Math 464
Homework: Due on 5/7

1) For \( j \in \mathbb{Z} \), let \( V_j \subset L^2(\mathbb{R}) \) be the space of all bandlimited functions for which the Fourier transform \( \hat{f} = 0 \) outside the interval \([-2^j, 2^j]\), that is, the support \( \hat{f} \) is included in \([-2^j, 2^j]\).

(a) Carefully justify why \( V_j \) satisfies the first four conditions in the definition of a multiresolution analysis.

(b) Let \( \phi(x) = \frac{\sin \pi x}{\pi x} \), show that \( \phi \) satisfies the last condition in the definition of a MRA.

(c) Find the two scales equation corresponding to this MRA, that is find the coefficients \( p_k \) such that

\[
\phi(x) = \sum_{k \in \mathbb{Z}} p_k \phi(2x - k).
\]

(d) Find an expansion for the wavelet \( \psi \) associated with \( \phi \).

2) Let \( h(x) = \max(0, 1 - |x|) \).

(a) Show that \( \hat{h}(\gamma) = \frac{\sin^2 \pi \gamma}{\pi^2 \gamma^2} \).

(b) Show that

\[
\frac{\pi^2}{\sin^2 \pi \gamma} = \sum_{k=-\infty}^{\infty} \frac{1}{(\gamma + k)^2}.
\]

[Hint: Use the Fourier transform domain of the orthonormality condition, applied to the Haar scaling function.]

(c) Deduce that

\[
\frac{3 - 2 \sin^2 \pi \gamma}{\sin^2 \pi \gamma} = \frac{3}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{(\gamma + k)^2}.
\]

[Hint: Differentiate twice both sides of the equation obtained in (b).]

(d) Prove that \( \{ \phi(x - k), k \in \mathbb{Z} \} \) form an orthonormal system, where \( \phi \) is given by

\[
\hat{\phi}(\gamma) = \frac{\sin^2 \pi \gamma}{\pi^2 \gamma^2 \sqrt{1 - \frac{2}{3} \sin^2 \pi \gamma}}.
\]

3) Let \( \{ V_j, j \in \mathbb{Z} \} \) be a MRA with scaling function \( \phi \) satisfying the two-scales equation

\[
\phi(x) = \sum_k p_k \phi(2x - k) \quad \text{with} \quad p_k = 2 \int_{-\infty}^{\infty} \phi(x)\overline{\phi(2x - k)} \, dx.
\]

Recall that the corresponding wavelet is given by

\[
\psi(x) = \sum_k (-1)^k \overline{p_{1-k}} \phi(2x - k).
\]

Let

\[
f_j(x) = \sum_k a_k^j \phi(2^j x - k) = f_{j-1} + w_{j-1} = \sum_k a_k^{j-1} \phi(2^{j-1} x - k) + \sum_k b_k^{j-1} \psi(2^{j-1} x - k)
\]

where \( f_{j-1} \in V_{j-1} \), and \( w_{j-1} \in W_{j-1} \). Prove that

\[
a_k^{j-1} = 2^{-1} \sum_k p_{k-2^j} a_k^j
\]

and

\[
b_k^{j-1} = 2^{-1} \sum_k (-1)^k p_{1-k+2^j} a_k^j.
\]