\big) \otimes_{\mathbf{A}_\tau} \mathbf{C}_\xi. \tag{7.2.3}$$

Combining 7.2.2 and 7.2.3, we obtain

$$\dim \mathbf{U}(L^{\Gamma}(\mathfrak{g})).w \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi} \geq \dim W(\tau) \otimes_{\mathbf{A}_{\tau}} \mathbf{C}_{\xi}.$$

On the other hand, since $\mathbf{U}(L^{\Gamma}(\mathfrak{g})).w \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi}$ is a cyclic $L^{\Gamma}(\mathfrak{g})$-module, generated by the highest weight space, we have

$$W^{\Gamma}(\lambda) \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi} \twoheadrightarrow \mathbf{U}(L^{\Gamma}(\mathfrak{g})).w \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi}. \tag{7.2.4}$$

So we obtain

$$\dim W^{\Gamma}(\lambda) \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi} \geq \dim \mathbf{U}(L^{\Gamma}(\mathfrak{g})).w \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi}.$$

Concluding we have

$$\dim W^{\Gamma}(\lambda) \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi} \geq \dim \mathbf{U}(L^{\Gamma}(\mathfrak{g})).w \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi} \geq \dim W(\tau) \otimes_{\mathbf{A}_{\tau}} \mathbf{C}_{\xi},$$

With 7.2.1, we conclude that we have equality throughout. That is,

$$W^{\Gamma}(\lambda) \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi} \cong_{L^{\Gamma}(\mathfrak{g})} \mathbf{U}(L^{\Gamma}(\mathfrak{g})).w \otimes_{\mathbf{A}_{\lambda}^{\Gamma}} \mathbf{C}_{\chi} \cong_{L^{\Gamma}(\mathfrak{g})} W(\tau) \otimes_{\mathbf{A}_{\tau}} \mathbf{C}_{\xi},$$

which completes the proof. $\square$

Conclusions

We make some final remarks here about the connections between Parts I and II, to place this manuscript in a broader context. Both sections are essentially aimed at understanding the global Weyl module, and the way it relates to other interesting structures such as the local Weyl modules. These modules have been useful in understanding the structure of various categories of representations of generalized loop algebras $\mathfrak{g} \otimes A$. All this is ultimately motivated by questions in the level–zero representation theory of the affine algebra associated to $\mathfrak{g}$. The work represented in Part I was an early step in a program (continued in [4] and [2]) to establish the global and local Weyl modules as "intermediate" modules between the simples and their projective covers. This is analogous to the role of the Verma module in the Bernstein–Gelfand–Gelfand Category $\mathcal{O}$, and so Part I is really an effort to form a comparison between global Weyl modules and Verma modules by abstracting some important homological properties of the Verma modules. Along the way, we are able to use a local Weyl module with a fundamental highest weight to reconstruct the global Weyl module of the same weight. This provides both a crucial tool in our proof, and some further insight into the relationship between local and global Weyl modules. In the case of an arbitrary dominant integral weight, it is not clear how one might pass from a local Weyl module back to the global, so any special cases one can construct should aid in understanding the general question.

On the other hand, the aim of Part II is to define the global Weyl module in the context of the twisted loop algebra $L^\Gamma(\mathfrak{g})$, which has until now been missing from the literature. The main goal is to prove that the twisted global Weyl module, which is straightforward to define by generators and relations, enjoys some nice relationship with its untwisted counterpart.

Intuitively, having already understood the connection between the twisted and untwisted local Weyl modules from the main result of [8] (that is, every twisted module may be found inside an appropriately chosen untwisted one), one might expect that the analogous result holds in the global case. However, for $\lambda \in P_0^+$, it is not true in general that $W^\Gamma(\lambda)$ embeds into $W(\mu)$ with μ satisfying $\bar{\mu} = \lambda$, which would be the most reasonable guess. It turns out that the canonical map, sending w_λ to w_μ, generally fails to be injective. In some cases, however, this canonical map is an embedding. Specifically, it can be shown that for $\omega_i \in P_0^+$, we have $W^\Gamma(\omega_i) \hookrightarrow W(\omega_j)$, for all $j \in \Gamma.i$. It follows that in this particular case, the results from Part I (and particularly Theorem 2) may provide insight into the homomorphisms between the twisted global Weyl modules we define in Part II.

Some remarks are also in order about more general Lie algebras than those of the form $\mathfrak{g}\otimes A$. Let Γ be any finite group, acting on $\mathfrak{g}$ and an affine scheme X by automorphisms. The Lie algebras of Γ–equivariant regular maps from X to $\mathfrak{g}$ are called the equivariant map algebras; they can also be realized as Lie algebras of fixed points $(\mathfrak{g}\otimes A)^\Gamma$, where A is the ring of regular functions on X and the action of Γ on A is diagonal. Many well–known infinite–dimensional Lie algebras are in fact examples of these algebras, including the generalized

Onsager algebra and the twisted loop and multiloop algebras. Their finite–dimensional simple modules were studied and classified in [24]. Local Weyl modules for an equivariant map algebra were defined and studied in [13] in the case when Γ is abelian and its action on X is free. Similarly to the case of twisted loop algebras in [8], it was shown there that any local Weyl module for $(\mathfrak{g} \otimes A)^\Gamma$ can be obtained by restriction from a local Weyl module for $\mathfrak{g} \otimes A$.

The contents of Part II can be viewed as a preliminary step toward a definition of the global Weyl module for the equivariant map algebras. In the twisted loop case, we still have a Cartan subalgebra, so we have weights and can define the global Weyl modules by generators and relations. In the general situation, there might be no non-zero Cartan subalgebra; however, the direct sum of Theorem 8 can be adapted in a way that is not dependent on having such a Cartan subalgebra. Thus, using our result for twisted loop algebras, one may attempt to define global Weyl modules for $(\mathfrak{g} \otimes A)^\Gamma$ as submodules in a direct sum of global Weyl modules for $\mathfrak{g} \otimes A$. Proving that they admit sufficient properties to justify the name *global Weyl module* would the main task in such a project.

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