HYPERPLANE ARRANGEMENTS AND INVARIANT THEORY

P. BELKALE, P. BROSNAN AND S. MUKHOPADHYAY

Abstract. In the first part of this paper, we consider, in the context of an arbitrary hyperplane arrangement, the map between compactly supported cohomology to the usual cohomology of a local system. A formula (i.e., an explicit algebraic de Rham representative) for a generalized version of this map is obtained.

These results are applied in the second part to invariant theory: Schechtman and Varchenko connect invariant theoretic objects to the cohomology of local systems on complements of hyperplane arrangements. The first part of this paper is then used, following Looijenga, to determine the image of invariants in cohomology. In suitable cases (e.g., corresponding to positive integral levels), the space of invariants is shown to acquire a mixed Hodge structure over a cyclotomic field. We investigate the Hodge filtration on the space of invariants, and characterize the subspace of conformal blocks in Hodge theoretic terms (as the smallest part of the Hodge filtration).

1. Introduction

Let W be a M-dimensional complex vector space, and consider an arbitrary weighted hyperplane arrangement $(W, \mathcal{C}, a)$ in W. This consists of the following data:

(1) A collection $\mathcal{C}$ of hyperplanes in W. We will assume that we are given polynomials $f_1, \dots, f_r$ on W of degree one such that the hyperplanes in the collection $\mathcal{C}$ are the zero loci $Z(f_1), \dots, Z(f_r)$. These are subsets of W, which need not pass through the origin.

(2) A vector $a = (a_1, \dots, a_r) \in \mathbb{C}^r$, which we will call a weighting of $\mathcal{C}$. The weight of the hyperplane $Z(f_i)$ is a_i.

Let $U = W - Z(f_1 f_2 \cdots f_r)$. The differential form corresponding to the weighting is

$$\eta = \sum_{i=1}^{r} a_i \operatorname{dlog} f_i = \sum_{i=1}^{r} a_i \frac{df_i}{f_i} \in H^0(U, \Omega^1).$$

Now set $\nabla = d + \eta : \mathcal{O}_{U^{\mathrm{an}}} \to \Omega_{U^{\mathrm{an}}}$ and write $\mathcal{L}(a)$ for the kernel of ∇ which is a rank one local system on U.

Definition 1. *The Aomoto complex of the weighted hyperplane arrangement $(W, \mathcal{C}, a)$ is the complex $(A^\bullet(U), \eta) = (A^\bullet(U), \eta \wedge \cdot)$. Here $A^j(U)$ are global log j-forms on U and the differential in the complex takes $\omega \to \eta \wedge \omega$ with $\omega \in A^j(U)$. By a result of Brieskorn* [Bri73], *$\oplus_j A^j(U) \subseteq \oplus_j H^0(U, \Omega^j)$ is the DG algebra generated over $\mathbb{C}$ by $\frac{df_i}{f_i}, i = 1, \dots, r$.*

To motivate the discussion, let us assume first that the weights a_i are " sufficiently small", for example assume that the absolute values $|\sum_{i=1}^r \epsilon_i a_i| \in \mathbb{C} - \mathbb{Z}_{>0}$ for all choices of $\epsilon_i \in \{0, 1\}$ (we need the weights and their negatives to be satisfy the conditions (Mon) in [ESV92])). In this case it is known by results of Esnault-Schechtman-Viehweg [ESV92] that the i-th cohomology group of the Aomoto complex represents $H^i(U, \mathcal{L}(a))$. It follows that $H^i(U, \mathcal{L}(a)) = 0$ for $i > M$.

Now consider the natural map

$$H^i_c(U,\mathcal{L}(a)) \to H^i(U,\mathcal{L}(a)). \tag{2}$$

The compactly supported cohomology $H^i_c(U,\mathcal{L}(a))$ is Poincaré-Verdier dual to $H^{2M-i}(U,\mathcal{L}(a)^*)$ and $\mathcal{L}(a)^* = \mathcal{L}(-a)$. By [ESV92] applied to $\mathcal{L}(-a)$, we get $H^{2M-i}(U,\mathcal{L}(-a))$ vanishes for $i < M$. Therefore $H^i_c(U,\mathcal{L}(a))$ also vanishes for $i < M$.

The map (2) is therefore interesting only when $i = M$. In this case it is "given" by an element (with a switch of factors)

$$\Sigma \in H^M(U,\mathcal{L}(a)^*) \otimes H^M(U,\mathcal{L}(a)) = H^{2M}(U\times U,\mathcal{L}(a)\boxtimes\mathcal{L}(-a)) \tag{3}$$

Now $H^{2M}(U\times U,\mathcal{L}(a)\boxtimes\mathcal{L}(-a))$ is computed by the Aomoto complex of a hyperplane arrangement in $W\times W$: The hyperplanes are of the form $H\times W$ and $W\times H$ with weights a_i and $-a_i$ for $H = Z(f_i)$. It therefore makes sense to ask for a formula of an explicit element in $A^{2M}(U\times U)$ which gives rise to the element Σ in (3).

Remark 4. *The question of determination of the image of the map (2) is in principle different from that of determining the map (2), see Section 3.5.1.*

Theorem 5. *Let*

$$S = \sum_{1\le i_1<\cdots<i_M\le r} \prod_{s=1}^{M} a_{i_s}\,\mathrm{dlog}\, f^{(1)}_{i_s}\,\mathrm{dlog}\, f^{(2)}_{i_s} \tag{6}$$

(here $f^{(1)}_i(x,y) = f_i(x)$ and $f^{(2)}_i(x,y) = f_i(y)$). There exists a non zero $c\in\mathbb{C}$ such that element $cS \in A^{2M}(U\times U)$ represents the cohomology class $\Sigma \in H^{2M}(U\times U,\mathcal{L}(a)\boxtimes\mathcal{L}(-a))$.

The Aomoto complex computes topological cohomological groups even when the weights a are not small, as shown by Looijenga [Loo99, Proposition 4.2]. We refer the reader to Section 1.2 for more details. The groups $H^i(U,\mathcal{L}(a))$ need to be replaced by hypercohomologies of complexes on suitable compactifications of U, and we have a generalization of the map (2) again in this set-up. Our main result, Theorem 48, gives an explicit form which represents (up to a constant multiplicative scalar) the generalized mapping (2) (see also the map (45)). In Remark 50 we note that Theorem 5 holds for arbitrary weights.

1.1. **The log form S.** The map $A^M(U)^* \to A^M(U)$ induced by the element S (in Theorem 5) appears in the work of Schechtman and Varchenko [SV91, Theorem 2.4 and Lemma 3.2.5]. There is an entire collection of such forms, our S corresponds to "S^M" in loc. cit.; all of these show up in the proof of Theorem 5 in roughly the following way: The element Σ is a suitable cohomology class of the diagonal in $U\times U$, and should therefore vanish on the open subsets $U\times U - \{(x,y)\in U\times U \mid F(x)=F(y)\}$, where F is an arbitrary linear form on W. Therefore one should have (for Theorem 5 to hold) S exact in the Aomoto cohomology that computes the cohomology of the local system corresponding to the hyperplane arrangement given by adding $\{(x,y)\in W\times W \mid F(x)=F(y)\}$ with weight 0 to the product weighted arrangement (with weights a on one factor and weights $-a$ on the other), see Corollary 82. The proof that S is exact features the other forms $S^{(b)}$ (one needs to also consider intersections of such open subsets). The above argument is carried out in Čech cohomology, which leads to a full proof of Theorem 5.

1.2. **The case of arbitrary weights.** We give a description of Looijenga's results from [Loo99, Proposition 4.2] sufficient for the statement of a generalization of (2).

Let P be any smooth projective compactification of U, with $P - U = \cup_\alpha E_\alpha$ a divisor with normal crossings. Let $V = P - \cup'_\alpha E_\alpha$, where the union is restricted to α such that a_α is not a strictly positive integer. Similarly let $V' = P - \cup'_\alpha E_\alpha$, where the union is restricted to α such that $a_\alpha = \operatorname{Res}_{E_\alpha} \eta$, the residue of η on E_α, is not an integer which is ≥ 0. Note that $V' \supseteq V$.

Let $q : U \to V'$ and $j : U \to V$ denote the inclusion map. The cohomology of the Aomoto complex $H^\bullet(A^\bullet(U), \eta)$ is equal to the cohomology $H^\bullet(V, j_! \mathcal{L}(a))$. The map (2) is replaced by the natural map (which is non-zero only for $\bullet = M$)

$$H^\bullet(V', q_! \mathcal{L}(a)) \to H^\bullet(V, j_! \mathcal{L}(a)) \tag{7}$$

The groups $H^M(V', q_! \mathcal{L}(a))$ are dual to Aomoto cohomology groups (for the weight vector $-a$), and this set-up generalizes the case of small weights (if the weights are small then $V = U$, and $H^M(V', q_! \mathcal{L}(a)) = H^M_c(U, \mathcal{L}(a))$, $H^M(V, j_! \mathcal{L}(a)) = H^M(U, \mathcal{L}(a))$). Theorem 48, which is stated in this context, "computes" (7). Note that Theorem 48 specializes to Theorem 5 in the case of small weights. In fact Theorem 5 is also true for all choices of weights, see Remark 50.

1.3. **Applications to invariant theory.** Consider a finite dimensional simple Lie algebra $\mathfrak{g}$ with a fixed Cartan decomposition and let R denote the set of positive simple roots. Let $(\ ,\ )$ be a normalized Killing form on $\mathfrak{h}$, the Lie algebra of the Cartan subgroup, such that $(\theta, \theta) = 2$ where $\theta \in \mathfrak{h}^*$ is the highest root (identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ using the Killing form). For a dominant integral weight $\lambda \in \mathfrak{h}^*$, let V_λ denote the corresponding irreducible representation.

Now suppose that we are given an n-tuple $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of dominant integral weights of $\mathfrak{g}$. The space of coinvariants

$$\mathbb{A}(\vec{\lambda}) = (V_{\lambda_1} \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_n})_{\mathfrak{g}} = \frac{(V_{\lambda_1} \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_n})}{\mathfrak{g}(V_{\lambda_1} \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_n})}$$

is a fundamental object of invariant theory. Note that the space of invariants maps isomorphically to the space of coinvariants. It is easy to see that $\mathbb{A}(\vec{\lambda})$ is zero if $\sum \lambda_i$ is not a positive sum of simple roots, and we will assume that this is indeed the case and write

$$\mu = \sum_{p=1}^{r} n_p \alpha_p, \quad n_p \in \mathbb{Z}_{\geq 0}.$$

Fix a map $\beta : [M] = \{1, \dots, M\} \to R$, so that

$$\mu = \sum_{b=1}^{M} \beta(b), \ M = \sum_{p=1}^{r} n_p \tag{8}$$

Fix a point $\vec{z} = (z_1, \dots, z_n)$ in the configuration space of n distinct points on $\mathbb{A}^1$. Let $W = \mathbb{C}^M$. The coordinate variables of W will be denoted by $t_1, \dots, t_M$. We will consider the variable t_b to be colored by the simple root $\beta(b)$. Consider the weighted hyperplane arrangement $(W, \mathcal{C}, a)$ in W given by the following collection of hyperplanes, and their attached weights (here κ is an arbitrary non-zero complex number):

(1) For $i \in [1, n]$ and $b \in [1, M]$, the hyperplane $t_b - z_i = 0$, with weight $\frac{(\lambda_i, \beta(b))}{\kappa}$.
(2) $b, c \in [1, M]$ with $b < c$, the hyperplane $t_b - t_c = 0$, with weight $-\frac{(\beta(b), \beta(c))}{\kappa}$.

Let U be the complement of the above hyperplane arrangement in W as before. The corresponding $\eta \in A^1(U)$ is

$$\eta = \frac{1}{\kappa}\Bigg(\sum_{i=1}^{n}\sum_{b=1}^{M}(\lambda_j, \beta(b))\frac{d(t_b - z_i)}{t_b - z_i} - \sum_{b,c\in[1,M],b<c}(\beta(b),\beta(c))\frac{d(t_b - t_c)}{t_b - t_c}\Bigg).$$

The basic connection between invariant theory and the topology of hyperplane arrangements arises from the following injective map constructed by Schechtman and Varchenko [SV91]:

$$\Omega^{SV} : V(\vec{\lambda})_0^* = (V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_n})_0^* \to A^M(U). \tag{9}$$

Here $V(\vec{\lambda})_0^*$ is the zero weight space (for $\mathfrak{h}$) in the dual of $V(\vec{\lambda}) = (V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_n})$, and $A^i(U)$ the space of logarithmic differential forms of degree i on U. Note that $\mathbb{A}(\vec{\lambda})^*$ is a subspace of $V(\vec{\lambda})_0^*$.

Remark 10. *The map in* [SV91] *is for Verma modules for the corresponding Lie algebra without Serre relations, and is in this context an isomorphism. The representations V_λ considered here are quotients of these Verma modules, and hence we get an injective map in* (9)*.*

Proposition 11. *The induced mapping $\mathbb{A}(\vec{\lambda})^* \to H^M(A^\bullet(U), \eta)$ is injective.*

We show that the injectivity in Proposition 11 follows from the unitarity results on conformal blocks [Ram09, Bel12].

Remark 12. *Proposition 11 above appears in* [Loo12, Lemma 3.3]*. The sketched proof of this proposition in* [Loo12] *is via* [Loo12, Proposition 2.18]*. Proposition 2.18 in loc. cit. is injection into a term in the Aomoto complex. But injection in a quotient of the top degree term in this complex is more subtle (indeed, the proof in our paper uses extension to compactifications).*

There is a natural symmetric group acting on $H^M(A^\bullet(U), \eta)$:

$$\Sigma_M = \{\sigma \in S_M \mid \beta(\sigma(b)) = \beta(b), b = 1, \ldots, M\} \tag{13}$$

We consider Σ_M as the "color preserving" symmetric group acting on variables $t_1, \ldots, t_M$. The map $\mathbb{A}(\vec{\lambda})^* \to H^M(A^\bullet(U), \eta)$ has its image in $H^M(A^\bullet(U), \eta)^\chi$ where χ is the sign character on Σ_M. Therefore one obtains an injective map

$$\mathbb{A}(\vec{\lambda})^* \hookrightarrow H^M(A^\bullet(U), \eta)^\chi \tag{14}$$

Note that as in Section 1.2, we may write $H^M(A^\bullet(U), \eta)^\chi = H^M(V, j_!\mathcal{L}(a))^\chi$. In the case of small weights, this becomes $H^M(A^\bullet(U), \eta)^\chi = H^M(U, \mathcal{L}(a))^\chi$. So the right hand side of (14) has a topological interpretation. Although this will not play a role in this paper, we note that the induced mapping

$$\mathbb{A}(\vec{\lambda})^* \to H^M(V, j_!\mathcal{L}(a))^\chi \tag{15}$$

is flat for connections as $\vec{z} = (z_1, \ldots, z_n)$ varies in the configuration space of n-distinct points on $\mathbb{A}^1$. Here the left hand side of (15) has the Knizhnik-Zamolodchikov (KZ) connection, and the right hand side the Gauss-Manin connection [SV91, Lemma 6.6.3], [Loo12, Lemma 3.9, Lemma 3.10].

1.4. **The image of the injective map (14).** Looijenga's strategy for determining this image is the following: For simplicity of exposition we will assume, in the introduction, that κ is sufficiently large in absolute value so that the weights above are small (this assumption is eventually dropped). Therefore (also similarly for $-a$), $H^M(A^\bullet(U),\eta) \xrightarrow{\sim} H^M(U,\mathcal{L}(a))$ and hence under the assumption of smallness of weights, there is an injection

$$\mathbb{A}(\vec{\lambda})^* \hookrightarrow H^M(U,\mathcal{L}(a))^\chi. \tag{16}$$

Consider the map (14) for the dual weights $\lambda_1^*,\dots,\lambda_n^*$, and $-\kappa$ for the value of κ. Since $\lambda^* = -w_0\lambda$, where w_0 is the longest element in the Weyl group, we can write

$$\sum_{i=1}^n \lambda_i = \sum_{b=1}^M \gamma(b)$$

where $\gamma(b) = -w_0\beta(b)$ are simple positive roots. Therefore we may use the same vector space and variables $t_1,\dots,t_M$, just the colors change for the dual. The weights are now

(1) For $i\in[1,n]$ and $b\in[1,M]$, the hyperplane $t_b - z_i = 0$, with weight $\frac{(-w_0\lambda_i,-w_0\beta(b))}{-\kappa} = \frac{(\lambda_i,\beta(b))}{-\kappa}$.
(2) $b,c\in[1,M]$ with $b<c$, the hyperplane $t_b - t_c = 0$, with weight $-\frac{(-w_0\beta(b),-w_0\beta(c))}{-\kappa} = \frac{(\beta(b),\beta(c))}{-\kappa}$.

Equalities of weights above hold because the Cartan-Killing form is invariant under the Weyl group. These weights are negatives of the weights assigned for $\lambda_1,\dots,\lambda_n$ and κ. We therefore have an injection

$$\mathbb{A}(\vec{\lambda}^*)^* \hookrightarrow H^M(A^\bullet(U),-\eta)^\chi = H^M(U,\mathcal{L}(-a))^\chi$$

Dualizing, we find a surjection (χ is a sign character, hence self dual)

$$H_c^M(U,\mathcal{L}(a))^\chi \twoheadrightarrow \mathbb{A}(\vec{\lambda}^*) \tag{17}$$

Now, there is a canonical isomorphism from invariants to coinvariants

$$\mathbb{A}(\vec{\lambda})^* \xrightarrow{\sim} \mathbb{A}(\vec{\lambda}^*) \tag{18}$$

Putting the maps (17) and (15) together with the inverse of (18), we get

$$H_c^M(U,\mathcal{L}(a))^\chi \twoheadrightarrow \mathbb{A}(\vec{\lambda}^*) \xrightarrow{\sim} \mathbb{A}(\vec{\lambda})^* \hookrightarrow H^M(U,\mathcal{L}(a))^\chi. \tag{19}$$

Therefore (following Looijenga [Loo12, Theorem 3.7]), we see that the image of the map (15) equals the image of the composite $H_c^M(U,\mathcal{L}(a))^\chi \to H^M(U,\mathcal{L}(a))^\chi$ in (19).

Now Looijenga assumes the following compatibility property:

- the composite map in (19) is the natural $H_c^M(U,\mathcal{L}(a))^\chi \to H^M(U,\mathcal{L}(a))^\chi$, induced by topology (see [Loo12, Proof of Theorem 3.7, page 33]).

and concludes the following (actually a generalized form, valid for arbitrary $\kappa\neq 0$ is proved in [Loo12], this result is Theorem 108 in our paper):

Theorem 20. *Assume $|\kappa|$ is sufficiently large. The image of invariants in the topological cohomological groups, i.e., the image of the injective map (16), coincides with the image of the map induced by topology $H_c^M(U,\mathcal{L}(a))^\chi \to H^M(U,\mathcal{L}(a))^\chi$. Therefore,*

$$\mathbb{A}(\vec{\lambda})^* = \text{Image}\ \ H_c^M(U,\mathcal{L}(a))^\chi \to H^M(U,\mathcal{L}(a))^\chi$$

Remark 21. *Looijenga's assumption is subtle, and needs a justification since the map* (19) *factors through a representation theoretic duality* (18)*, and also uses Poincaré-Verdier duality and the maps* (9)*. Therefore Looijenga's proof assumes a compatibility property between topological and representation theoretic dualities, as well as a compatibility with Schechtman-Varchenko maps* (9)*.*

The results of the first part of the paper were motivated by the problem of proving this interesting compatibility property. In Theorems 48 and 5, which are valid for arbitrary weighed hyperplane arrangements, a "formula" for the (topological) Σ_M equivariant mapping $H^M_c(U,\mathcal{L}(a)) \to H^M(U,\mathcal{L}(a))$ (and a generalization) is obtained. By this we mean a de Rham representative for the corresponding element (which will be Σ_M-invariant for the action on $U\times U$) of the space (3).

This formula is compared with a formula for the actual composite (19) which is obtained using the work of Schechtman and Varchenko [SV91, Theorem 6.6]. These formulas coincide (up to a non-zero scalar), and one proves the assumption implicit in Looijenga's proof of [Loo12, Theorem 3.7].

1.5. **The case of rational weights.** In Section 10.4, we show that our results imply that $\mathbb{A}(\vec{\lambda})^*$ carries a mixed Hodge structure over a cyclotomic field extension of $\mathbb{Q}$ if κ is an integer (or even a rational number). If $\kappa = \ell + g^*$ where ℓ is a positive integer and g^* the dual Coxeter number of $\mathfrak{g}$, then the F^M (Hodge) part of $\mathbb{A}(\vec{\lambda})^*$ coincides with the space of conformal blocks at level ℓ, for $\mathfrak{g}$ classical or G_2.

In Section 11 we give an example where the mixed Hodge structure on $\mathbb{A}(\vec{\lambda})^*$ is not pure, by showing that the monodromy of the KZ system is not semisimple.

2. The Aomoto complex

The aim of this section is to make explicit what the Aomoto complex represents.

2.1. **Resolution of singularities.** Let (X,D) be a pair of a smooth variety X and a divisor D. There exists [BVP15] a canonical resolution of singularities of (X,D): a birational projective morphism $\pi:\widetilde{X}\to X$ with the following properties

(a) $\widetilde{X}$ is smooth.
(b) $\widetilde{D}=\pi^{-1}(D)$ is a divisor with simple normal crossings.
(c) $\pi:\widetilde{X}-\widetilde{D}\to X-D$ is an isomorphism.
(d) For any $p\in D$ such that D is simple normal crossings at p, the map π is an isomorphism over a neighborhood of p.
(e) Automorphisms of the pair (X,D), extend to the pair $(\widetilde{X},\widetilde{D})$.

We will have occasion to use the above result when D is not locally a hyperplane arrangement. Property (d) is used in an essential manner in this paper (Section 6); Property (e) can be avoided, but leads to a more satisfying picture.

2.2. **Compactifications.** Let P be any smooth projective compactification of U, with $P - U = \cup_\alpha E_\alpha$ a divisor with normal crossings. The higher cohomology $H^j(P,\Omega^i_P(\log E))$, $j>0$ vanishes by [ESV92, Section 2], and $H^0(P,\Omega^i_P(\log E))$ is the space of log forms $A^i(U)$. The global hypercohomology of the complex $(\Omega^\bullet_P(\log E), d+\eta)$ coincides with the cohomology of the Aomoto complex. In Lemma 22 and 24, we recall known results on the stalks of the Aomoto complex. These are used to give a topological characterization of the Aomoto complex in Lemma 25.

Lemma 22. *Suppose $p \in E_\beta \subseteq P$. Assume a_β is not an integer which is ≤ 0. Then the stalk of the hypercohomology at p of the complex $(\Omega_P^\bullet(\log E), d+\eta)$ is zero.*

Note that in the above statement we allow p to be in the intersection of several E_α; For some (but not all) of these, a_α could be an integer ≤ 0.

Proof. (Standard) We can replace P by the open polydisc D^n. For simplicity assume that D is the union of coordinate hyperplanes (it can be assumed that it is a union of some coordinate hyperplanes, we will assume that all appear for ease of exposition).

Assume that $\eta = \sum_i a_i \operatorname{dlog} z_i$, where a_1 (corresponding to β) is not an integer ≤ 0. Let A_k be the denote the set of all subsets of $\{1, \dots, n\}$ of cardinality k. The stalk of $\Omega^k_{D^{*n}}(\log E)$ at 0 is given by

$$\bigoplus_{(i_1 < \cdots < i_k) \in A_k} \mathbb{C}\{z_1, \dots, z_n\} \operatorname{dlog} z_{i_1} \dots \operatorname{dlog} z_{i_k},$$

where $\mathbb{C}\{z_1, \dots, z_n\}$ is the set of convergent power series. Thus to complete the proof, we need to show that the logarithmic de Rham complex is exact.

The aim is to construct homotopies using the one variable case. Let

$$\alpha = z_1^{m_1} \dots z_n^{m_n} \operatorname{dlog} z_{i_1} \dots \operatorname{dlog} z_{i_k}$$

set $\delta_k(\alpha) = 0$ if $i_1 \neq 1$, and if $i_1 = 1$,

$$\delta_k(\alpha) = \frac{1}{m_1 + a_1} z_1^{m_1} \dots z_n^{m_n} \operatorname{dlog} z_{i_2} \dots \dots \operatorname{dlog} z_{i_k}.$$

We claim that $\delta_k(\alpha)$ extends by linearity to $\Omega^k_{D^{*n}}(\log E)$. This is a convergence issue, and reduces to the one variable case: If $f(z) = \sum_m \alpha_m z^m$ is a holomorphic function of z near zero, then $\sum_m \frac{\alpha_m}{m+\alpha} z^m$ also converges since the radius of convergence only improves.

We now show $(d+\eta)\delta_k + \delta_{k+1}(d+\eta))(\alpha) = \alpha$. The desired exactness follows immediately. We divide the proof into two cases

- If $i_1 \neq 1$, set $\beta = z_2^{m_2} \dots z_n^{m_n} \operatorname{dlog} z_{i_1} \dots \operatorname{dlog} z_{i_k}$. Clearly $(d+\eta)\delta_k(\alpha) = 0$. Now, $\delta_{k+1}(d+\eta)(\alpha) = \delta_{k+1}(m_1 z_1^{m_1} \operatorname{dlog} z_1 \beta + a_1 z_1^{m_1} \operatorname{dlog} z_1 \beta) = \alpha$.
- If $i_1 = 1$, set $\beta = z_2^{m_2} \dots z_n^{m_n} \widehat{\operatorname{dlog} z_1} \dots \operatorname{dlog} z_{i_k}$ then $(d+\eta)\delta_k(\alpha)$ is equal to $\frac{1}{m_1+\alpha_1}$ times $z_1^{m_1} d\beta + \eta'\beta + (m_1 + a_1)\alpha$, here $\eta' = \sum_{i>1} a_i \operatorname{dlog} z_i$. Now the terms in $(d+\eta)(\alpha)$ which contain $\operatorname{dlog} z_1$ are of the form $-(\operatorname{dlog} z_1) z_1^{m_1} d\beta - (\operatorname{dlog} z_1) z_1^{m_1} \eta'\beta$. The desired equality follows.

□

Definition 23. *Let X be a complex algebraic variety. We define the constructible derived category $D_c^b(X)$ to be the subcategory of the bounded derived category of the category of sheaves of $\mathbb{C}$ vector spaces on X which are cohomologically constructible.*

The following is a part of [Del70, Proposition 3.13]:

Lemma 24. *Suppose $p \in P$. Assume that for every E_β passing through p, a_β is not a strictly positive integer. Then the local hypercohomology at p of the complex $(\Omega_P^\bullet(\log E), d+\eta)$ is isomorphic to the hypercohomology at p of $Rk_*\mathcal{L}(a)$ where $k : U \to P$ (via the canonical map to $Rk_*\mathcal{L}(a)$ obtained by adjunction).*

Let $V = P - \cup'_\alpha E_\alpha$, where a_α is not a strictly positive integer. Let $j : U \to V$, $k : V \to P$ denote the inclusion maps. The following is a result of Looijenga.

Lemma 25. *The complex $\Omega^\bullet_\eta = (\Omega^\bullet_P(\log E), d + \eta)$ equals $Rk_*j_!\mathcal{L}(a)$ as objects in $D^b_c(P)$.*

Recall that $j_!$ is an exact functor and $j_! = Rj_!$.

Proof. ([Loo99, Proposition 4.2]) There is a canonical isomorphism $\mathcal{L}(a) \to \Omega^\bullet_\eta$ on U. By adjunction this gives rise to a map $j_!\mathcal{L}(a) \to \Omega^\bullet_\eta$ on V which is verified to be an quasi-isomorphism using Lemma 22. We get isomorphisms

$$Rk_*j_!\mathcal{L}(a) \to Rk_*\Omega^\bullet_\eta$$

Now there is by adjunction a canonical map $\Omega^\bullet_\eta \to Rk_*\Omega^\bullet_\eta$. We claim this is an isomorphism, and hence complete the proof of Lemma 25. Once this claim is proved, the isomorphism morphism $Rk_*j_!\mathcal{L}(a)$ to $\Omega^\bullet_\eta$ is obtained in two steps: The map $Rk_*j_!\mathcal{L}(a) \xrightarrow{\sim} Rk_*\Omega^\bullet_\eta$, composed with the inverse of $\Omega^\bullet_\eta \xrightarrow{\sim} Rk_*\Omega^\bullet_\eta$.

We prove this claim by comparing local cohomologies at points $p \in P$. Let $p \in P - V$. If p does not lie on an E_β with a_β a positive integer, the desired isomorphism follows from Lemma 24.

Now assume that $p \in E_\beta$ and a_β a positive integer. Then the local cohomology at p of $\Omega^\bullet_\eta$ is zero by Lemma 22. Therefore to conclude the proof we only have to verify that the stalk at p of $Rk_*j_!\mathcal{L}(a)$ is zero. This stalk is isomorphic to the stalk of $Rk_*\Omega^\bullet_\eta$, which can be computed analytically, and shown to be zero by the same method as Lemma 25. We can also proceed topologically to show that the stalk at p of $Rk_*j_!\mathcal{L}(a)$ is zero as in the Lemma below (which for simplicity we prove only for two factors). This finishes the proof of Lemma 25. ∎

Lemma 26. *Consider $\eta = a_1 \operatorname{dlog} z_1 + a_2 \log z_2$ on D^2. Let $U = D^* \times D^*$, $V = D^* \times D$ (i.e., $z_1 \neq 0$). Let $j : U \to V$ and $k : V \to D^2$. Then the local cohomology at 0 of $Rk_*j_!\mathcal{L}(a)$ vanishes. Here there are no conditions on a_1 and a_2*[1].

Proof. We need to compute $H^\bullet(V, j_!\mathcal{L}(a))$. Let $p : V \to D^*$ be the projection to the first factor, and show that $Rp_*j_!\mathcal{L}(a) = 0$, to do this, which is local on D^*, we may assume $a_1 = 0$, and compute fiberwise along fibers of p, we need $H^1(D^2, j_!\mathcal{L}(a_2)) = 0$, which is clear. ∎

2.3. **A variation.** We could have taken $\hat{V} = P - \cup E_\alpha$ all α such that a_α is either a strictly positive integer, or a non-integer. Let $j' : U \to \hat{V}$ and $k' : \hat{V} \to P$. Then $\Omega^\bullet_\eta$ is represented by $k'_!Rj'_*\mathcal{L}(a)$. This comes about by combining the isomorphisms

$$k'_!\Omega^\bullet_\eta \xrightarrow{\sim} k'_!Rj'_*\mathcal{L}(a) \quad \text{and} \quad k'_!\Omega^\bullet_\eta \xrightarrow{\sim} \Omega^\bullet_\eta$$

2.4. Let $E_1, \dots, E_n$ be an enumeration of the irreducible components of $P - U$. The index set of α is therefore $\{1, \dots, n\}$. Let $U_0 = P$, $U_1 = P - E_1$, $U_2 = P - E_1 \cup E_2$, etc, and $U = U_n = P = E_1 \cup E_2 \cup \cdots \cup E_n$. Let $j_i : U_i \to U_{i-1}$.

Color the divisors E_α by four colors (the divisor is colored, not the points on it!):

- If a_α is not an integer, then color the divisor green,
- If a_α is a positive integer then color the divisor white,
- If a_α is a negative integer, color the divisor black,
- If a_α equals zero, color the divisor blue.

[1]The local cohomology at 0 of $k_!Rj_*\mathcal{L}(a)$ is obviously zero as well.

Now form an extension of the sheaf $\mathcal{L}(a)$ on U to all of P as follows: The extension is

$$Rj_{1,?}Rj_{2,?}\dots Rj_{n,?}\mathcal{L}(a) \tag{27}$$

Here the ? in $Rj_{i,?}$ is ! if the color on E_i is white, and $*$ if black or blue, and either ! or $*$ if the color is green (both produce the same answer). Note that $j_!$ is exact and $j_! = Rj_!$.

We claim that the resulting object in the derived category is independent of the ordering of divisors. To prove this we consider $V = P - \cup' E_\alpha$, with E_α either green or white. All of the sheaves produced have zero local cohomology at points of $\cup' E_\alpha$, therefore all the sheaves are extension by zero from V. Restricted to V, all are derived lower star extensions from U which obviously commute.

Definition 28. *Let* $\underline{\mathcal{L}}(a) = (\Omega^\bullet_P(\log E), d+\eta)$ *as an object in* $D^b_c(P)$*. This object is canonically quasi-isomorphic to any of the elements* (27) *as above.*

2.5. **Independence from choices of compactifications.** We use standard adjunction properties in this section, see Remark 35. Suppose P' is another compactification of U with $P' - U = \cup_\beta E'_\beta$ a divisor with normal crossings. Assume that there is a map $\pi : P' \to P$ which is identity over U. Let

$$V' = P' - \cup' E'_\beta, \quad V = P - \cup' E_\alpha$$

with the union restricted to β and α with a'_β and a_α in the set $\mathbb{R} - \{1, 2, 3, \dots\}$. Let $j : U \to V$ and $j' : U \to V'$.

Now $H^i(P, \underline{\mathcal{L}}(a))$ and $H^i(P', \underline{\mathcal{L}'}(a))$ are both isomorphic to Aomoto cohomology and hence isomorphic. We want these to be isomorphic via a natural morphism

$$R\pi_* \underline{\mathcal{L}'}(a) \to \underline{\mathcal{L}}(a) \tag{29}$$

It suffices to construct such a morphism over V, by adjunction properties of Rk_* where $k : V \to P$. By Lemma 30 below, it is lower shriek extension through out in V and $\pi^{-1}(V)$, and the map (29) is evident over V(compare stalks of both sides on $V - U$, and show they are both zero by proper base change).

Lemma 30. $\pi^{-1}(V) \subseteq V'$

Proof. We claim that for any divisor $E'_{\alpha'}$ which intersects V', $a_{\alpha'}$ is a positive integer. Now $\pi(E_{\alpha'})$ has its generic point in V. Let p' be a generic point of E'_α and $p = \pi(p')$. Assume we have coordinate systems $u_1, \dots, u_M$ on P' and $z_1, \dots, z_M$ near p' and p respectively so that $u_1 = 0$ is E'_{α_1}, and $z_j, 1 \le j \le s$ are the divisors $E_\bullet$ passing though p. Therefore z_j for $1 \le j \le s$ pull back to functions divisible by u_1 (in a neighborhood of p'). Also the zeros of $z_j = 0$ for $1 \le j \le s$ are contained in $E'_{\alpha'}$. Hence we may write for $1 \le i \le s$

$$z_i = u_1^{m_i} f(z')$$

with $f(u)$ invertible near p' and $m_i > 0$. The residue of the pullback of such $\operatorname{dlog} z_i$ is m_i, and η pulls back to a form with residue along $E'_{\alpha'}$ given by a linear combination $\sum_{i=1}^s m_i a_i$ where a_i is the residue of η along $z_i = 0$ $(1 \le i \le s)$. This linear combination is clearly a positive integer as desired. ☐

There is another way of getting canonical morphisms in this picture: Let

$$\hat{V}' = P' - \cup' E'_\beta, \quad \hat{V} = P - \cup' E_\alpha$$

with the union restricted to β and α with a'_β and a_α in the set $\mathbb{R} - \{0, -1, -2, -3, \dots\}$. Let $j : U \to \hat{V}$ and $j' : U \to \hat{V}'$. Using this set-up, we may create a morphism which goes in a direction opposite to (29).

$$\underline{\mathcal{L}}(a) \to R\pi_* \underline{\mathcal{L}'}(a) \tag{31}$$

Using adjunction properties of $j_!$ (see Section 2.3), it suffices to construct such a morphism over $\hat{V}$. Over $\hat{V}$, the residues over divisors in P' are sums of elements in $\{0, -1, -2, -3, \dots\}$ (by using the same argument as in Lemma 30), and hence we extend using Rk_* from U in P', therefore we are again done using adjunction.

2.5.1. *Compatibility with Aomoto cohomology.* We know that $(\Omega^\bullet_P(\log E), d + \eta)$ represents $\underline{\mathcal{L}}(a)$. There is an evident morphism

$$(\Omega^\bullet_P(\log E), d + \eta) \to (\pi_*\Omega^\bullet_{P'}(\log E'), d + \eta) \to R\pi_*(\Omega^\bullet_{P'}(\log E'), d + \eta)$$

and hence a morphism

$$\underline{\mathcal{L}}(a) \to R\pi_* \underline{\mathcal{L}'}(a)$$

which has to coincide with the canonical morphism (31) by adjunction properties of morphisms. By [FC90, Lemma VI.3.4], for any i, $\Omega^i_{P'}(\log E')$ is acyclic for the functor π_* (i.e., higher direct images vanish), and $\pi_*\Omega^i_{P'}(\log E') = \Omega^i_P(\log E)$. Therefore,

Proposition 32. *The morphism* (31) *is a quasi-isomorphism.*

The following determines the inverse of the quasi-isomorphism (31).

Proposition 33. *The morphism* (29) *is the inverse of* (31).

Proof. It suffices to show that the composition

$$\underline{\mathcal{L}}(a) \to R\pi_* \underline{\mathcal{L}'}(a) \to \underline{\mathcal{L}}(a) \tag{34}$$

is an quasi-isomorphism. By various adjunction properties,

$$\mathrm{Hom}_P(\underline{\mathcal{L}}(a), \underline{\mathcal{L}}(a)) = \mathrm{Hom}_U(\mathcal{L}(a), \mathcal{L}(a))$$

therefore any homomorphism $\underline{\mathcal{L}}(a) \to \underline{\mathcal{L}}(a)$ is determined by its restriction to U and the proposition follows. □

Remark 35. *If $k : V \subset P$ is any open subset and $F \in D^b_c(V)$ and $G \in D^b_c(P)$ then we have adjunctions* [KS94, Proposition 2.6.4 and 3.1.12] $\mathrm{Hom}_P(k_!F, G) = \mathrm{Hom}_V(F, G_V)$ *and* $\mathrm{Hom}_P(G, Rk_*F) = \mathrm{Hom}_V(G_V, F)$*, where G_V is the pull back of G to the open subset V.*

Remark 36. *A log M form on U gives elements in $H^M(P, \underline{\mathcal{L}}(a))$ and $H^M(P', \underline{\mathcal{L}'}(a))$. By Proposition 33 and the above, these elements correspond under the map on global cohomology induced by* (29) *and* (31).

3. The main morphism

Definition 37. *Let X be a complex algebraic variety. For an element $F \in D^b_c(X)$, the Verdier dual DF is defined as follows*

$$DF = \mathcal{R}hom(F, a^!_X\mathbb{C})$$

where $a_X : X \to \mathrm{Spec}(\mathbb{C})$.

Recall that for any open inclusion $j : V \to X$, and $F \in D^b_c(V)$, $D(j_! F) = Rj_* D(F)$ and $D(Rj_* F) = j_! D(F)$. Moreover if $\mathcal{L}$ is a local system on V, $D(\mathcal{L})[-2M] = \mathcal{L}^*$ where M is the dimension of V. Using the description (27) of $\underline{\mathcal{L}}(-a)$, we see that the Verdier dual $D(\underline{\mathcal{L}}(-a))[-2M]$ is almost the same as $\underline{\mathcal{L}}(a)$, the only difference is extension over the blue divisors (divisors E_α such that $a_\alpha = 0$)): In $\underline{\mathcal{L}}(a)$, the extension is Rj_* and in $D(\underline{\mathcal{L}}(-a))$, it is $j_!$. Therefore, in the notation of Section 2.4 (with colors on divisors for a)

$$D(\underline{\mathcal{L}}(-a))[-2M] = Rj_{1,?} Rj_{2,?} \dots Rj_{n,?} \mathcal{L}(a) \tag{38}$$

Here the ? in $Rj_{i,?}$ is ! if the color on E_i is white or blue, and $*$ if black, and either ! or $*$ if the color is green (both produce the same answer). Therefore there is an obvious map

$$D(\underline{\mathcal{L}}(-a))[-2M] \to \underline{\mathcal{L}}(a). \tag{39}$$

Recall that M is the complex dimension of P.

Lemma 40.

$$\mathrm{Hom}_U\big(D(\underline{\mathcal{L}}(-a)), \underline{\mathcal{L}}(a)\big) = \mathbb{C}. \tag{41}$$

Therefore, up to scale there is only possible morphism between the objects that appear in (39).

Proof. We first note the following: X be a complex algebraic variety, j:$V \to X$ an open inclusion, and $F, G \in D^b_c(V)$, Then $j_!$ and Rj_* are fully faithful: $\mathrm{Hom}_X(j_! F, j_! G) = \mathrm{Hom}_V(F, G)$ and similarly for Rj_*. These properties follow from adjunction (see Remark 35). In addition we also have (again by adjunction) $\mathrm{Hom}_X(j_! F, Rj_* G) = \mathrm{Hom}_V(F, G)$.

By the description (38) and (27) of $D(\underline{\mathcal{L}}(-a))$ and $\underline{\mathcal{L}}(a)$ respectively, we now get

$$\mathrm{Hom}_P(D\big(\underline{\mathcal{L}}(-a)\big)[-2M], \underline{\mathcal{L}}(a)) = \mathrm{Hom}_U\big(D(\underline{\mathcal{L}}(-a))[-2M], \underline{\mathcal{L}}(a)\big) \tag{42}$$

But $D(\underline{\mathcal{L}}(-a))[-2M]$ restricted to U equals the local system $\mathcal{L}(a)$, and hence the last space in (42) equals

$$\mathrm{Hom}_U(\mathcal{L}(a), \mathcal{L}(a)\big) = \mathbb{C}$$

□

Remark 43. *By* [EV86, Appendix A]*,*

$$D(\underline{\mathcal{L}}(-a))[-2M] \xrightarrow{\sim} (\Omega^\bullet_P(\log E)(-E), d + \eta)$$

and the map (39) *is the natural map,*

$$(\Omega^\bullet_P(\log E)(-E), d + \eta) \to (\Omega^\bullet_P(\log E), d + \eta) \tag{44}$$

3.1. **Small weights.** If the weights a are sufficiently small, then $H^M(P, \underline{\mathcal{L}}(a)) = H^M(U, \mathcal{L}(a))$ because there $j_!$ extensions are not used. Similarly, $H^M(P, D(\underline{\mathcal{L}}(-a))[-2M]) = H^M_c(U, \mathcal{L}(a))$.

3.2. The map (39) in global cohomology.

$$H^i(P, D(\underline{\mathcal{L}}(-a))[-2M]) = H^{2M-i}(P, \underline{\mathcal{L}}(-a))^* \to H^i(P, \underline{\mathcal{L}}(a)) \tag{45}$$

This map is non-zero only for $i = M$: the RHS vanishes for $i > M$ since it is computed by Aomoto cohomology, and the LHS vanishes for $i < M$. Therefore the only map of interest is

$$H^M(P, D(\underline{\mathcal{L}}(-a))[-2M]) \to H^M(P, \underline{\mathcal{L}}(a)) \tag{46}$$

Therefore we have a canonical element

$$\Sigma \in H^M(P, \underline{\mathcal{L}}(-a)) \otimes H^M(P, \underline{\mathcal{L}}(a)) = H^{2M}(P \times P, \underline{\mathcal{L}}(a) \boxtimes \underline{\mathcal{L}}(-a)) \tag{47}$$

Now $A^{2M}(U \times U) = A^M(U) \otimes A^M(U)$ surjects onto $H^{2M}(P \times P, \underline{\mathcal{L}}(a) \boxtimes \underline{\mathcal{L}}(-a))$, and we will show that Σ is represented by an explicit form as follows:

Theorem 48. *Let (as in (6), with $f_i^{(1)}(x,y) = f_i(x)$ and $f_i^{(2)}(x,y) = f_i(y)$)*

$$S = \sum_{1 \le i_1 < \cdots < i_M \le r} \prod_{s=1}^{M} a_{i_s} \operatorname{dlog} f_{i_s}^{(1)} \operatorname{dlog} f_{i_s}^{(2)} \in A^{2M}(U \times U) \tag{49}$$

Then, there exists a non-zero constant $c \in \mathbb{C}$ such that the image of cS in $H^{2M}(P \times P, \underline{\mathcal{L}}(a) \boxtimes \underline{\mathcal{L}}(-a))$ equals Σ.

In the case of small weights, Theorem 48 specializes to Theorem 5.

Remark 50. *We can, for arbitrary weights, also consider the map $H_c^M(U, \mathcal{L}(a)) \to H^M(U, \mathcal{L}(a))$ which is easily see to factor through (46). The form S induces an element in $H^{2M}(U \times U, \mathcal{L}(a) \boxtimes \mathcal{L}(-a))$, which by Theorem 48, represents $H_c^M(U, \mathcal{L}(a)) \to H^M(U, \mathcal{L}(a))$. Therefore, Theorem 5 holds for arbitrary weights.*

3.3. **A restatement of Theorem 48.** Consider the diagram

$$\begin{array}{ccccc} H^M(P, D(\underline{\mathcal{L}}(-a))[-2M]) & \xrightarrow{\sim} & H^M(P, \underline{\mathcal{L}}(-a))^* & \xrightarrow{\sim} & \left(\frac{A^M(U)}{\eta \wedge A^{M-1}(U))}\right)^* \\ & \searrow^{\alpha} & \downarrow & & \downarrow S \\ & & H^M(P, \underline{\mathcal{L}}(a)) & \xleftarrow[\sim]{} & \frac{A^M(U)}{\eta \wedge A^{M-1}(U)} \end{array} \tag{51}$$

Here α is the map (39). The triangle on the left commutes because of the commutative diagram (60). Theorem 48 implies that the square on the right commutes. The vertical map on the right is the composite:

$$\left(\frac{A^M(U)}{\eta \wedge A^{M-1}(U))}\right)^* \to A^M(U)^* \xrightarrow{S} A^M(U) \to \left(\frac{A^M(U)}{\eta \wedge A^{M-1}(U))}\right) \tag{52}$$

3.4. **Action of symmetries.** Suppose a finite group G acts on the hyperplane arrangement, permuting the hyperplanes, and preserving the corresponding weights. Then G acts on $P \times P$ as well, and preserves the degree $2M$ form S (since any two forms commute) in the statement of Theorem 48. All objects and maps in (51) are preserved under the action of this finite group G.

3.5. **Change of compactification.** Let P' be another compactification of P, and we assume as before that there is a regular birational morphism $\pi : P' \to P$. We use notation from Section 2.5 in this section. There is a natural commutative diagram, note D commutes with $R\pi_*$, and maps (29) and (34). The commutativity is because of adjunction properties again.

$$\begin{array}{ccc} D(\underline{\mathcal{L}}(-a))[-2M]) & \longrightarrow & \underline{\mathcal{L}}(a) \\ \downarrow & & \uparrow \\ R\pi_*(D(\underline{\mathcal{L}'}(-a))) & \longrightarrow & R\pi_*(\underline{\mathcal{L}'}(a)) \end{array} \tag{53}$$

Therefore the image of the map (45) does not depend upon the compactification chosen, under the identification of $H^\bullet(P, \underline{\mathcal{L}}(a))$ with the cohomology of the Aomoto complex $(A^\bullet(U), \eta\wedge)$. The following is now immediate,

Lemma 54. *The image of the map* (45) *inside the cohomology of the Aomoto complex is*

$$\{x \in \frac{A^M(U)}{\eta \wedge A^{M-1}(U)} \mid \exists\, \tau \in A^M(U)^*,\ \tau(\eta \wedge A^{M-1}(U)) = 0,\ x = [S(\tau)]\}$$

Therefore the image, under this identification, does not change if each a_i is multiplied by multiplied by the same non-zero scalar.

3.5.1. *Representation of image by algebraic forms.*

Question 55. *Is the image of* (45) *for $i = M$ (the image is zero otherwise), the linear span of $[\Omega]$ with $\Omega \in A^M(U)$ such that Ω does not have poles on divisors E_α with $a_\alpha = 0$?*

Lemma 56. *The image of* (45) *for $i = M$ contains the linear span of $[\Omega]$ with $\Omega \in A^M(U)$ such that Ω does not have poles on divisors E_α with $a_\alpha = 0$.*

Proof. Let Ω be as in the statement of the lemma. By Lemma 54, we are allowed to replace the numbers a_i by ca_i where $|c|$ is very small. We then want to show the following:

- The element $[\Omega] \in H^M(U, \mathcal{L}(a))$ is in the image of the mapping $H^M_c(U, \mathcal{L}(a)) \to H^M(U, \mathcal{L}(a))$.

Let $V = P - \cup E'_\alpha$ where the union is over α such that $a_\alpha \neq 0$. Let $j : U \to V$. Then $H^M_c(U, \mathcal{L}(a)) = H^M(V, j_!\mathcal{L}(a))$ and $H^M(U, \mathcal{L}(a)) = H^M(V, Rj_*\mathcal{L}(a))$. By [EV86], and Remark 43 there is a factorization

$$H^M(V, j_!\mathcal{L}(a)) \xrightarrow{\sim} H^M(V, \Omega^\bullet(\log E)(-E)) \to H^M(V, \Omega^\bullet(\log E)) \xrightarrow{\sim} H^M(V, Rj_*\mathcal{L}(a)).$$

Now any Ω with the pole property above gives a class in $H^0(V, \Omega^M(\log E)(-E))$ which is $d+\eta$ closed, and hence an element in $H^M(V, \Omega^\bullet(\log E)(-E))$. The corresponding element of $H^M(V, \Omega^\bullet(\log E)) = H^M(V, Rj_*\mathcal{L}(a))$ is the image of $[\Omega]$ under the isomorphism $\frac{A^M(U)}{\eta\wedge A^{M-1}(U)} \to Rj_*\mathcal{L}(a)$. □

In the case of arrangements coming from representation theory, the reverse implication in the above question is also true (for suitable isotypical components), see Section 10.3 and Proposition 115.

4. Steps in the proof of Theorem 48

4.1. **The product hyperplane arrangement.** On $W \times W$ define polynomials of degree one $f_i^{(1)}(x,y) = f_i(x)$ and $f_i^{(2)}(x,y) = f_i(y)$. This defines a hyperplane arrangement $\mathcal{C}'$ in $W \times W$ given by hyperplanes $f_i^{(b)} = 0, b = 1,2,\ i = 1,\dots,r$. We get a weighted arrangement by assigning the weights $\tilde{a}$: assign a_i to $f_i^{(1)}(x,y) = 0$ and $-a_i$ to $f_i^{(2)}(x,y) = 0$. The associated differential form is

$$\eta' = \sum_{i=1}^r a_i(\operatorname{dlog} f_i^{(1)} - \operatorname{dlog} f_i^{(2)}) = \eta^{(1)} - \eta^{(2)}, \tag{57}$$

where $\eta^{(b)} = \sum_{i=1}^r a_i \operatorname{dlog} f_i^{(b)},\ b = 1,2$.

Note that $P\times P - U\times U$ is a divisor with normal crossings E'. The resulting object in the derived category of $P \times P$, i.e., $\underline{\mathcal{L}}(\tilde{a})$ equals $\underline{\mathcal{L}}(a) \boxtimes \underline{\mathcal{L}}(-a)$, this can be seen also by showing the corresponding Aomoto cohomology for $P \times P$ equals the external product of the Aomoto complexes.

There is a switching factors automorphism σ of $P \times P$ which preserves the arrangements, but is -1 on weights of hyperplanes: $\sigma(x,y) = (y,x)$.

4.2. **Main steps.** Let $\mathcal{K} = \underline{\mathcal{L}}(a) \boxtimes \underline{\mathcal{L}}(-a) \in D^b_c(P \times P)$

(1) We will construct a non-zero class
$$\delta_\alpha \in H^{2M}_\Delta(P \times P, \mathcal{K})$$
which maps to $\Sigma \in H^{2M}(P \times P, \mathcal{K})$. The details can be found in Section 5. Recall that Σ was defined in Section 3.2 (see (47)). We also explain in Section 5 that it suffices to show that the image of δ_α in $H^{2M}(P \times P, \mathcal{K})$ coincides with the class $[S]$.

(2) In Lemma 68, we show that $H^{2M}_\Delta(P \times P, \mathcal{K}) = \mathbb{C}$, and that δ_α is a generator of this group.

(3) We construct a *non-zero* element $S_\Delta \in H^{2M}_\Delta(P \times P, \mathcal{K})$ which maps to $[S] \in H^{2M}(P \times P, \mathcal{K})$, where S is as in Theorem 48. We show that S_Δ is a non-zero multiple of δ_α. We refer to Section 7 for details.

The following consistency check is a good way of viewing the third step above. For such a S_Δ to exist, the image of $[S]$ in $H^{2M}(P \times P - \Delta, \mathcal{K})$ needs to be zero since we have an exact sequence (see the exact sequence on cohomology induced by the distinguished exact triangle in [KS94, Exercise I.2.5] with $S = S_{k-1}$ and $S_k = \emptyset$)
$$H^{2M}_\Delta(P \times P, \mathcal{K}) \to H^{2M}(P \times P, \mathcal{K}) \to H^{2M}(P \times P - \Delta, \mathcal{K})$$

For a linear function F on W, we can form $U_F = P \times P - \overline{Z(h)} \subseteq P \times P - \Delta$ where $h(x, y) = F(x) - F(y)$, and $Z(h) \subset U \times U$ is the zero set of h. Therefore the image of $[S]$ in $H^{2M}(U_F, \mathcal{K})$ should be zero. This would be true if the image of
$$S \in A^{2M}\big(U_F \cap (U \times U)\big) \subseteq H^0(U_F, \Omega^{2M}_{P\times P}(\log E'))$$
is exact, note that $U_F - E' = U_F \cap (U \times U)$. This follows from Corollary 82, which implies (see (77) for the definition of $S^{(M-1)}$)
$$S = (\eta^{(1)} - \eta^{(2)}) \wedge \operatorname{dlog} h \wedge S^{(M-1)} \tag{58}$$
Varying F, one could hope that U_F form an open cover of $P \times P - \Delta$, and that a generalization of (58) would show that $[S]$ vanishes in $H^{2M}(P \times P - \Delta, \mathcal{K})$. The element S_Δ can then be hoped to arise from a cone construction.

But it seems to be difficult to ensure that U_F form an open cover of $P \times P - \Delta$. Instead we use a slightly different strategy: We suitably blow up $P \times P$ outside of $U \times U$ and use a similar argument.

5. Some generalities

Let $\alpha : F \to G$ be a morphism in $D^b_c(P)$. We will use the considerations of this section with α the mapping (39). First note that α produces a cohomology class
$$\delta_\alpha \in H^0\big(P, \Gamma_\Delta(\mathcal{R}\mathrm{hom}\,(p_1^{-1}F, p_2^!G))\big) \tag{59}$$
by Proposition 3.1.14 of [KS]. Clearly δ_α also produces an element in H^0 of $\mathcal{R}\mathrm{hom}\,(p_1^{-1}F, p_2^!G)$, which is the object on the top right of the following diagram (the remaining objects and maps

are explained below):

(60)
$$\begin{array}{ccc}
R\Gamma(P\times P, DF\boxtimes G) & \longrightarrow & \mathrm{Rhom}(p_1^{-1}F, p_2^!G) \\
\uparrow & & \downarrow \\
R\Gamma(P, DF)\otimes R\Gamma(G) & & \mathrm{Rhom}((Ra_P)_*F, (Ra_P)_*G) \\
\uparrow & \nearrow & \\
R\Gamma(P, F)^*\otimes R\Gamma(G) & &
\end{array}$$

In the rest of this section we recall the maps in (60) which are all isomorphisms. The vertical map on the right in (60) is the morphism as in Proposition 3.1.15 of [KS]:

Lemma 61. *There is a natural isomorphism*

(62)
$$\mathrm{Rhom}(p_1^{-1}F, p_2^!G)\xrightarrow{\sim}\mathrm{RHom}((Ra_P)_!F, (Ra_P)_*G)$$

The map (62) arises as follows

$$\mathrm{Rhom}(p_1^{-1}F, p_2^!G) = (Ra_P)_*(Rp_2)_*\mathcal{R}\mathrm{hom}\,(p_1^{-1}F, p_2^!G)\to(Ra_P)_*\mathcal{R}\mathrm{hom}\,((Rp_2)_!p_1^{-1}F, G)$$

which in turn maps to

$$(Ra_P)_*\mathcal{R}\mathrm{hom}\,(a_P^{-1}(Ra_P)_!F, G)\to\mathrm{Rhom}((Ra_P)_!F, (Ra_P)_*G)$$

We note that the image of δ_α (defined in (59)) under (62) is the map induced by α:

$$(Ra_P)_!F\xrightarrow{\alpha}(Ra_P)_!G\to(Ra_P)_*G.$$

Lemma 63.

(64)
$$DF\boxtimes G\xrightarrow{\sim}\mathcal{R}hom\,(p_1^{-1}F, p_2^!G)$$

We recall how the morphism in Lemma 63 is constructed

$$DF\boxtimes G\to p_1^{-1}\mathcal{R}\mathrm{hom}\,(F, a_P^!\mathbb{C})\otimes p_2^{-1}G$$

Using equation (2.6.27), page 114 of [KS],

(65)
$$p_1^{-1}\mathcal{R}\mathrm{hom}\,(F, a_P^!\mathbb{C})\otimes p_2^{-1}G\to\mathcal{R}\mathrm{hom}\,(p_1^{-1}F, p_1^{-1}a_P^!\mathbb{C})\otimes p_2^{-1}G\to\mathcal{R}\mathrm{hom}\,(p_1^{-1}F, p_1^{-1}a_P^!\mathbb{C}\otimes p_2^{-1}G)$$

Using prop 3.1.9, (iii), page 146 of [KS] we have a map induced by adjunction:

$$p_1^{-1}a_P^!\mathbb{C}\to p_2^!a_P^{-1}(\mathbb{C})\to p_2^!\mathbb{C}$$

By Proposition 3.1.11, on page 147 of [KS], there is a natural map $p_2^!\mathbb{C}\otimes p_2^{-1}G\to p_2^!G$. We compose (65) with this morphism to complete the argument. Proposition 3.4.4 in [KS] shows that the constructed map is an isomorphism.

Lemma 66. *Let $F' = DF, G\in D_c^b(P)$. Then*

$$R\Gamma(P, F')\otimes R\Gamma(G)\xrightarrow{\sim}R\Gamma(P\times P, F'\boxtimes G)$$

and $R\Gamma(P, F') = R\Gamma(P, T)^*$.

The morphism in Lemma 66 arises as follows:

$$(Ra_P)_*((Rp_2)_* p_1^{-1} F' \otimes G) \xrightarrow{\sim} (Ra_P)_*(Rp_2)_*(p_1^{-1}F' \otimes p_2^{-1}G) \xrightarrow{\sim} R\Gamma(F' \boxtimes G)$$

Now $Ra_P^*(Ra_P)_*F' \to (Rp_2)_* p_1^{-1}F'$. Therefore we have an isomorphism map

$$(Ra_P)_*(Ra_P^*(Ra_P)_*F' \otimes G) \to R\Gamma(P\times P, F' \boxtimes G)$$

and hence

$$(Ra_P)_*F' \otimes (Ra_P)_*G \to R\Gamma(F' \boxtimes G)$$

Therefore all maps in (60) have been constructed, the key claim (standard) is

Lemma 67. *The diagram (60) commutes.*

Now let α be the mapping (39). We therefore have a class

$$\delta_\alpha \in H^{2M}_{\Delta}(P\times P, \underline{\mathcal{L}}(a)\boxtimes \underline{\mathcal{L}}(-a))$$

which maps to $\Sigma \in H^{2M}(P\times P, \underline{\mathcal{L}}(a)\boxtimes\underline{\mathcal{L}}(-a))$, which in turn induces the mapping (45).

Lemma 68. (a) *Setting $\tilde{k}:\Delta\cap(U\times U)\to\Delta$.*

$$i^!_\Delta(\underline{\mathcal{L}}(a)\boxtimes\underline{\mathcal{L}}(a)) = R\tilde{k}_*\mathbb{C}[2M]$$

(b) $H^{2M}_{\Delta}(P\times P, \underline{\mathcal{L}}(a)\boxtimes\underline{\mathcal{L}}(-a)) = \mathbb{C}$.

(c) *Under the isomorphism in (b), δ_α corresponds to $1\in\mathbb{C}$.*

Proof. We claim that for all points $p\in\Delta - U\times U$, there is a β such that $p\in E'_\beta$ with $a'_\beta\in\mathbb{C}-\{1,2,\dots\}$.

This is true because σ fixes p, and if $p\in E'_\beta$ then $p\in E'_{\sigma(\beta)}$. Therefore the claim follows from the identities, $a'_{\sigma(\beta)} = -a'_\beta$ ($\sigma(\eta')=-\eta'$, and $\sigma(E'_\beta)=E'_{\sigma(\beta)}$ by definition of $\sigma(\beta)$).

We can use standard base change properties (Proposition 3.1.9 on page 145, and Proposition 3.1.1 on page 147 of [KS94]). We also use the fact that $\widetilde{\mathcal{L}}(a)$ restricted to $\Delta\cap(U\times U)$ is trivial. This implies (a), and (b) follows from (a). To prove (c) we may localize in P, and reduce to a standard compatibility. We only need that the factor is non-zero which is clear because otherwise the map α in question, i.e., the map (39) would be zero on U. □

6. Cohomology class of the diagonal

Let $\mathcal{K} = \underline{\mathcal{L}}(a)\boxtimes\underline{\mathcal{L}}(-a)\in D^b_c(P\times P)$. Our aim in this section (see Proposition 85) is to construct an element $S_\Delta\in H^{2M}_\Delta(P\times P,\mathcal{K})$ which maps to $[S]\in H^{2M}(P\times P,\mathcal{K})$ (see Theorem 48 for the definition of S).

In this section we construct a suitable $\widetilde{P}$ birational to $P\times P$. Set $\widetilde{\Delta}\subset\widetilde{P}$ the strict transform of Δ, and let $\widetilde{\underline{\mathcal{L}}}(a)$ denote the object in the derived category of sheaves on $\widetilde{P}$ for the product arrangement (with weights a and $-a$). By Section 2.5, we know that $\widetilde{\underline{\mathcal{L}}}(a)$ is represented by the Aomoto complex. We show

1. $[S]\in H^{2M}(\widetilde{P},\widetilde{\underline{\mathcal{L}}}(a))$ goes to zero when restricted to $\widetilde{U}=\widetilde{P}-\widetilde{\Delta}$.
2. Carrying out the previous step keeping track of forms that appear in the vanishing, we construct an element in $H^{2M}_{\widetilde{\Delta}}(\widetilde{P},\widetilde{\underline{\mathcal{L}}}(a))$.
3. $H^{2M}_{\widetilde{\Delta}}(\widetilde{P},\widetilde{\underline{\mathcal{L}}}(a))$ maps to $H^{2M}_{\Delta}(P\times P,\mathcal{K})$. We then define S_Δ to be the image of the element in the previous step.

6.1. **Functorial Resolution and the diagonal.** Let $F_1 = 0, \dots, F_M = 0$ be linear hyperplanes in W such that $dF_1, \dots, dF_M$ are linearly independent. The diagonal in $W \times W$ is cut out by $h_j(x, y) = F_j(x) - F_j(y) = 0$ with $j = 1, \dots, M$. These meet transversally in $W \times W$. Let $Z(h_j)$ be the variety of zeroes of h_j on $U \times U$, and $\overline{Z(h_j)} \subset P \times P$ its closure.

Remark 69. *The intersection $\cap_{j=1}^{M} \overline{Z(h_j)} \subset P \times P$ may be bigger than the diagonal $\Delta \subset P \times P$, and therefore the complements $P \times P - \overline{Z(h_j)}$ may not cover $P \times P - \Delta$. We pass to a blowup of $P \times P$ such that the strict transforms of $\overline{Z(h_j)}$ meet properly. It then follows that the strict transform meet in a locus which maps to $\Delta \subset P \times P$.*

Let $\pi : \widetilde{P} \to P \times P$ be a functorial resolution of singularities [BVP15], following Section 2.1, of the pair $(P \times P, E' \cup \bigcup_{j=1}^{M} \overline{Z(h_j)})$, where $E' = P \times P - U \times U$. Since the zero loci $Z(h_j)$ meet transversally in $U \times U$, by property (d) in Section 2.1, π is an isomorphism over $U \times U$. Set

$$\widetilde{Z}(h_j) = \overline{\{(x, y) \in U \times U : F_j(x) = F_j(y)\}} \subseteq \widetilde{P}.$$

Let $\widetilde{E} = \pi^{-1}(E' \cup \bigcup_{j=1}^{M} \overline{Z(h_j)})$, a divisor with simple normal crossings. Write $\widetilde{E}$ as a union $\cup_\beta \widetilde{E}_\beta$, let $\tilde{a}_\beta$ be the residue of $\eta' = \eta^{(1)} - \eta^{(2)}$ along $\widetilde{E}_\beta$. As will be clear from what follows, $\widetilde{P}$ is a much better place for actual computations.

Definition 70. *It is clear that $\widetilde{Z}(h_j)$ are irreducible components of $\widetilde{E}$. Write*

$$\widetilde{E} = \widetilde{E}_0 \cup \left(\bigcup_{j=1}^{M} \widetilde{Z}(h_j)\right)$$

where $\widetilde{E}_0$ is the union of the other irreducible components.

- The automorphism σ of switching the two factors in $P \times P$ lifts to $\widetilde{P}$, so that each $\widetilde{E}_\beta$ goes to $\widetilde{E}_{\beta'}$ with $\tilde{a}_{\beta'} + \tilde{a}_\beta = 0$ (canonical resolution of singularities).
- Let $\widetilde{\Delta}$ be the closure of $\Delta \cap (U \times U)$ in $\widetilde{P}$. It is easy to see that $\cap_{j=1}^{M} \widetilde{Z}(h_j) = \widetilde{\Delta}$ using transversality (and dimension counting). $\widetilde{\Delta}$ maps to Δ under $\widetilde{P} \to P \times P$.

Let $\widetilde{\underline{\mathcal{L}}}(a)$ denote the object in the derived category of sheaves on $\widetilde{P}$ for the arrangement, (without the hyperplanes $h_j = 0$), and the form $\sum_{i=1}^{r} a_i(\operatorname{dlog} f_i^{(1)}(x, y) - \operatorname{dlog} f_i^{(2)}(x, y))$, this is quasi-isomorphic to $(\Omega^\bullet_{\widetilde{P}}(\log \widetilde{E}_0), d + \eta^{(1)} - \eta^{(2)})$. Recall the notation Let $\mathcal{K} = \underline{\mathcal{L}}(a) \boxtimes \underline{\mathcal{L}}(-a)$.

By Section 29, we have a quasi-isomorphism

$$R\pi_* \widetilde{\underline{\mathcal{L}}}(a) \to \mathcal{K} \tag{71}$$

$$\begin{array}{ccccc}
H^{2M}_{\widetilde{\Delta}}(\widetilde{P}, \widetilde{\underline{\mathcal{L}}}(a)) & \longrightarrow & H^{2M}(\widetilde{P}, \widetilde{\underline{\mathcal{L}}}(a)) & \longrightarrow & H^{2M}(\widetilde{U}, \widetilde{\underline{\mathcal{L}}}(a)) \\
\downarrow & & \downarrow & & \downarrow \\
H^{2M}_{\Delta}(P \times P, \mathcal{K}) & \longrightarrow & H^{2M}(P \times P, \mathcal{K}) & \longrightarrow & H^{2M}(P \times P - \Delta, \mathcal{K})
\end{array} \tag{72}$$

Recall that $\widetilde{U} = \widetilde{P} - \widetilde{\Delta}$.

The middle vertical arrow is an isomorphism. The vertical arrow on the far right is obtained by restricting (71) to $P \times P - \Delta$, using the restriction map

$$H^{2M}(\widetilde{U}, \widetilde{\underline{\mathcal{L}}}(a)) \to H^{2M}(\pi^{-1}(P \times P - \Delta), \widetilde{\underline{\mathcal{L}}}(a)).$$

The vertical map on the far left is induced by the isomorphism [KS94, Proposition 3.1.9]

$$R\pi'_* i^!_{\pi^{-1}\Delta}\widetilde{\underline{\mathcal{L}}}(a) \to i^!_\Delta R\pi_*\widetilde{\underline{\mathcal{L}}}(a)$$

and the inclusion $\widetilde{\Delta} \subseteq \pi^{-1}\Delta$. The following lemma, which uses property (e) of resolution of singularities from Section 2.1, shows that this vertical map is an isomorphism. This result is not used, but gives a more satisfying picture.

Lemma 73. *$i^!_{\widetilde{\Delta}}\widetilde{\underline{\mathcal{L}}}(a)) = Rk'_*\mathbb{C}[2M]$ where $k' : \Delta \cap (U \times U) \to \widetilde{\Delta}$.*

Proof. We claim that for all points $p \in \widetilde{\Delta} - U \times U$, there is a β such that $p \in \widetilde{E}_\beta$ with $\tilde{a}_\beta \in \mathbb{R} - \{1, 2, \dots\}$.

This is true because $\sigma : \widetilde{P} \to \widetilde{P}$ fixes p, and if $p \in \widetilde{E}_\beta$ then $p \in \widetilde{E}_{\sigma(\beta)}$. Therefore the claim follows from the identity

$$\tilde{a}_{\sigma(\beta)} = -\tilde{a}_\beta$$

($\sigma(\eta') = -\eta'$, and $\sigma(\widetilde{E}_\beta) = \widetilde{E}_{\sigma(\beta)}$ by definition of $\sigma(\beta)$).

We have used standard base change properties (Proposition 3.1.9 on page 145, and Proposition 3.1.1 on page 147 of [KS94]), also that $\widetilde{\underline{\mathcal{L}}}(a)$ restricted to $\Delta \cap (U \times U)$ is trivial. □

Proposition 74. *$S \in H^M(\widetilde{P}, \widetilde{\underline{\mathcal{L}}}(a))$ goes to zero in $H^M(\widetilde{U}, \widetilde{\underline{\mathcal{L}}}(a))$, where $\widetilde{U} = \widetilde{P} - \widetilde{\Delta}$.*

Now, $\widetilde{U} = \widetilde{P} - \widetilde{\Delta}$ has an open covering by open subsets of the form $\widetilde{P} - \widetilde{Z}(h_j)$. An intersection of these open subsets has the form

$$\widetilde{P}_J = \widetilde{P} - \bigcup_{j\in J}\widetilde{Z}(h_j) = \bigcap_{j\in J}\big(\widetilde{P} - \widetilde{Z}(h_j)\big).$$

where $J \subset \{1, \dots, M\}$.

6.2. **Čech complexes.** For a sheaf $\mathcal{F}$ on $\widetilde{U} = \widetilde{P} - \widetilde{\Delta}$, let

$$C^p(\mathcal{F}) = \prod_{j_0<\cdots<j_p} \mathcal{F}(\widetilde{P}_{\{j_0,\dots,j_p\}})$$

Elements $\alpha \in C^p(\mathcal{F})$ are determined by giving elements

$$\alpha_{j_0,\dots,j_p} \in \mathcal{F}(\widetilde{P}_{\{j_0,\dots,j_p\}}).$$

Define $\delta : C^p \to C^{p+1}$ by

$$(\delta\alpha)_{j_0,\dots,j_{p+1}} = \sum_{k=0}^{p+1}(-1)^k\alpha_{j_0,\dots,\hat{j}_k,\dots j_{p+1}}$$

Also form the sheaf version (with a corresponding differential)

$$\mathcal{C}^p(\mathcal{F}) = \prod_{j_0<\cdots<j_p}(k_J)_*\mathcal{F} \mid \widetilde{P}_{\{j_0,\dots,j_p\}}$$

where k_J denotes the inclusion of $\widetilde{P}_{\{j_0,\dots,j_p\}} \to \widetilde{P} - \widetilde{\Delta}$. It is known that there is a quasi-isomorphism $\mathcal{F} \to (\mathcal{C}^p(\mathcal{F}), \delta)$ (see [Har77, Lemma III.4.2]).

Now suppose $(\mathcal{F}^\bullet, d_{\mathcal{F}})$ is a complex of sheaves. Let $(\mathcal{C}^\bullet(\mathcal{F}^\bullet, d_{\mathcal{F}}), D)$ be the complex with

$$\mathcal{C}^n(\mathcal{F}^\bullet, d_{\mathcal{F}}) = \oplus_{p+q=n}\mathcal{C}^p(\mathcal{F}^q)$$

and differential given by $D = d_{\mathcal{F}} + \delta$. It is again known that there is a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{C}^\bullet(\mathcal{F}^\bullet)$.

6.3. **Čech complex for twisted log de Rham complex.** Let, as before, $\widetilde{U} = \widetilde{P} - \widetilde{\Delta}$. Now, $\widetilde{U}$ has a covering $\widetilde{P} - \widetilde{Z}(h_j)$. Let $\mathcal{F}^\bullet = (\Omega^\bullet_{\widetilde{P}}(\log \widetilde{E}_0), d + \eta^{(1)} - \eta^{(2)})$ which is quasi-isomorphic to $\underline{\widetilde{\mathcal{L}}}(a)$. ($\widetilde{E}_0$ was defined in Definition 70). Let $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism where $\mathcal{I}^\bullet$ is a complex of injectives. We obtain a commutative diagram of complexes

$$\begin{array}{ccc} & & \Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{F}^\bullet)) \\ & \nearrow & \uparrow \\ \Gamma(\widetilde{P}, \mathcal{F}^\bullet) & \longrightarrow & \Gamma(\widetilde{U}, \mathcal{F}^\bullet) \end{array}$$

Here $\mathcal{C}^\bullet(\mathcal{F}^\bullet)$ is the Čech complex of $\mathcal{F}^\bullet$ on $\widetilde{U}$, see Section 6.2 for the definition. The above diagram maps to a similar diagram associated to $\mathcal{I}^\bullet$: The vertical map in the diagram below is a quasi-isomorphism, and $\mathcal{C}^\bullet(\mathcal{I}^\bullet))$ is a complex of injectives.

$$\begin{array}{ccc} & & \Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{I}^\bullet)) \\ & \nearrow & \uparrow \\ \Gamma(\widetilde{P}, \mathcal{I}^\bullet) & \longrightarrow & \Gamma(\widetilde{U}, \mathcal{I}^\bullet) \end{array}$$

Hence we obtain

$$\begin{array}{ccc} \Gamma(\widetilde{P}, \mathcal{F}^\bullet) & \longrightarrow & \Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{F}^\bullet)) \\ \downarrow & & \downarrow \\ \Gamma(\widetilde{P}, \mathcal{I}^\bullet) & \longrightarrow & \Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{I}^\bullet)) \end{array} \tag{75}$$

We are given $[S] \in H^{2M}(\Gamma(\widetilde{P}, \mathcal{F}^\bullet)) = H^{2M}(\Gamma(\widetilde{P}, \mathcal{I}^\bullet))$. Therefore, to prove Proposition 74, it suffices to show that the image of $[S]$ in $H^{2M}(\Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{F}^\bullet)))$ is zero. Note that $H^{2M}\Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{I}^\bullet)) = H^{2M}(\widetilde{U}, \mathcal{F}^\bullet)$.

Let

$$D^n = \oplus_{p+q=n} \oplus_J A^q(U_J)$$

where J runs through $J = \{j_0 < \cdots < j_p\}$ and

$$U_J = \widetilde{P}_J \cap (U \times U) = \{(x, y) \mid \forall i \in J, F_i(x) \neq F_i(y)\}.$$

Note that $D^n \subset \mathcal{C}^n(\mathcal{F}^\bullet)$: This is because for any J,

$$A^q(U_J) \subset H^{0,\mathrm{an}}(\widetilde{P}_J, \Omega^q_{\widetilde{P}}(\widetilde{E}_0))$$

The last inclusion needs an explanation: $\widetilde{P}$ is a compactification of U_J, with complement

$$D_J = \widetilde{E}_0 \cup \bigcup_{j \in J} \widetilde{Z}(h_j).$$

Therefore, $A^q(U_J) = H^0(\widetilde{P}, \Omega^q_{\widetilde{P}}(D_J))$, justifying the inclusion above.

6.4. **Some log forms.** The following Shapovolov form, a log form on $U \times U$ was defined in equation (6)

$$S = \sum_{1 \le i_1 < \cdots < i_M \le r} \prod_{s=1}^{M} a_{i_s} \,\mathrm{dlog}\, f_{i_s}^{(1)} \,\mathrm{dlog}\, f_{i_s}^{(2)} \tag{76}$$

In fact for any $0 \le b \le M$, we may define log forms on $U \times U$ of degree b:

$$S^{(b)} = \sum_{1 \le i_1 < \cdots < i_b \le r} \prod_{s=1}^{b} a_{i_s} \,\mathrm{dlog}\, f_{i_s}^{(1)} \,\mathrm{dlog}\, f_{i_s}^{(2)} \tag{77}$$

so that $S^{(0)} = 1$ and $S^{(M)} = S$.

Further for every $\{q_1 < \cdots < q_w\} \subseteq [1, M]$, and b such that $w + b = M$, we define a log form on $U \times U - \cup_{j=1}^{w} Z(F_{q_j}^{(1)} - F_{q_j}^{(2)})$. Here $F_j^{(1)}(x, y) = F_j(x)$ and $F_j^{(2)}(x, y) = F_j(y)$, note that we had earlier defined $h_j(x, y) = F_j(x) - F_j(y) = F_j^{(1)}(x, y) - F_j^{(2)}(x, y)$.

$$S_{q_1 \ldots q_w} := S^{(b)} \prod_{j=1}^{w} \mathrm{dlog}(F_{q_j}^{(1)} - F_{q_j}^{(2)})$$

a form of degree $2b + w = 2(M - w) + w = 2M - w$. When $w = 0$, we recover S.

Remark 78. *The form $S_{1,2,\ldots,M}$ corresponds to $w = M$ and $b = 0$, hence*

$$S_{1,2,\ldots,M} := \prod_{j=1}^{M} \mathrm{dlog}(F_j^{(1)} - F_j^{(2)}).$$

This form does not depend upon the weights a.

6.5. **Proof of Proposition 74.** The Shapovolov element $S|U_{\{i\}}$, and zero in all intersections gives a closed element in D^{2M}. We claim that this is zero in cohomology. For this we need elements $\alpha = \{\alpha_J\}$ in $D^{2M-1} = \oplus_{s+q=2M-1} \oplus_J A^q(U_J)$ with $J = \{j_0 < \cdots < j_s\}$ where $U_J = \widetilde{P}^J \cap (U \times U)$ (defined in Section 6.3). Define

$$\alpha_J = S_{j_0, j_1 \ldots, j_s} \tag{79}$$

which is a degree $q = 2M - (s + 1)$ form. In particular when $J = \{1 < 2 < \cdots < M\}$, we get $\alpha_J = \prod_{i=1}^{M} \mathrm{dlog}(F_j^{(1)} - F_j^{(2)})$. Proposition 80 below shows that the differential of α is the image of $[S]$ in $H^M(\Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{F}^\bullet)))$, proving Proposition 74.

Proposition 80. *Let $1 \le k \le M$, then*

$$\sum_{j=1}^{k} (-1)^{j+1} S_{q_1, \ldots \hat{q}_j, \ldots q_k} = (\eta^{(1)} - \eta^{(2)}) \wedge S_{q_1 \ldots q_k} \tag{81}$$

We give a proof of Proposition 80 in Section 8. We note the following corollary.

Corollary 82. *For any $q = 1, \ldots, M$, $S = (\eta^{(1)} - \eta^{(2)}) \wedge S_q$*

6.6. **Construction of the class supported on the diagonal from S.** Our aim in this section is to construct $S' \in H^M_{\widetilde{\Delta}}(\widetilde{P}, \underline{\widetilde{\mathcal{L}}}(a))$ which maps to $S \in H^M(\widetilde{P}, \underline{\widetilde{\mathcal{L}}}(a)) = H^M(P \times P, \mathcal{K})$. The element $S_\Delta \in H^M_\Delta(P \times P, \mathcal{K})$ will then be the image of S', see diagram (72).

Using notation from Section 6.3, consider the exact sequences:

$$0 \to \Gamma_{\widetilde{\Delta}}(\widetilde{P}, \mathcal{I}^\bullet) \to \Gamma(\widetilde{P}, \mathcal{I}^\bullet) \xrightarrow{f} \Gamma(\widetilde{U}, \mathcal{I}^\bullet) \to 0$$

The hypercohomology groups $H^\bullet_{\widetilde{\Delta}}(\widetilde{P}, \mathcal{F}^\bullet)$ are therefore canonically isomorphic to cohomology of the cone of f, which in turn is canonically isomorphic to the cohomology of the cone of $\Gamma(\widetilde{P}, \mathcal{I}^\bullet) \to \Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{I}^\bullet))$. We therefore obtain, using the diagram (75),

Lemma 83. *The cohomology of the cone of* $\Gamma(\widetilde{P}, \mathcal{F}^\bullet) \to \Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{F}^\bullet))$ *admits a mapping to* $H^\bullet_{\widetilde{\Delta}}(\widetilde{P}, \mathcal{F}^\bullet)$ *consistent with various exact sequences.*

Remark 84. *We have used the following standard fact about cones of morphisms: Suppose*

$$0 \to A^\bullet \to B^\bullet \xrightarrow{f} C^\bullet \to 0$$

is a short exact sequence of complexes (exact in each degree), then $A^\bullet$ *(with a shift) canonically maps to the cone of* f*, and this map is a quasi-isomorphism (see e.g.,* [Wei94, Exercise 1.5.7]*).*

The elements S and α from Section 6.5 give an explicit element in the cohomology of the cone of $\Gamma(\widetilde{P}, \mathcal{F}^\bullet) \to \Gamma(\widetilde{U}, \mathcal{C}^\bullet(\mathcal{F}^\bullet))$. We therefore get an element in in $H^\bullet_{\widetilde{\Delta}}(\widetilde{P}, \mathcal{F}^\bullet)$. Let $S_\Delta \in H^{2M}_\Delta(P \times P, \mathcal{K})$ be the image of this element under the vertical map on the left in the diagram (72). Therefore,

Proposition 85. *There is an element* $S_\Delta \in H^{2M}_\Delta(P \times P, \mathcal{K})$ *which maps to* $[S] \in H^{2M}(P \times P, \mathcal{K})$*. Here* $\mathcal{K} = \underline{\mathcal{L}}(a) \boxtimes \underline{\mathcal{L}}(-a) \in D^b_c(P \times P)$.

7. Proof of Theorem 48

7.1. **Some standard facts.** Consider the natural map $H^{2M-1}(\mathbb{A}^M - \{0\}, \mathbb{C}) \to H^{2M}_0(\mathbb{A}^M, \mathbb{C})$ coming from the standard supports sequence. We have a natural map $H^{2M}_0(\mathbb{A}^M, \mathbb{C}) = H^{2M}(i^!_0\mathbb{C}) = \mathbb{C}$ because of the standard isomorphism $i^!_0\mathbb{C} = \mathbb{C}[-2M]$. We hence obtain a map $H^{2M-1}(\mathbb{A}^M - \{0\}, \mathbb{C}) \to \mathbb{C}$.

We also have a map $H^{2M-1}(\mathbb{A}^M - \{0\}, \mathbb{C}) \to \mathbb{C}$ given by integration on the $2M-1$ sphere. This coincides with the map (up to a sign) with the map in the previous paragraph.

There is a map $H^0(\mathbb{A}^M - \cup_{i=1}^M D_i, \Omega^M) \to H^{2M-1}(\mathbb{A}^M - \{0\}, \mathbb{C})$, coming from the Čech covering of $\mathbb{A}^M - \{0\}$ by $\mathbb{A}^M - D_i$ where $D_i = (z_1, \ldots, z_M) \mid z_i = 0\}$, and the holomorphic de Rham resolution of $\mathbb{C}$. The composite map $H^0(\mathbb{A}^M - \cup_{i=1}^M D_i, \Omega^M) \to H^{2M-1}(\mathbb{A}^M - \{0\}, \mathbb{C}) \to \mathbb{C}$ is $\frac{1}{(2\pi\sqrt{-1})^M}$ times the Grothendieck residue. We record the following consequence (see [GH94, Page 651]):

Lemma 86. *The topological map* $H^0(\mathbb{A}^M - \cup_{i=1}^M D_i, \Omega^M) \to \mathbb{C}$ *sends* $\frac{dz_1}{z_1} \ldots \frac{dz_M}{z_M}$ *to* $(2\pi\sqrt{-1})^M$.

7.2. **Completion of the Proof of Theorem 48.** Recall from Section 4.2 the outline of the proof of Theorem 48. To complete the proof of Theorem 48, we need to show that S_Δ (constructed in Proposition 85) is a non-zero multiple of the generator δ_α of $H^M_\Delta(P \times P, \underline{\mathcal{L}}(a) \boxtimes \underline{\mathcal{L}}(-a))$.

The construction of S_Δ can be localized, to any open subset of $U \times U \subset \widetilde{P}$ of the form $A \times A$ where A is an open polydisc contained in U. Therefore S_Δ gives an element in the $2M$-th cohomology cone $\mathrm{Cone}_A(f)$ of

$$\Gamma(A \times A, \mathcal{F}^\bullet) \to \Gamma(A \times A - \Delta_A, \mathcal{C}^\bullet(\mathcal{F}^\bullet))$$

Consider the mapping $A \to A \times A$ given $\vec{z} \mapsto (\vec{z}, 0)$ where the forms F_i correspond to the coordinates. This map pulls back the diagonal to $0 \in A$. There is therefore a map from the H^{2M} of $\mathrm{Cone}_A(f)$ to H^{2M} of the cone $\mathrm{Cone}_0(f_0)$ of

$$\Gamma(A, \mathcal{F}'^\bullet) \to \Gamma(A - \{0\}, \mathcal{C}^\bullet(\mathcal{F}'^\bullet))$$

where $\mathcal{F}'^\bullet$ is the corresponding Aomoto complex of A. This map between the H^{2M} of the cones is clearly an isomorphism, since the cones compute topological cohomology groups with supports.

Therefore we need to show that the forms in the construction of S_Δ when pulled back to A together give a non-zero element in $H^{2M}_0(A, \mathcal{L}(a))$. It is clear that $\operatorname{dlog} f_i^{(2)}$ pull back to zero, and hence the pull back element in the complex $\mathrm{Cone}_0(f_0)$ sits in only one degree (see Remark 78 and Equation (79)), and is $\prod_{i=1}^M \operatorname{dlog} z_i \in H^M(A - \cup_{i=1}^M \{z_i = 0\}, \Omega^M)$. We need to multiply this function by a local generator of the local system $\mathcal{L}(a) \times \mathcal{L}(-a)_0$ and view it as an image of an element in $H^{2M-1}(A - \{0\}, \mathbb{C})$ (computed in Čech cohomology of the holomorphic de Rham complex for the coordinate covering $A - \{z_i = 0\}$) of $A - \{0\}$).

Let g is a holomorphic function on A which is a local generator of $\mathcal{L}(a)$, i.e., $dg + g\eta = 0$. The corresponding element of $H^{2M-1}(A - \{0\}, \mathbb{C})$ is the image of $gg(0)^{-1} \prod_{i=1}^M \operatorname{dlog} z_i \in H^M(A - \cup_{i=1}^M \{z_i = 0\}, \Omega^M)$. Here the local generator of $\mathcal{L}(a) \boxtimes \mathcal{L}(-a)_0$ is the function $gg(0)^{-1}$ (the second coordinate is a constant), which is one plus a function that vanishes at zero.

To see that $gg(0)^{-1} \prod_{i=1}^M \operatorname{dlog} z_i$ produces a non-zero element in $H^{2M-1}(A - \{0\}, \mathbb{C})$, we may apply Grothendieck residues [GH94, Page 651], and hence the proof of Theorem 48 is complete. In fact we only need to show that $\prod \operatorname{dlog} z_i$ produces a non-zero element in $H^{2M-1}(A - \{0\}, \mathbb{C})$, which is standard.

8. Proof of Proposition 80

For the proof of the proposition assume, without loss of generality, that $q_1 = 1, q_2 = 2, \ldots, q_k = k$. The left hand side of (81) is (with $(k-1) + b = M$, i.e., $k + b = M + 1$),

$$(87)\quad \sum_{c=1}^{k} (-1)^{c+1} S_{1,\ldots,\hat{c},\ldots,k}$$
$$= \sum_{c=1}^{k} (-1)^{c+1} \prod_{q=1, q\neq c}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \Bigg(\sum_{1 \le i_1 < \cdots < i_b \le r} \prod_{s=1}^{b} a_{i_s} \operatorname{dlog} f_{i_s}^{(1)} \operatorname{dlog} f_{i_s}^{(2)} \Bigg)$$

The right hand side of (81) is as follows

$$(88)\quad (\eta^{(1)} - \eta^{(2)}) \wedge S_{1\ldots,k}$$
$$= (\eta^{(1)} - \eta^{(2)}) \wedge \prod_{q=1}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \Bigg(\sum_{1 \le j_1 < \cdots < j_{b-1} \le r} \prod_{s=1}^{b-1} a_{j_s} \operatorname{dlog} f_{j_s}^{(1)} \operatorname{dlog} f_{j_s}^{(2)} \Bigg)$$

Let us compare the coefficients of $a_{i_1} \dots a_{i_b}$ on both sides of (81). Both coefficients are easily seen to be zero if $i_j = i_{j'}$ for some $j \neq j'$. We will assume that this is not the case.

Since $k + b = M + 1$, we may write a linear dependence equation of form $\sum_{q=1}^{k} c_q F_q + \sum_{s=1}^{b} b_s f_{i_s}$ equals a constant with at least one coefficient not zero. We may assume that this constant is zero, and $c_1 = 1$ (if the f_i are affinely dependent then both sides are zero). So after adjusting signs,

$$F_1 = \sum_{q=2}^{k} c_q F_q + \sum_{s=1}^{b} b_s f_{i_s}$$

We will now simply replace $F_1^{(1)} - F_1^{(2)}$ by

$$\sum_{q=2}^{k} c_q (F_q^{(1)} - F_q^{(2)}) + \sum_{s=1}^{b} b_s (f_{i_s}^{(1)} - f_{i_s}^{(2)}) \tag{89}$$

wherever it appears (which is all terms of RHS of (81), and in all but one term on LHS of (81)).

The left hand side of (81) is

$$\sum_{c=1}^{k} (-1)^{c+1} \prod_{q=1, q\neq c}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \prod_{s=1}^{b} \operatorname{dlog} f_{i_s}^{(1)} \operatorname{dlog} f_{i_s}^{(2)} \tag{90}$$

which is a sum

$$\prod_{q=2}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \prod_{s=1}^{b} \operatorname{dlog} f_{i_s}^{(1)} \operatorname{dlog} f_{i_s}^{(2)} \tag{91}$$

plus sum over $c = 2, \dots, k$ of

$$(-1)^{c+1} \operatorname{dlog}(F_1^{(1)} - F_1^{(2)}) \prod_{q=2, q\neq c}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \prod_{s=1}^{b} \operatorname{dlog} f_{i_s}^{(1)} \operatorname{dlog} f_{i_s}^{(2)} \tag{92}$$

The expression in (92) equals (using (89))

$$-\frac{\sum_{q=2}^{k} c_q (F_q^{(1)} - F_q^{(2)})}{F_1^{(1)} - F_1^{(2)}} \cdot \prod_{q=2}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \prod_{s=1}^{b} \operatorname{dlog} f_{i_s}^{(2)} \operatorname{dlog} f_{i_s}^{(2)}$$

Therefore the coefficient of $a_{i_1} \dots a_{i_b}$ on the LHS of (81) is

$$(1 - \frac{\sum_{q=2}^{k} c_q (F_q^{(1)} - F_q^{(2)})}{F_1^{(1)} - F_1^{(2)}}) \cdot \prod_{q=2}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \prod_{s=1}^{b} \operatorname{dlog} f_{i_s}^{(1)} \operatorname{dlog} f_{i_s}^{(2)} \tag{93}$$

On the right hand side of (81) the desired coefficient of $a_{i_1} \dots a_{i_b}$ is sum over $\ell = 1, \dots, b$ of

$$(\operatorname{dlog} f_{i_\ell}^{(1)} - \operatorname{dlog} f_{i_\ell}^{(2)}) \wedge \left(\prod_{q=1}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \prod_{s=1, s\neq \ell}^{b} \operatorname{dlog} f_{j_s}^{(1)} \operatorname{dlog} f_{j_s}^{(2)} \right) \tag{94}$$

We replace $\operatorname{dlog}(F_1^{(1)} - F_1^{(2)})$ by (89) and calculate

$$(\operatorname{dlog} f - \operatorname{dlog} g) \wedge (df - dg) = (f - g) \operatorname{dlog} f \wedge \operatorname{dlog} g \tag{95}$$

The quantity (94) is therefore equal to

$$\frac{\sum_{\ell=1}^{b} b_\ell(f_{i_\ell}^{(1)} - f_{i_\ell}^{(2)})}{F_1^{(1)} - F_1^{(2)}} \prod_{q=2}^{k} \operatorname{dlog}(F_q^{(1)} - F_q^{(2)}) \prod_{s=1}^{b} \operatorname{dlog} f_{j_s}^{(1)} \operatorname{dlog} f_{j_s}^{(2)} \tag{96}$$

It is now easy to see that (93) and (96) are equal and we are done.

9. Applications to invariant theory

We will use the notational set-up as in the introduction (Section 1.3).

9.1. **Conformal Blocks.** Let ℓ be a positive integer and consider an n-tuple of dominant integral weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ such that each $(\lambda_i, \theta) \le \ell)$ for each $1 \le i \le n$. Let $\vec{z} = (z_1, \dots, z_n)$ be an n-tuple of distinct points of $\mathbb{A}^1 \subset \mathbb{P}^1$. Associated to this data there is the space of dual conformal blocks $\mathbb{V}_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$ which is a quotient of $\mathbb{A}(\vec{\lambda})$ (see the survey [Sor96]). The following is an explicit description of the quotient:

Define an operator

$$T_{\vec{z}} \in \operatorname{End}(V(\vec{\lambda})),\ V(\vec{\lambda}) = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n},\ T_{\vec{z}} = \sum_{i=1}^{n} z_i e_\theta^{(i)}$$

with $e_\theta^{(i)}$ acting on the i-th tensor summand and θ the highest root. Let $C_{\vec{z}}$ denote the image of $T_{\vec{z}}^{\ell+1}$. The space of dual conformal blocks $\mathbb{V}_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$ is the cokernel of the natural map $C_{\vec{z}}$ to $\mathbb{A}(\vec{\lambda})$ [Bea96, FSV95]. The dual vector spaces $\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$ are known as conformal blocks.

9.2. **The Schechtman-Varchenko map.** As in Section 1.3, let $V(\vec{\lambda})_0^*$ denote the zero $\mathfrak{h}$-weight space of $V(\vec{\lambda})^*$. Suppose

$$\psi \in V(\vec{\lambda})_0^* = (V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_n})_0^*.$$

Let v_i be a highest weight vector in V_{λ_i}, $i = 1, \dots, n$. Then,

$$\Omega_{\vec{\lambda}}^{SV}(\psi) = \Omega^{SV}(\psi) = \psi(v(\vec{t}, \vec{z}))dt_1 \ldots dt_M \in A^M(U)$$

where for a fixed $\vec{z}$

$$v(\vec{t}, \vec{z}) = \sum_{\text{part}} \prod_{i=1}^{n} \langle\langle \prod_{b \in I_i} f_{\beta(b)}(t_b) v_i \rangle\rangle \in (V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_n})_0,$$

where $\beta : [1, \dots, M] \to R$ is a map to the positive roots as in Section 1.3 and

$$\langle\langle f_{\gamma_1}(u_1) f_{\gamma_2}(u_2) \ldots f_{\gamma_q}(u_q) v_i \rangle\rangle = \sum_{\text{perm}} \frac{1}{(u_1 - u_2)(u_2 - u_3) \ldots (u_q - z_i)} (f_{\gamma_1} f_{\gamma_2} \ldots f_{\gamma_q} v_i)$$

Here, $\sum_{\text{part}}$ part stands for the summation over all partitions of $I = \{1, \dots, M\}$ into n disjoint parts $I = I_1 \cup I_2 \cup \cup I_n$ and σ_{perm} perm the summation over all permutations of the elements of $\{1, \dots, q\}$. The operators f_γ are the standard f operators corresponding to simple roots γ.

9.3. **Proof of Proposition 11.** [2] Recall (see Section 1.3) that we need to show that

$$\mathbb{A}(\vec{\lambda})^* \hookrightarrow H^M(A^\bullet(U), \eta\wedge).$$

The cohomology of the Aomoto complex is independent of κ, and the map in Proposition 11 is linear in $\frac{1}{\kappa}$. Therefore an element in the kernel of the map for one value of κ, is in the kernel for any value of κ. We may therefore assume that $\kappa = \ell + g^*$ where ℓ is a sufficiently large integer, and g^* is the dual Coxeter number. We take ℓ large so that $\mathbb{A}(\vec{\lambda})^*$ coincides with the space of conformal blocks

$$\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$$

for the data $\vec{\lambda}$ at level ℓ (with the marked curve equal to $\mathbb{P}^1$, and the marked points $z_1, \dots, z_n$). Note that there is always a surjective map $\mathbb{A}(\vec{\lambda}) \to \mathbb{V}_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$.

It is a consequence of [Ram09, Bel12], and results of Deligne [Del71] that the map

$$\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n) \to H^M(U, \mathcal{L}(a)) \tag{97}$$

is injective. But (97) factors through the map in Proposition 11, and therefore Proposition 11 follows.

For any choice of weights, $H^M(A^\bullet(U), \eta\wedge) = H^M(P, \underline{\mathcal{L}}(a))$, and so the

Corollary 98. *The induced mapping $\mathbb{A}(\vec{\lambda})^* \to H^M(P, \underline{\mathcal{L}}(a))$ is injective for arbitrary κ.*

Remark 99. *Since κ can be taken to be an integer in the proof of Proposition 11, there is a (cyclic) unramified cover $\pi : \widehat{U} \to U$ so that $\mathcal{L}(a)$ is an isotypical component of $\pi_*\mathbb{C}$. Let $\widehat{P}$ be a smooth projective compactification of U'. It is shown in [Ram09, Bel12] that the corresponding injective map*

$$\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n) \to H^0(\widehat{U}, \Omega^M) \tag{100}$$

is induced by an injective map (i.e. the forms in the image of (100) extend to compactifications)

$$\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n) \to H^0(\widehat{P}, \Omega^M) = H^{M,0}(\widehat{P}, \mathbb{C}).$$

By results of Deligne (see [Ram09]), the map $H^{M,0}(\widehat{P}) \to H^M(\widehat{U}, \mathbb{C})$ is injective. Therefore the map

$$\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n) \to H^M(\widehat{U}, \mathbb{C}) \tag{101}$$

is injective. Now (101) factors maps to an isotypical component of $H^M(\widehat{U}, \mathbb{C})$ which equals $H^M(U, \mathcal{L}(a))$. Therefore (97) is injective.

9.4. **Proof of a generalization of Theorem 20.** We consider the maps (9) for two sets of data:

(1) The representations $(\lambda_1, \dots, \lambda_n)$, and κ.
(2) The representations $(\lambda_1^*, \dots, \lambda_n^*)$, and $-\kappa$.

[2] We thank A. Varchenko for a useful discussion that led to the following proof.

As in the introduction (see Section 1.3), these two give rise to the same hyperplane arrangement, but with weights that are negatives of each other. We therefore find a compactification $P \supseteq U$, and two objects (see Section 2.4) in the derived category $D^b_c(P)$: $\underline{\mathcal{L}}(a)$ and $\underline{\mathcal{L}}(-a)$.

The map (14), and Proposition 11 gives rise to the following two injective maps (it should really be $-\eta$ in the second equation but the sign does not affect the quotient):

$$\mathbb{A}(\vec{\lambda})^* \hookrightarrow \big(\frac{A^M(U)}{\eta \wedge A^{M-1}(U)}\big)^{\chi}. \tag{102}$$

and

$$\mathbb{A}(\vec{\lambda}^*)^* \hookrightarrow \big(\frac{A^M(U)}{\eta \wedge A^{M-1}(U)}\big)^{\chi}. \tag{103}$$

Recall the form $S \in A^{2M}(U \times U)$ as in the statement of Theorem 5. We will use S to form a diagram, which will be shown to commute:

$$\begin{array}{ccc} \big(\big(\frac{A^M(U)}{\eta\wedge A^{M-1}(U)}\big)^{\chi}\big)^* & \twoheadrightarrow & \mathbb{A}(\vec{\lambda}^*) \\ \downarrow{\scriptstyle \tilde{S}} & & \downarrow{\scriptstyle \sim} \\ \big(\frac{A^M(U)}{\eta\wedge A^{M-1}(U)}\big)^{\chi} & \hookleftarrow & \mathbb{A}(\vec{\lambda})^* \end{array} \tag{104}$$

For a representation V of the group Σ_M, the natural map

$$(V^*)^{\chi} \to (V^{\chi})^* \tag{105}$$

is an isomorphism. The map $\tilde{S}$ in (104) arises as follows. From S, one obtains a Σ_M-equivariant map $A^M(U)^* \to A^M(U)$, taking χ- isotypical components, we get $(A^M(U)^*)^{\chi} \to A^M(U)^{\chi}$. Composing with the inverse of the natural map (105), we get the map $\tilde{S}$ in (104). The vertical map on the right of (104) is the inverse of the natural isomorphism (18) from invariants $\mathbb{A}(\vec{\lambda})^*$ to coinvariants $\mathbb{A}(\vec{\lambda}^*)$.

The following result is a direct consequence of [SV91, Theorem 6.6], as we will explain in the next section.

Proposition 106. *The diagram* (104) *commutes up to a non-zero multiplicative constant.*

Putting together the χ isotypical component of (51) together with (104), we get a diagram which commutes up to a non-zero factors (commutes "projectively"):

(107)

$$\begin{array}{ccccccc} H^M(P, D(\underline{\mathcal{L}}(-a))[-2M])^{\chi} & \xrightarrow{\sim} & (H^M(P,\underline{\mathcal{L}}(-a))^*)^{\chi} & \xrightarrow{\sim} & \big(\big(\frac{A^M(U)}{\eta\wedge A^{M-1}(U)}\big)^{\chi}\big)^* & \twoheadrightarrow & \mathbb{A}(\vec{\lambda}^*) \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle \sim} \\ & & H^M(P,\underline{\mathcal{L}}(a))^{\chi} & \xleftarrow[\sim]{} & \big(\big(\frac{A^M(U)}{\eta\wedge A^{M-1}(U)}\big)^{\chi}\big) & \hookleftarrow & \mathbb{A}(\vec{\lambda})^* \end{array}$$

This leads to the following description, of the image of the map in Corollary 98:

Theorem 108. *The image of the injective mapping* $\mathbb{A}(\vec{\lambda})^* \to H^M(P, \underline{\mathcal{L}}(a))^{\chi}$ *coincides with the image of the topological map* $H^M(P, D(\underline{\mathcal{L}}(-a))[-2M])^{\chi} \to H^M(P, \underline{\mathcal{L}}(a))^{\chi}$ *(see the map* (45)*). Therefore,*

$$\mathbb{A}(\vec{\lambda})^* = \text{Image} : H^M(P, D(\underline{\mathcal{L}}(-a))[-2M])^{\chi} \to H^M(P, \underline{\mathcal{L}}(a))^{\chi}. \tag{109}$$

Theorem 108 specializes to Theorem 20 for large $|\kappa|$.

10. Proof of Proposition 106

There is another way of obtaining the Schechtman-Varchenko maps (9) with the role of e's replaced by f's and the highest weight vectors replaced by lowest weight vectors: Let

$$\psi \in V(\vec{\lambda})_0^* = (V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_n})_0^*.$$

The lowest weight vector in V_{λ_i} is $w_0(v_i)$ where v_i is the highest weight vector in V_{λ_i}, here w_0 is a lifting of the longest element of the Weyl group to G. Now, the weight of $w_0(v_i)$ is $w_0(\lambda_i)$ and

$$\sum_{i=1}^{n} w_0(\lambda_i) = \sum_{b=1}^{n} w_0(\beta(b))$$

We can form a map

$$\Omega^{SV-} : V(\vec{\lambda})_0^* \to A^M(U)$$

again with e_i replaced by $f_{i'}$ where $\alpha_{i'} = w_0(\alpha_i)$. The following is easy to check:

Lemma 110.

$$\Omega^{SV-}(\psi) = \Omega^{SV}(w_0(\psi))$$

and hence if ψ is Weyl group invariant then $\Omega^{SV-}(\psi) = \Omega^{SV}(\psi)$.

Let

$$\mathrm{Sh} : V(\vec{\lambda}^*)_0^* \to V(\vec{\lambda})_0^*$$

be the Shapovolov isomorphism in representation theory induced from an isomorphism without the outer duals and weight spaces. Let $\psi \in V(\vec{\lambda}^*)_0^*$: Then it follows that

$$\Omega^{SV-}_{\vec{\lambda}^*}(\psi) = \Omega^{SV}_{\vec{\lambda}}(\mathrm{Sh}(\psi))$$

10.1. We will use Ω^{SV-} for the Schechtman-Varchenko map for $(\lambda_1^*, \ldots, \lambda_n^*)$. This does not change (106) since we have restricted the maps (9) always to invariants in forming that diagram, and Lemma 110. We will also identify $V(\vec{\lambda}^*)_0^*$ and $V(\vec{\lambda})_0^*$, as well as $\mathbb{A}(\vec{\lambda}^*)$ and $\mathbb{A}(\vec{\lambda})$ by the Shapovolov form.

Consider the following diagram. Here the objects on the top row are to be considered as objects for the dual weights, using the above identification. Therefore the object on the top right is also $\mathbb{A}(\vec{\lambda}^*)$. The vertical map $\mathbb{A}(\vec{\lambda}) \to \mathbb{A}(\vec{\lambda})^*$ is the inverse of the evident isomorphism $\mathbb{A}(\vec{\lambda})^* \to \mathbb{A}(\vec{\lambda})$.

(111)
$$\begin{array}{ccccc} (A^M(U)^{\chi})^* & \xrightarrow{\Omega^{SV*}_{\vec{\lambda}}} & V(\vec{\lambda})_0 & \longrightarrow & \mathbb{A}(\vec{\lambda}) \\ \downarrow{\scriptstyle \tilde{S}} & & \downarrow{\scriptstyle \mathrm{Sh}} & & \downarrow \\ A^M(U)^{\chi} & \xleftarrow[\mathrm{SV}_{\lambda}]{} & V(\vec{\lambda})_0^* & \longleftarrow & \mathbb{A}(\vec{\lambda})^* \end{array}$$

(1) The square on the right "almost commutes", i.e., the two maps $V(\vec{\lambda})_0 \to V(\vec{\lambda})_0^*$ are not the same. They differ by a map to $(n_+V(\vec{\lambda})^*)_0$.

(2) The square on the left commutes: This is [SV91, Theorem 6.6], noting that the flag complex of degree M in [SV91] is dual to the space of logarithmic forms $A^M(U)$ (see Theorem 2.4 in loc. cit, and below).

The "almost commutativity" of the big square resulting from the above picture above allows us to conclude that (106) commutes: The map (9) is $V(\vec{\lambda})_0^* \to A^M(U)$ has the following property (see [SV91, Lemma 6.8.10])

$$(n_+V^*(\vec{\lambda}))_0 \mapsto \eta \wedge A^{M-1}(U).$$

10.2. **Flag forms.** We now recall the notion of flag forms [SV91, Section 3] $\mathcal{F}^p$ for $0 \leq p \leq M$. Let as before $\mathcal{C}$ be an arrangement given by linear polynomials $f_1, \dots, f_r$ and let $H_i = Z(f_i)$ and $\mathcal{C}^i$ denote the set of edges of codimension i in $\mathbb{A}^M$. For $0 \leq p \leq M$, let $\widetilde{\mathcal{F}}^p$ be the set of flags $(L^0 \supset L^1 \dots \supset L^p)$, where $L^i \in \mathcal{C}^i$. Let $\mathcal{F}^p$ be the quotient of the free abelian group generated by the set of flags $\widetilde{\mathcal{F}}^p$ by relations of the following form:

$$\sum_{F \supset \widehat{F}} F = 0,$$

where $\widehat{F} = (L^0 \supset \dots L^{i-1} \supset L^{i+1} \supset \dots \supset L^p)$ and $F = (\tilde{L}^0 \supset \dots \supset \tilde{L}^p)$ such that $\tilde{L}^j = L^j$ for all $j \neq i$.

Now for each p, following [SV91], we define a map $\varphi^p : A^p(U) \to \mathcal{F}^{p*}$. For a p-tuple of hyperplanes $(H_1, \dots, H_p)$ in general position, we define $F(H_1, \dots, H_p) = (H_1 \supset H_{12} \supset H_{123} \supset \dots \supset H_{12\dots p}) \in \mathcal{F}^p$, where $H_{1\dots i} = H_1 \cap \dots \cap H_i$. These flag forms are dual to the Aomoto log forms [SV91, Theorem 2.4] by the map φ^p is given by the following formula

$$\varphi^p(H_1, \dots, H_p) = \sum_{\sigma \in S_p} (-1)^{sgn(\sigma)} \delta_{F(H_{\sigma(1)}, \dots H_{\sigma(p)})}, \tag{112}$$

where δ_F for a flag F is the delta functional for a flag F.

10.2.1. *Quasi-classical contravariant form.* Consider the arrangement $\mathcal{C}$ and a be a set of weights. Let $\bar{H} = (H_1, \dots, H_p) \in A^p(U)$, we say H is adjacent to a flag F if we can find a permutation $\sigma \in S_p$ such that $F = F(H_{\sigma(1)}, \dots, H_{\sigma(p)})$. For any two flags F and F', Schechtman-Varchenko ([SV91, Equation (3.3.1)]) defines a bilinear form on $\mathcal{F}^p$.

$$S^p(F, F') = \frac{1}{p!} \sum_{\bar{H}} (-1)^{\sigma_{\bar{H}}(F)\sigma_{\bar{H}}(F')} a(H_1) \dots a(H_p), \tag{113}$$

where the sum is taken over all $\bar{H} = (H_1, \dots, H_p)$ adjacent to the flags F and F'. We will use the following fact that is an easy check using Equation (113)

Proposition 114. *Under this identification, the quasi-classical contravariant form on $\mathcal{F}^M$ equals the Shapovolov form $S^{(M)}$ given by (77).*

Proof. This proceeds as follows: Pick two flags F and F': The two values on the pair (F, F') are sums over $(H_1, \dots, H_M)$. Start with the values assigned by our S: a term

$$\Omega = (H_1^{(1)} H_1^{(2)} \dots H_M^{(1)} H_M^{(2)})$$

acts on $F \otimes F'$ with non-zero coefficient only if F is of the form

$$F = (H_{\sigma(1)} \supset H_{\sigma(1)} \cap H_{\sigma(2)} \supset \dots),$$

similarly F' for a permutation σ. In the formula for Ω, $H_i^{(1)} = \mathrm{dlog}\, f_i^{(1)}$ where f_i is the defining equation for H_i, similarly for $H_i^{(2)}$. This coefficient is so because a log form acts on flag forms via the formula [SV91, Equation 2.3.2]. The corresponding coefficient is the product of the weights $a(H_i)$ times (-1) raised to the sum of signs of σ and σ'.

For the form defined in [SV91, Equation (3.3.1)], the contributions are again over the same σ and σ', and the signs are the same. □

10.3. **Aomoto Representatives.** Consider the image Ω_ψ of the map (9) on an element $\psi \in \mathbb{A}(\vec{\lambda})^* = (V(\vec{\lambda})^*)^{\mathfrak{g}}$. We claim that Ω_ψ does not have poles on any E_α with $a_\alpha = 0$. This is because we may raise the level (and assume it to be integral) so that ψ lives in the space of conformal blocks. Then, for a suitable master function $\mathcal{R}$ the form $\mathcal{R}\Omega_\psi$ extends to compactifications of ramified covers of P by the main results of [Ram09, Bel12], and is therefore square integrable on P. Thus on any E_α on which $\mathcal{R}$ has order zero, the form Ω_ψ is regular. The claim follows immediately because a_α is the order of $\mathcal{R}$ along E_α.

The image of the mapping $\mathbb{A}(\vec{\lambda})^* \to H^M(P, \underline{\mathcal{L}}(a))^\chi$, which coincides with the image of $H^M(P, D(\underline{\mathcal{L}}(-a))[-2M])^\chi \to H^M(P, \underline{\mathcal{L}}(a))^\chi$ (by Theorem 108), is therefore contained in the span of $[\Omega]$ where Ω runs through elements of $A^M(U)^\chi$ which do not have poles on divisors E_α with $a_\alpha = 0$. Such elements $[\Omega]$ are always in the image of the topological mapping $H^M(P, D(\underline{\mathcal{L}}(-a))[-2M])^\chi \to H^M(P, \underline{\mathcal{L}}(a))^\chi$ (see Lemma 56). Therefore,

Proposition 115. *The image of the Schechtman-Varchenko mapping* (9)*:*

$$\mathbb{A}(\vec{\lambda})^* \to H^M(P, \underline{\mathcal{L}}(a))^\chi$$

equals the set of $[\Omega]$ *where* Ω *runs through elements of* $A^M(U)^\chi$ *which do not have poles on divisors* E_α *with* $a_\alpha = 0$*.*

The classes $[\Omega]$ in this proposition can be zero even if Ω is not zero.

10.4. **Mixed Hodge structures.** Assume that $\kappa \neq 0$ is an integer. Recall (109):

$$\mathbb{A}(\vec{\lambda})^* = \mathrm{Image} : H^M(P, D(\underline{\mathcal{L}}(-a))[-2M])^\chi \to H^M(P, \underline{\mathcal{L}}(a))^\chi$$

Note that as in Section 1.2, we may write

$$H^M(P, \underline{\mathcal{L}}(a))^\chi = H^M(V, j_!\mathcal{L}(a))^\chi,\ H^M(P, D(\underline{\mathcal{L}}(-a))[-2M])^\chi = H^M(V', q_!\mathcal{L}(a)).$$

Let $\pi : \widehat{U} \to U$ be an unramified cyclic cover such that $\mathcal{L}(a)$ is an isotypical component of $\pi_*(\mathbb{C})$. Pick an equivariant compactification $\widehat{U} \subseteq \widehat{P}$ of $\widehat{U}$, such that π extends to a a mapping $\pi : \widehat{P} \to P$, and set $\widehat{V} = \pi^{-1}(V)$.

A group of the form $\Sigma_M \times \mu_{C\kappa}$ acts on $\widehat{P} \to P$ (see [BM14]), where $\mu_{C\kappa}$ is the group of $C\kappa$ roots of unity in $\mathbb{C}^*$ and Σ_M is defined in (13). It is easy to see that $H^M(V, j_!\mathcal{L}(a))^\chi$ is an isotypical component $H^M(\widehat{V}, D; \mathbb{C})^\tau$ where $D \subset \widehat{V}$ is the inverse image $\pi^{-1}(V - U)$ (a closed subset), and hence acquires a mixed Hodge structure over the cyclotomic field $\mathbb{Q}(\mu_{C\kappa})$; here $\tau : \Sigma_M \times \mu_{C\kappa} \to \mathbb{C}^*$ is a character.

Similarly $H^M(V', q_!\mathcal{L}(a))^\chi$ is isomorphic to $H^M(\widehat{V'}, D'; \mathbb{C})^\tau$ where $D' \subset \widehat{V'} = \pi^{-1}(V')$ is the inverse image $\pi^{-1}(V' - U)$, and hence acquires a mixed Hodge structure (MHS) over a cyclotomic field. Therefore,

- By Theorem 108, $\mathbb{A}(\vec{\lambda})^*$ the image of a morphism of mixed Hodge structures, acquires a mixed Hodge structure defined over a cyclotomic field.

Section 11 gives an example where this mixed Hodge structure is not pure.

There is an exact sequence of topological cohomology groups with $\mathbb{C}$ coefficients,

$$H^{M-1}(D) \to H^M(\widehat{V}, D) \to H^M(\widehat{V}) \to H^M(D)$$

In the following, $F^\bullet$ is the Hodge filtration and $M = \dim V$. We record some facts about the Hodge filtration

(i) $F^{M+i}(H^{M-1}(D)) = 0$, $F^{M+i}(H^M(D)) = 0$, $i = 0, 1$, and hence exactness of the functors F^k implies
(ii) $F^M(H^{M+i}(\widehat{V}, D)) = F^{M+i}(H^M(\widehat{V})), i = 0, 1$.
(iii) $F^M(H^M(\widehat{V})) = H^{M,0}(\widehat{P})$ and $F^M(H^{M+1}(\widehat{V})) = 0$
(iv) $F^{M+1}H^M(\widehat{V}, D) = 0$.

For (i), by [Del74, page 45 (e)], $h^{p,q}$ for $H^n(D)$ vanishes if $p > \dim D = M - 1$.

The above applies also to $(\widehat{V'}, D')$ and hence both $H^M(\widehat{V'}, D'; \mathbb{C})^\tau$ and $H^M(\widehat{V}, D; \mathbb{C})^\tau$ have F^M equal to $H^{M,0}(\widehat{P})^\tau$. Therefore,

Proposition 116. *The above MHS on $\mathbb{A}(\vec{\lambda})^*$ has F^M isomorphic to $H^{M,0}(\widehat{P})^\tau$.*

10.5. Now consider the case $\kappa = \ell + g^*$ with ℓ a positive integer. The space of invariants $\mathbb{A}(\vec{\lambda})^*$ has a subspace given by the space of conformal blocks $\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$. The subspace of conformal blocks $\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$ injects into $H^{M,0}(\widehat{P})^\tau$ by [Ram09, Bel12]. The image of conformal blocks is all of $H^{M,0}(\widehat{P})^\tau$ for classical groups and G_2 [BM14]. Therefore

Proposition 117. *For classical groups and G_2,*

$$F^M(\mathbb{A}(\vec{\lambda})^*) = \mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n) \tag{118}$$

For ℓ sufficiently large, $\mathbb{A}(\vec{\lambda})^*$ coincides with $\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$ and therefore $\mathbb{A}(\vec{\lambda})^*$ carries a pure Hodge structure of type $(M, 0)$ over a cyclotomic field.

Remark 119. *If $\ell > -1 + \frac{1}{2}(\lambda_i, \theta)$, then $\mathbb{A}(\vec{\lambda})^* = \mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$ (see* [BGM16] *where this is derived from* [Bea96, FSV95]*).*

10.6. **Hodge filtration.** In general, for positive ℓ, we have a guess for the Hodge filtration on $\mathbb{A}^*$. By the previous section, for $\mathfrak{g}$ classical or G_2,

$$F^M(\mathbb{A}^*) = \mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n)$$

Question 120. *Are the other steps in the filtration F^{M-p} given by conformal blocks at higher levels, $\mathbb{V}^*_{\mathfrak{g},\vec{\lambda},\ell+p}(\mathbb{P}^1, z_1, \dots, z_n)$?*

The only justification we have for this question is that the KZ connection behaves consistently as we see below.

By the description of conformal blocks given in Section 9.1,

$$\mathbb{V}_\ell \mid_{\vec{z}} = \mathbb{V}_{\mathfrak{g},\vec{\lambda},\ell}(\mathbb{P}^1, z_1, \dots, z_n) = \frac{V(\vec{\lambda})}{\mathfrak{g}V(\vec{\lambda}) + \operatorname{im} T^{\ell+1}_{\vec{z}}}$$

10.6.1. *Connections.* For each ℓ, the KZ connection is a connection $\nabla^{(\ell)}$ on $(V_{\lambda_1}\otimes\ldots\otimes V_{\lambda_n})$ of the form (see (123)) $\frac{\partial}{\partial z_i}\mapsto \nabla_i^{(\ell)}=\frac{\partial}{\partial z_i}+\frac{A_i}{\ell+g^*}$ where $A_i=A_i(\vec{z})$ are operators on $V(\vec{\lambda})$ which are independent of ℓ.

It is known that $\nabla^{(\ell)}$ descends to a connection operator on $\mathbb{A}(\vec{\lambda})$ and on its quotient $\mathbb{V}_\ell$. Therefore $\frac{\partial}{\partial z_i}+\frac{A_i}{\ell+g^*}$ preserves the subbundle $\mathfrak{g}(V(\vec{\lambda}))+\operatorname{im}T_{\vec{z}}^{\ell+1}$ of $V(\vec{\lambda})$.

This means that

Lemma 121. *$[\frac{\partial}{\partial z_i}+\frac{A_i}{\ell+g^*}](T_{\vec{z}}^{\ell+1}w)$ is in $\mathfrak{g}V(\vec{\lambda})+\operatorname{im}T_{\vec{z}}^{\ell+1}$.*

But $\frac{\partial}{\partial z_i}T_{\vec{z}}^{\ell+1}w$ is a multiple of $z_iT_{\vec{z}}^{\ell}e_\theta^{(i)}w$. Therefore the result of the action of A_i on the vector $T_{\vec{z}}^{\ell+1}w$ is in $\mathfrak{g}V(\vec{\lambda})+\operatorname{im}T_{\vec{z}}^{\ell+1}$ up to a multiple of $z_iT_{\vec{z}}^{\ell}e_\theta^{(i)}w$. Therefore

Proposition 122. (1) *$\nabla^{(k)}$ carries the subbundle $\mathfrak{g}V(\vec{\lambda})+\operatorname{im}T_{\vec{z}}^{\ell+1}$ of $V(\vec{\lambda})$ to $\mathfrak{g}V(\vec{\lambda})+\operatorname{im}T_{\vec{z}}^{\ell}$ for any $k>0$.*
(2) *Dualizing, the connection $\nabla^{(k)}$ on $\mathbb{A}(\lambda)^*$ carries subbundle $\mathbb{V}_\ell^*$ to the "larger" $\mathbb{V}_{\ell+1}^*$ for any ℓ. For $k=\ell$, $\nabla^{(k)}$ carries $\mathbb{V}_k^*$ to itself.*
(3) *The filtration of $\mathbb{A}(\vec{\lambda})^*$ by the filtration of conformal blocks behaves like the Hodge filtration with respect to connections (Griffiths transversality).*

11. Non semi-simple monodromy for $\mathfrak{sl}(2)$

Consider the representation $\mathbb{C}^2$ of $\mathfrak{sl}(2)$ with highest weight ω_1. Let $\vec{\lambda}=(\omega_1,\omega_1,\omega_1,\omega_1)$, and consider the space of coinvariants $\mathbb{A}(\vec{\lambda})$. Let $\vec{z}=(z_1,\ldots,z_4)\in\mathbb{C}^4$ be a tuple of distinct points. As $\vec{z}$ varies in the configuration space of points in $\mathbb{A}^1$, we know that the trivial vector bundle $\mathcal{A}(\vec{\lambda})$ with fibers $\mathbb{A}(\vec{\lambda})$ is endowed with a flat connection known as the Knizhnik-Zamolodchikov (KZ) connection. In this section, we show that the monodromy of the KZ connection on $\mathbb{A}(\vec{\lambda})$ with $\kappa=3$ is not semi-simple.

Consider the standard basis v_1 and v_2 of $\mathbb{C}^2$. We know that $\dim\mathbb{A}(\vec{\lambda})$ is two. Let $v=v_1\otimes v_1\otimes v_2\otimes v_2$ and $w=v_1\otimes v_2\otimes v_1\otimes v_2$. It is easy to check that the classes $[v]$ and $[w]$ in $\mathbb{A}(\vec{\lambda})$ form a basis. Let e, f and h be standard elements of $\mathfrak{sl}(2)$ and let $\Omega=\frac{1}{2}h^2+e.f+f.e$ denote the Casimir operator considered as an element of the universal enveloping algebra of $\mathfrak{sl}(2)$.

Associated to the data $\vec{\lambda}$, the equations of the flat sections of KZ connections as operators on $\mathbb{A}(\vec{\lambda})$ are given by the following connection equations: Here $u\in\mathcal{A}(\vec{\lambda})=\mathbb{A}(\vec{\lambda})\otimes\mathcal{O}$,

$$(\frac{\partial}{\partial z_j}+\frac{1}{\kappa}\sum_{k=1,k\neq j}^{4}\frac{\Omega_{jk}}{z_j-z_k})u=0, \tag{123}$$

We now compute the matrices Ω_{jk} where κ is a complex number, Ω_{jk} is the Casimir operator acting on j-th and k-th component with respect to the chosen basis $\{[v],[w]\},$. They are given as follows:

$$\Omega_{1,2}=\begin{pmatrix}1/2 & -1\\ 0 & -3/2\end{pmatrix},\ \Omega_{1,3}=\begin{pmatrix}-3/2 & 0\\ -1 & 1/2\end{pmatrix},\ \Omega_{1,4}=\begin{pmatrix}-1/2 & 1\\ 1 & -1/2\end{pmatrix}$$

$$\Omega_{2,3}=\begin{pmatrix}-1/2 & 1\\ 1 & -1/2\end{pmatrix},\ \Omega_{2,4}=\begin{pmatrix}-3/2 & 0\\ -1 & 1/2\end{pmatrix},\ \Omega_{3,4}=\begin{pmatrix}1/2 & -1\\ 0 & -3/2\end{pmatrix}$$

There is a non-constant rank one sub-bundle $\mathcal{K}(\vec{\lambda})$ of $\mathcal{A}(\vec{\lambda})$ given fiberwise by the kernel of the map $\mathbb{A}(\vec{\lambda})$ to the dual conformal blocks $\mathbb{V}_{\mathfrak{g},\vec{\lambda},1}(\mathbb{P}^1, z_1, \ldots, z_n)$. Since the KZ connection on the bundle $\mathcal{A}(\vec{\lambda})$ descends to the bundle of conformal blocks, it follows that the bundle $\mathcal{K}$ is preserved by the KZ connection. Hence the monodromy representation of KZ connection of $\mathbb{A}(\vec{\lambda})$ is reducible. We can describe the dual conformal blocks $\mathbb{V}_{\mathfrak{g},\vec{\lambda},1}(\mathbb{P}^1, z_1, \ldots, z_n)$ as the class of $[v]$ or $[w]$ with the following relation:

$$[v] = -\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}[w] \tag{124}$$

Further the fiber of $\mathcal{K}$ at $\vec{z}$ is given by $\Phi(\vec{z}) = (z_1 - z_3)(z_2 - z_4)v + (z_1 - z_2)(z_3 - z_4)w$ as an element of $\mathbb{A}(\vec{\lambda})$. The following lemma can be checked by a direct calculation

Lemma 125. *With the above notations, we get*

1. *The multi-valued section of* $\widetilde{\Phi}(\vec{z}) = \prod_{1\leq i<j\leq 4}(z_i - z_j)^{-\frac{1}{6}}\Phi(\vec{z})$ *is flat for the KZ connection on* $\mathcal{A}(\vec{\lambda})$.
2. *The multi-valued section* $\widetilde{[v]} := f[v]$ *gives a flat section of the KZ connection in the dual conformal blocks bundle, where*
$$f(\vec{z}) = (z_1 - z_2)^{-1/2}(z_1 - z_3)^{1/2}(z_1 - z_4)^{1/2}(z_2 - z_3)^{1/2}(z_2 - z_4)^{1/2}(z_3 - z_4)^{-1/2}.$$
3. *The sections* $\widetilde{v} = f(\vec{z}).v$ *and* $\widetilde{\Phi}(\vec{z})$ *are related by the following:*
$$\nabla_{\frac{\partial}{\partial z_j}}\widetilde{v} = -\frac{1}{3}\prod_{k=1,k\neq j}^{4}(z_j - z_k)^{-1}\prod_{1\leq i<k\leq 4}(z_i - z_k)^{\frac{1}{6}}.f(\vec{z}).\widetilde{\Phi}(\vec{z}),$$
where $f(\vec{z})$ *is as above.*

The following proposition is the main result of this section

Proposition 126. *The monodromy representation of the KZ connection on* $\mathbb{A}(\vec{\lambda})$ *with* $\kappa = 3$ *is not semisimple.*

Proof. We know that the monodromy representation is reducible and two dimensional. Assume that the representation is semi-simple, and hence abelian. We choose the point $(-1/2, 0, 1/2, 1)$ as our base point and consider a Pochhammer loop γ, where the point z_1 moves and the other coordinates remain fixed.

We know that the Pochhammer loop is an element of the commutator of the fundamental group. Now if the monodromy is abelian, then the image of γ under the monodromy representation must be the 2×2-identity matrix. We have already shown that the monodromy is upper triangular. Hence the Pochhammer contour γ maps to an unipotent matrix of the form
$$\begin{pmatrix} 1 & 0 \\ a_{2,1} & 1 \end{pmatrix}.$$
If we can show that the constant $a_{2,1}$ is non-zero, we will be done. We need to find a flat section $\Psi(\vec{z})$ of the form $\widetilde{v} + g(\vec{z})\widetilde{\Phi}(\vec{z})$. If such a section exists, then by Lemma 125, we will get that
$$\nabla_{\frac{\partial}{\partial z_1}}\Psi(\vec{z}) = \xi(\vec{z})\widetilde{\Phi}(\vec{z}) + \frac{\partial}{\partial z_1}g(\vec{z})\widetilde{\Phi}(\vec{z}) = 0, \text{ i.e } \frac{\partial}{\partial z_1}g(\vec{z}) = -\xi(\vec{z})$$

where

$$\xi(\vec{z}) = -\frac{1}{3}(z_1 - z_2)^{-4/3}(z_1 - z_3)^{-1/3}(z_1 - z_4)^{-1/3}(z_2 - z_3)^{2/3}(z_2 - z_4)^{2/3}(z_3 - z_4)^{-1/3}.$$

The entry a_{21} is exactly the monodromy of $g(\vec{z})$ about the Pochhammer contour γ. Since we are interested in the monodromy as z_1 changes, the terms $(z_2 - z_3)^{2/3}(z_2 - z_4)^{2/3}(z_3 - z_4)^{-1/3}$ in $\xi(\vec{z})$ have no contribution to monodromy. Thus, we can just focus on the term $(z_1 - z_2)^{-4/3}(z_1 - z_3)^{-1/3}(z_1 - z_4)^{-1/3}$. Let ${}_2F_1(a, b, c|u)$ denote the Gauss hypergeometric function.

Now for any $b, b - c \notin \mathbb{Z}$, we have

$${}_2F_1(a, b, c|u) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{1}{(1 - e^{2\pi i b})(1 - e^{2\pi i(c-b)})} \int_\gamma t^{b-1}(1-t)^{c-b-1}(1-tu)^{-a} dt,$$

where Γ is the Euler Gamma-function and γ is the Pochhammer contour. Thus, if put $(z_2, z_3, z_4) = (0, 1/2, 1)$, and $(a, b, c|u) = (1/3, -1/3, 1/3|2)$, we see that $a_{1,2}$ is given up to a constant by ${}_2F_1(1/3, -1/3, 1/3|2)$ which can easily be checked to be non-zero using Wolfram Mathematica. This gives a contradiction.

□

Remark 127. *Proposition 126 therefore implies that the variation of Hodge structures induced on $\mathbb{A}(\vec{z})$ by Theorem 20 is not always pure. Now if we consider the case $\ell = 2$, i.e. $\kappa = 4$, the situation is different. We know that the dimension of the conformal block $\mathbb{V}_{\mathfrak{sl}(2),\vec{\lambda},2}(\mathbb{P}^1, z_1, \ldots, z_n)$ is two and is equal to the space $\mathbb{A}(\vec{\lambda})$. Since the KZ connection on conformal blocks is unitary, it follows that the monodromy representation of the KZ connection on $\mathbb{A}(\vec{\lambda})$ is semisimple for $\kappa = 4$.*

11.1. **Acknowledgements.** We thank N. Fakhruddin, G. Pearlstein, E. Looijenga, A. Nair, M. Nori and A. Varchenko for useful discussions and communication. P. Brosnan and S. Mukhopadhyay were partially supported by NSF grant DMS 1361159 (PI: P. Brosnan). S. Mukhopadhyay was also supported by a Simons Travel Grant.

References

[Bea96] A. Beauville, *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 1996, pp. 75–96. ↑24, 30

[Bel12] P. Belkale, *Unitarity of the KZ/Hitchin connection on conformal blocks in genus 0 for arbitrary Lie algebras*, J. Math. Pures Appl. (9) **98** (2012), no. 4, 367–389. ↑4, 25, 29, 30

[BGM16] P. Belkale, A. Gibney, and S. Mukhopadhyay, *Nonvanishing of conformal blocks divisors on $\overline{M}_{0,n}$*, Transform. Groups **21** (2016), no. 2, 329–353. ↑30

[BM14] P. Belkale and S. Mukhopadhyay, *Conformal blocks and cohomology in genus 0*, Ann. Inst. Fourier (Grenoble) **64** (2014), no. 4, 1669–1719. ↑29, 30

[BVP15] E. Bierstone and F. Vera Pacheco, *Desingularization preserving stable simple normal crossings*, Israel J. Math. **206** (2015), no. 1. ↑6, 17

[Bri73] E. Brieskorn, *Sur les groupes de tresses [d'après V. I. Arnol′d]*, Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, 1973, pp. 21–44. Lecture Notes in Math., Vol. 317. ↑1

[Del70] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970. ↑7

[Del71] ———, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57. ↑25

[Del74] ———, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77. ↑30

[EV86] H. Esnault and E. Viehweg, *Logarithmic de Rham complexes and vanishing theorems*, Invent. Math. **86** (1986), no. 1, 161–194. ↑11, 13

[ESV92] H. Esnault, V. Schechtman, and E. Viehweg, *Cohomology of local systems on the complement of hyperplanes*, Invent. Math. **109** (1992), no. 3, 557–561. ↑1, 2, 6

[FC90] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990. With an appendix by David Mumford. ↑10

[FSV95] B. Feigin, V. Schechtman, and A. Varchenko, *On algebraic equations satisfied by hypergeometric correlators in WZW models. II*, Comm. Math. Phys. **170** (1995), no. 1, 219–247. ↑24, 30

[GH94] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original. ↑21, 22

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. ↑18

[KS94] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1994. ↑10, 14, 16, 18

[Loo99] E. Looijenga, *Arrangements, KZ systems and Lie algebra homology*, Singularity theory (Liverpool, 1996), 1999, pp. xvi, 109–130. ↑2, 3, 8

[Loo12] ———, *The KZ system via polydifferentials*, Arrangements of hyperplanes—Sapporo 2009, 2012, pp. 189–231. ↑4, 5, 6

[Ram09] T. R. Ramadas, *The "Harder-Narasimhan trace" and unitarity of the KZ/Hitchin connection: genus 0*, Ann. of Math. (2) **169** (2009), no. 1, 1–39. ↑4, 25, 29, 30

[SV91] V. Schechtman and A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), no. 1, 139–194. ↑2, 4, 6, 26, 28, 29

[Sor96] C. Sorger, *La formule de Verlinde*, Astérisque **237** (1996), Exp. No. 794, 3, 87–114. Séminaire Bourbaki, Vol. 1994/95. ↑24

[Var95] A. Varchenko, *Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups*, Advanced Series in Mathematical Physics, vol. 21, World Scientific Publishing Co., Inc., River Edge, NJ, 1995. ↑

[Wei94] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. ↑21

P. Belkale: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA
email: belkale@email.unc.edu

P. Brosnan: Department of Mathematics, University of Maryland, College Park, MD 20742, USA
email: pbrosnan@gmail.com

S. Mukhopadhyay: Department of Mathematics, University of Maryland, College Park, MD 20742, USA
email: swarnava@umd.edu