JUMPS IN THE ARCHIMEDEAN HEIGHT

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ABSTRACT. We introduce a pairing on local intersection cohomology groups of variations of pure Hodge structure, which we call the asymptotic height pairing. Our original application of this pairing was to answer a question on the Ceresa cycle posed by R. Hain and D. Reed. (This question has since been answered independently by Hain.) Here we show that a certain analytic line bundle, called the biextension line bundle, defined in terms of normal functions, always extends to any smooth partial compactification of the base. We then show that the asymptotic height pairing on intersection cohomology governs the extension of the natural metric on this line bundle studied by Hain and Reed (as well as, more recently, by several other authors). We also prove a positivity property of the asymptotic height pairing, which generalizes results of a recent preprint of J. Burgos Gill, D. Holmes and R. de Jong, along with a continuity property of the pairing in the normal function case. Moreover, we show that the asymptotic height pairing arises in a natural way from certain Mumford-Grothendieck biextensions associated to normal functions.

CONTENTS

1. INTRODUCTION

Let $\mathcal{H}$ be a weight -1 polarizable variation of pure Hodge structure with integral coefficients over a complex manifold S. Assume that the local system $\mathcal{H}_{\mathbb{Z}}$ associated to $\mathcal{H}$ is torsion-free. The elements of the group $\mathrm{NF}(S, \mathcal{H})$ of normal functions into $\mathcal{H}$ can be thought of either as certain holomorphic sections of the Griffiths intermediate Jacobian $\mathcal{J}(\mathcal{H}) \to S$, or as Yoneda extensions of $\mathbb{Z}$ by $\mathcal{H}$ in the category VMHS(S) of variations of mixed Hodge structure over S, i.e., elements of $\mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbb{Z},\mathcal{H})$. Define $\mathrm{NF}(S,\mathcal{H})^\vee = \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathcal{H},\mathbb{Z}(1))$ and note that via duality (cf. Proposition (111))

$$\mathrm{NF}(S,\mathcal{H})^\vee \cong \mathrm{NF}(S,\mathcal{H}^\vee)$$

where $\mathcal{H}^\vee = \mathrm{Hom}(\mathcal{H},\mathbb{Z}(1))$.

A biextension variation of type $\mathcal{H}$ is a variation of mixed Hodge structure $\mathcal{V}$ over S equipped with isomorphisms

$$Gr^W_0(\mathcal{V}) \cong \mathbb{Z}(0), \qquad Gr^W_{-1}(\mathcal{V}) \cong \mathcal{H}, \qquad Gr^W_{-2}(\mathcal{V}) \cong \mathbb{Z}(1). \tag{1}$$

To any biextension variation $\mathcal{V}$ over S there is an associated C^∞ function [cf. (41)]

$$h : S \to \mathbb{R} \tag{2}$$

originally studied by Hain [19]. The function h can be defined by Deligne's δ-splitting (Theorem (2.20), [9]), which measures how far the fibers $\mathcal{V}_s$ are from being $\mathbb{R}$-split mixed Hodge structures.

Given $\nu \in \mathrm{NF}(S,\mathcal{H})$ and $\omega \in \mathrm{NF}(S,\mathcal{H})^\vee$, the set of biextension variations of type (ν,ω) consists of all biextension variations over S such that

$$\begin{aligned}(W_0/W_{-2})\mathcal{V} &\cong \nu; \\ (W_{-1}/W_{-3})\mathcal{V} &\cong \omega.\end{aligned} \tag{3}$$

For any open subset U of S, $B(\nu,\omega)(U)$ is defined to be the set of isomorphism classes of biextension variations of type (ν,ω) over U.

Hain and Reed noticed that $\mathcal{B}(\nu,\omega)$ is naturally an $\mathcal{O}_S^\times$-torsor over S. Therefore the sections of $\mathcal{B}(\nu,\omega)$ can be identified with the non-vanishing sections of an analytic line bundle $\mathcal{L} = \mathcal{L}(\nu,\omega)$ over S which Hain and Reed have called the *biextension line bundle* [21].

In [21], Hain and Reed further observed that $\mathcal{L}(\nu,\omega)$ comes with a natural metric defined by the requirement that

$$|\mathcal{V}| = \exp(-h(\mathcal{V})) \tag{4}$$

for any $\mathcal{V} \in B(\nu,\omega)$. In the case where ν and ω arise from a family of homologically trivial algebraic cycles, the function (2) is the archimedean height pairing.

Suppose that $j : S \hookrightarrow \overline{S}$ is an inclusion of S as a Zariski open subset of a complex manifold $\overline{S}$. Let $\mathcal{U}$ be a weight -1 polarizable variation of pure Hodge structure with integral coefficients on S. Assume that $\mathcal{U}_{\mathbb{Z}}$ is torsion free. Then, M. Saito [37] defined the group of admissible normal functions $\mathrm{ANF}(S,\mathcal{U})_{\bar{S}}$ to be set of extensions of $\mathbb{Z}(0)$ by $\mathcal{U}$ in the category of admissible variations of mixed Hodge structure, i.e.

$$\mathrm{ANF}(S,\mathcal{U})_{\bar{S}} := \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathbb{Z}(0),\mathcal{U})$$

Normal functions of geometric origin (e.g. arising from families of algebraic cycles) are admissible.

Remark 5. A choice of polarization Q of $\mathcal{U}$ determines a morphism $a_Q : \mathcal{U} \to \mathcal{U}^\vee$ given by $\beta \mapsto Q(\beta, \_)$. This induces a map $\mathrm{ANF}(S,\mathcal{U})_{\bar{S}} \to \mathrm{ANF}(S,\mathcal{U}^\vee)_{\bar{S}}$ which we denote by $\nu \mapsto \nu^\vee$.

Let $\nu \in \mathrm{ANF}(S,\mathcal{H})_{\bar{S}}$ and $\omega \in \mathrm{ANF}(S,\mathcal{H}^\vee)_{\bar{S}}$. Then, one can ask two questions.

Q 6. *Does the line bundle $\mathcal{L}$ extend to a line bundle $\overline{\mathcal{L}}$ on $\overline{S}$? If fact, what we really seek is an extension $\overline{\mathcal{L}}$ satisfying the following property: For U an open subset of $\overline{S}$, the restrictions of the non-vanishing sections in $\overline{\mathcal{L}}(U)$ to $\mathcal{L}(U)$ are biextension variations which are admissible as variations of mixed Hodge structure.*

Q 7. *Does the metric $|\ |$ extend to $\overline{\mathcal{L}}$?*

Note that, in the (most interesting) case where $\bar{S}$ is a smooth, projective variety, Q 6 can be rephrased (via the GAGA principle [39]) as asking whether or not $\mathcal{L}$ is algebraic. In Theorem 233 we show that Q 6 has a positive answer. While there are, in general, many extensions of $\mathcal{L}$ to $\bar{S}$, we show in §13 that there is a canonical meromorphic extension. Moreover, there is a canonical choice of extension $\overline{\mathcal{L}}_{\mathrm{can}} \in \operatorname{Pic} \bar{S} \otimes \mathbb{Q}$. (See Remark 234.)

On the other hand, we will give a criterion in terms of the local intersection cohomology of ν and ω for Q 7 to have a negative answer and, using this criterion, we will give an example where the metric does not extend.

Remark 8. Note that there are many examples of analytic line bundles over smooth complex algebraic varieties S which do not extend to $\bar{S}$ with $\bar{S}$ as in Q6. For example, in [40], Serre gives uncountably many examples of analytic line bundles on $\mathbb{C}^2 \setminus \{(0,0)\}$ which do not extend to $\mathbb{C}^2$. Consequently, it is not obvious that Q6 has a positive answer even in the case that the codimension of $\bar{S} \setminus S$ in $\bar{S}$ is greater than 1.

D. Lear's 1991 University of Washington (unpublished) PhD thesis treats special cases of Q6 and Q7. For Q6, Lear handles the case where S is a curve and the local monodromy is unipotent with Jordan blocks of size at most 2. In this case, there are local monodromy invariant splittings of the local systems underlying the variations of Hodge structure corresponding to the normal functions ν and ω. While Lear does not show explicitly that the extension $\overline{\mathcal{L}}$ satisfies the property in Q6, we believe that his arguments could be used to establish this. However, we remark that in this case, the monodromy invariant splitting makes the arguments that we use in §4 much simpler.

We now sketch our results concerning intersection cohomology and obstructions to Q7. Let $\bar{S} = \Delta^r$ be a polydisk with local coordinates $(s_1, \ldots, s_r)$ and $S = \Delta^{*r}$ be the complement of the divisor $s_1 \ldots s_r = 0$. Let $\mathcal{H}$ be a torsion free variation on S with unipotent monodromy and let $\mathrm{IH}^1(\mathcal{H})$ denote the local intersection cohomology group attached to $\mathcal{H}$. If $i : \{0\} \to \Delta^r$ is the inclusion of the origin, then, in terms of the intersection complex $\mathrm{IC}(\mathcal{H})$ on Δ^r associated to $\mathcal{H}$, $\mathrm{IH}^1(\mathcal{H}) = H^{-r+1}(i^* \mathrm{IC}(\mathcal{H}))$. The group $\mathrm{IH}^1(\mathcal{H})$ is a subgroup of $H^1(\Delta^{*r}, \mathcal{H})$. Moreover, since $\mathcal{H}$ is a variation of weight -1, the natural map (cf. Theorem (110))

$$\mathrm{sing} : \mathrm{ANF}(\Delta^{*r}, \mathcal{H})_{\Delta^r} \to H^1(\Delta^{*r}, \mathcal{H})$$

factors through $\mathrm{IH}^1(\Delta^r, \mathcal{H})$ [3]. We then define, for each $t \in \mathbb{Q}^r_{\geq 0}$, a pairing

$$h(t) : \mathrm{IH}^1(\mathcal{H}) \otimes \mathrm{IH}^1(\mathcal{H}^*) \to \mathbb{Q}, \tag{9}$$

which we call the asymptotic height pairing.

Remark 10. It will be clear from context if h refers to the height $h(\mathcal{V})$ of a variation of mixed Hodge structure, or the asymptotoic height pairing $h(t)$.

We will show that the pairing h has the following properties (cf. Theorem (135), Proposition (136) and Theorem 163).

(i) It is homogeneous of degree 1 in t: $h(\lambda t)(\alpha, \beta) = \lambda h(t)(\alpha, \beta)$ for $\lambda \in \mathbb{Q}_{\geq 0}$.
(ii) If $(\nu, \omega) \in \mathrm{ANF}(\Delta^{*r}, \mathcal{H}) \times \mathrm{ANF}(\Delta^{*r}, \mathcal{H}^\vee)$, then $h(\nu, \omega)(t)$ is a rational function which extends continuously to the cone $\mathbb{R}^r_{\geq 0}$.
(iii) For each fixed t, $h(t)$ induces a morphism of Hodge structures with $\mathbb{Q}$ given the pure Hodge structure of weight 0. Thus $h(t)$ factors through a pairing

$$\bar{h}(t) : \mathrm{Gr}^W_0 \mathrm{IH}^1(\mathcal{H}) \otimes \mathrm{Gr}^W_0 \mathrm{IH}^1(\mathcal{H}^*) \to \mathbb{Q}(0), \tag{11}$$

which is a morphism of Hodge structures.
(iv) Suppose $\phi : \mathcal{H} \otimes \mathcal{H} \to \mathbb{Q}(1)$ is a polarization. Then, for $t \in \mathbb{Q}^r_+$, the pairing on $\mathrm{Gr}^W_0 \mathrm{IH}^1(\mathcal{H})$ induced by ϕ and (11) is a polarization.

Remark 12. In section 7 we show that $h(t)$ can be generalized to a pairing (147) on the higher intersection cohomology groups $\mathrm{IH}^p(\mathcal{H})$ as well.

Returning to Q 7, it follows from Theorem (22) below that the metric $|\ |$ on $\mathcal{L}$ does not extend from S to $\bar{S}$ if $h(t)$ is not linear in t.

To illustrate the consequences of the results listed in (iii) and (iv) above, we give the following corollary.

Corollary 13. *$h(t)(\nu, \nu^\vee) \geq 0$ for all ν, t. Moreover, $h(t)(\nu, \nu^\vee) = 0$ for all t if and only if* $\mathrm{sing}(\nu) = 0$.

Proof. $\mathrm{IH}^1(\mathcal{H})$ carries a mixed Hodge structure such that $W_0 \mathrm{IH}^1(\mathcal{H}) = \mathrm{IH}^1(\mathcal{H})$. Furthermore, the map $\mathrm{sing} : \mathrm{ANF}(\Delta^{*r}, \mathcal{H})_{\Delta^r} \to \mathrm{IH}^1(\mathcal{H})$ factors through the subgroup $\mathrm{Hdg}\, \mathrm{IH}^1(\mathcal{H})$ consisting of classes in the image of morphisms in $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(0), \mathrm{IH}^1(\mathcal{H}))$. Therefore (cf. (11)), it follows from (iii) and (iv) above that $h(t)(\nu, \nu^\vee) \geq 0$. Moreover, $h(t)(\nu, \nu^\vee) = 0$ for all t if and only if $\mathrm{sing}(\nu) = 0$. □

Singularities of normal functions can also be defined in the non-normal crossing case (see [3] or (109) below). Let S is a complex manifold and $j : S \hookrightarrow \bar{S}$ is an inclusion of S as a Zariski open subset of a complex manifold $\bar{S}$. We say that $\nu \in \mathrm{ANF}(S, \mathcal{H})_{\bar{S}}$ is *singular* at $p \in \bar{S} - S$ if $\mathrm{sing}_p(\nu) \neq 0$.

1.1. Applications.

Griffiths–Green Program. Let X be a complex projective manifold of dimension $2n$ and $L \to X$ be a very ample line bundle. Let $\bar{P} = |L| = \mathbb{P}(H^0(X, L))$, and let $\mathcal{X} \subset X \times \bar{P}$ be the incidence variety consisting of pairs $(x, \sigma) \in X \times \bar{P}$ such that $\sigma(x) = 0$. Let $\pi : \mathcal{X} \to \bar{P}$ denote the projection map $\pi(x, \sigma) = \sigma$. For $\sigma \in \bar{P}$ let $X_\sigma = \pi^{-1}(\sigma)$. Let $P \subset \bar{P}$ denote the locus of points over which π is smooth, and $\hat{X} = \bar{P} - P$.

Let ζ be a primitive integral, non-torsion Hodge class on X of type (n, n). Then, via Deligne cohomology, ζ determines an admissible normal function

$$\nu_\zeta : P \to J(\mathcal{H})$$

where $\mathcal{H}$ is the variation of Hodge structure on the variable cohomology of the smooth hyperplane sections of X.

Conjecture 14. *For every triple (X, L, ζ) as above, there exists an integer $d > 0$ such that after replacing L by L^d, the normal function ν_ζ is singular at some point of $\hat{X}$.*

Theorem 15. [18, 3, 12] *The Hodge conjecture is equivalent to Conjecture* (14).

By Theorem (7.18) of [3], Conjecture (14) is equivalent to the following statement:

Conjecture 16. *For every triple (X, L, ζ) as above, there exists $d > 0$ and a resolution of singularities $\bar{S} \to \bar{P}$ of $\hat{X}$ for $\bar{P} = |L^d|$ such that $f^*\nu_\zeta$ is singular at some point of $\bar{S} - S$.*

So, using Corollary 13, the Hodge conjecture would imply that there exists a $d > 0$ and an $\bar{S}$ as above such that the biextension metric on $\mathcal{L}(\nu_\zeta, \nu_\zeta^\vee)$ does not extend to $\overline{\mathcal{L}}(\nu_\zeta, \nu_\zeta^\vee)$.

Ceresa Cycle. Let C be a smooth projective curve of genus $g > 2$. Pick a point $p \in C$. Then, the maps $x \mapsto x - p$ and $x \mapsto p - x$ give two embeddings of C into $\mathrm{Jac}(C)$. The difference, denoted $C - C^-$, is a homologically trivial algebraic 1-cycle on $\mathrm{Jac}(C)$ which has a well defined image in the intermediate Jacobian of the primitive part of the cohomology of $\mathrm{Jac}(C)$, independent of p.

Let $\mathcal{M}_g$ denote the moduli space of smooth projective curves of genus $g > 2$. Then, application of the previous construction determines a normal function $\nu = \nu_{C-C^-}$ over $\mathcal{M}_g$. In Theorem (8.3) of [22], Richard Hain showed that is the "basic" normal function over $\mathcal{M}_g$.

In [21], Hain and Reed showed explicitly that $\mathcal{L}(\nu, \nu^\vee)$ extended to a line bundle over $\overline{\mathcal{M}_g}$, and asked if the metric also extends to $\overline{\mathcal{M}_g}$. Our calculations show that it does not. (Hain has since given an independent proof of this result. See [20].)

1.2. Existence and Regularity.

Existence of $\mathcal{L}$, Local Case. In the case where $\mathcal{H}$ is a torsion free variation of pure Hodge structure of weight -1 over a punctured disk Δ^{*r}, it follows from the results of §4 that $\mathcal{L}$ has an analytic extension over Δ^r. (As mentioned above, this is proved in greater generality in Theorem 233.)

Moreover, assuming the monodromy of $\mathcal{H}$ is unipotent, the biextension metric has a continuous extension on the complement of a codimension 2 subset after a suitable renormalization. In the case where $\omega = \nu^\vee$ the biextension metric has a plurisubharmonic extension as described below. More precisely, given a biextension variation $\mathcal{V}$ such that $Gr^W_{-1}(\mathcal{V}) = \mathcal{H}$ has unipotent monodromy, it follows from (Theorem (5.37) [34]) that there exists a unique element

$$\mu(\mathcal{V}) \in V_r \tag{17}$$

with the following property:

Property 18. *Let $\phi : \Delta \to \Delta^r$ be a holomorphic map such that $\phi(0) = 0$ and $\phi(\Delta^*) \subset \Delta^{*r}$. Suppose that $\phi(s) = (\phi_1(s), \dots, \phi_r(s))$ with $\phi_i(s) = a_i s^{m_i}$ + higher order terms and $m_j > 0$. Then*

$$h(\mathcal{V}_{\phi(s)}) \sim -\mu(\mathcal{V})(m_1, \dots, m_r) \log |s| \tag{19}$$

for s close to 0. (In other words, the difference between the left side and the right is bounded near 0.)

Remark 20. The value of $\mu(m)$ depends only on the local monodromy logarithm $N(m) = \sum_j m_j N_j$, the weight filtration $W(V_\mathbb{Q})$ and the choice of positive generators $1 \in Gr^W_0(V_\mathbb{Q})$ and $1^\vee \in Gr^W_{-2}(V_\mathbb{Q})$. More precisely, as long as $m_1, \dots, m_r \geq 0$, the admissibility of $\mathcal{V}$ implies the existence of a rational direct sum decomposition

$$V_\mathbb{Q} = \mathbb{Q}v_0 \oplus U_\mathbb{Q} \oplus \mathbb{Q}v_{-2} \tag{21}$$

such that

— v_0 projects to $1 \in Gr^W_0$ and $N(m)(v_0) \in W_{-2}$;
— $U_\mathbb{Q}$ is an $N(m)$-invariant subspace of W_{-1};
— $v_{-2} \in W_{-2}$ projects to $1^\vee$.

Then, by Theorem (5.19) of [34], for any such decomposition (21) we have $N(m)v_0 = \mu(m)v_{-2}$. Indeed, in the language of [34], such a decomposition of $V_\mathbb{Q}$ defines a grading Y' of W (cf. §2) such that $[Y', N]$ lowers W by 2. For future use, we define $\mu_j = \mu(\epsilon_j)$ where ϵ_j is the j'th unit vector.

Theorem 22. *If $\mathcal{V}$ is a biextension variation of type (ν, ω) over Δ^{*r} with unipotent monodromy then $h(m)(\nu, \omega) = \sum_j m_j \mu_j - \mu(\mathcal{V})$.*

Proof. See Theorem (155) □

Regularity of the Biextension Metric. Let $\mathcal{V}$ be a biextension variation with unipotent monodromy defined on the complement of the divisor $D = \{s_1 \cdots s_k = 0\}$ in a polydisk Δ^r with local coordinates $(s_1, \dots, s_r)$. In this case, it follows by Theorem (5.8) of [25] that $\log |\mathcal{V}|$ is locally L^1.

Let E denote the singular locus of D and $D_j \subset D$ denote the divisor defined by $s_j = 0$. Given a point $p \in (D - E) \cap D_j$ let $\Delta(p) \subset \Delta^r$ denote the disk defined by $s_i = s_i(p)$ for $i \neq j$. Then, $\mathcal{V}$ restricts to an admissible biextension variation on $\Delta^*(p) = \Delta(p) - \{p\}$ with unipotent monodromy, and hence

$$h(\mathcal{V}|_{\Delta^*(p)}) \sim -\mu_j \log |s_j|$$

in the notation of Remark (20).

Define

$$\bar{h}(\mathcal{V}) = h(\mathcal{V}) + \sum_{j=1}^{k} \mu_j \log |s_j| : \Delta^r - D \to [-\infty, \infty) \tag{23}$$

Then, for any point $p \in D - E$, by restriction to a small polydisk containing p, we can reduce the study of the regularity of $h(\mathcal{V})$ at p to the case $k = 1$.

Assume therefore that $k = 1$ and $\mathcal{V}$ has unipotent monodromy about $s_1 = 0$.

Theorem 24. *Let $\mathcal{V}$ be a biextension variation with unipotent monodromy defined on the complement of the divisor $s_1 = 0$ in Δ^r. Then, $\bar{h}(\mathcal{V})$ extends continuously to Δ^r.*

Proof. See Theorem (79). □

Remark 25. The proof of Theorem (24) involves showing that in the case of a biextension over Δ^*

$$\lim_{s \to 0} h(\mathcal{V}) + \mu \log(s) = ht(N, F, W)$$

where $(e^{zN}F, W)$ is the associated nilpotent orbit. See Theorem (76) for details.

Returning now to the general case $\Delta^r - D$ where D is given by the vanishing of $s_1 \cdots s_k = 0$ we first recall the following result in the case where $\omega = \nu^\vee$:

Theorem 26. [33] *If $\mathcal{V}$ is a biextension variation of type $(\nu, \nu^\vee)$ on a complex manifold S then $h(\mathcal{V})$ is plurisubharmonic function on S.*

To continue, we recall the following [See (7.1) and (7.2) [31]]: Let $\{f_j\}$ be a sequence of plurisubharmonic functions on a domain $\Omega \subseteq \mathbb{C}^r$ which are locally bounded from above. Given $z \in \Omega$ let

$$f(z) = \limsup_{j \to \infty} f_j(z)$$

and define $f^*(z) = \limsup_{w \to z} f(w)$. Then, f^* is plurisubharmonic.

Theorem 27. *Let $\mathcal{V}$ be a biextension of type $(\nu, \nu^\vee)$ with unipotent monodromy on the complement of $D = \{s_1 \cdots s_k = 0\}$ in the polydisk Δ^r. Suppose that $\mu_1, \dots, \mu_k \geq 0$. Then, $\bar{h}(\mathcal{V})$ has a unique plurisubharmonic extension to Δ^r.*

Proof. By Theorem (24) it follows that $g = \bar{h}(\mathcal{V}) \in C^0(\Delta^r - D)$ extends to $\bar{g} \in C^0(\Delta^r - E)$. Moreover, since $h(\mathcal{V})$ is plurisubharmonic on $\Delta^r - D$ and all $\mu_j \geq 0$ by assumption, it follows that g is plurisubharmonic on $\Delta^r - D$. In the paragraphs below, we shall prove that $\bar{g}$ is plurisubharmonic on $\Delta^r - E$. Since E has codimension 2, it then follows from a theorem of Grauert and Remmert[17] that $\bar{g}$ has a unique plurisubharmonic extension to Δ^r.

To prove that $\bar{g}$ is plurisubharmonic on $\Delta^r - E$, consider the sequence of functions

$$f_j = \bar{g} + \frac{1}{j} \log |s_1 \dots s_k| : \Delta^r - E \to [-\infty, \infty)$$

where $f_j = -\infty$ on $D - E$. Clearly, each f_j is plurisubharmonic on $\Delta^r - E$ since

— g and $\log |s_1 \dots s_k|$ are plurisubharmonic on $\Delta^r - D$;

— f_j is upper semicontinuous on $\Delta^r - E$ as the sum of the continuous function $\bar{g}$ and the upper semicontinuous function $\frac{1}{j}\log|s_1 \dots s_k|$.
— f_j has the subaveraging property at any point of $D - E$;

Moreover, since $\bar{g}$ is continuous and all $\mu_j \geq 0$ it follows that each f_j is locally bounded above. Accordingly, f^* is plurisubharmonic. Finally, since $\bar{g}$ is continuous on $\Delta^r - E$ it follows form the definition of the sequence f_j that $f^* = \bar{g}$. □

Having constructed an extension of $\bar{h}(\mathcal{V})$ to Δ^r, we can inquire as to its value at $(0,\dots,0) \in \Delta^r$. To this end, let $m = (m_1,\dots,m_r) \in \mathbb{Z}^r_{>0}$ and $\phi : \Delta \to \Delta^r$ be the test curve defined by $s_j = s^{m_j}$, $j = 1,\dots,r$. Then, by Theorem 22

$$\begin{aligned}\phi^*(\bar{h}(\mathcal{V})) &= \phi^*(h(\mathcal{V})) + (\sum_j m_j\mu_j)\log|s| \sim (-\mu(\mathcal{V})(m) + \sum_j m_j\mu_j)\log|s| \\ &\sim h(m)(\nu,\nu^\vee)\log|s|\end{aligned}$$

By Corollary (13), we have the following two cases:

(a) If $\mathrm{sing}_0(\nu) \neq 0$ then $h(m) > 0$ and hence $\lim_{s\to 0} \phi^*(\bar{h}(\mathcal{V})) = -\infty$.
(b) If $\mathrm{sing}_0(\nu) = 0$ then $h(m) = 0$ and hence $\phi^*(\bar{h}(\mathcal{V}))$ is bounded near zero. Since $\bar{h}(\mathcal{V})$ plurisubharmonic it follows from upper semicontinuity that $\bar{h}(\mathcal{V})$ is bounded near zero.

1.3. **Mumford-Grothendieck Biextensions.** There is a close relationship between the concept of biextension variation from (1) and the concept of a biextension introduced by Mumford and studied extensively by Grothendieck in [41]. Essentially, as we vary the normal functions ν and ω, the set of all biextension variations $\mathcal{V}$ forms a biextension (in the Mumford-Grothendieck sense) of $\mathrm{NF}(S,\mathcal{H}) \times \mathrm{NF}(S,\mathcal{H}^\vee)$ by the sheaf $\mathcal{O}_S^\times$. (See Corollary 177.) Starting in §10 we exploit this connection, and we use it in §12 to define another pairing on intersection cohomology with values in $\mathbb{Q}/\mathbb{Z}$ which we use to answer Question 6.

Unfortunately, the two terminologies clash. We think that we have solved the problems by writing the sections involving Mumford-Grothendieck biextensions in such a way that it is always clear which type of biextension we mean. But we would like to warn the reader (and apologize for the fact) that it will sometimes be necessary to decide this based on context.

The Mumford-Grothendieck biextensions we deal with come out of another notion studied by Grothendieck in [41], the notion of *extensions panachées*. These are objects X in an abelian category equipped with a two step filtration. For example, biextension variations are *extensions panachèes* of $\mathbb{Z}$ by $\mathcal{H}$ by $\mathbb{Z}(1)$. In §10, we work out the (slightly subtle) conditions under which the set of isomorphism classes of *extension panachées* give rise to a Mumford-Grothendieck biextension. Then we apply these to certain Mumford-Grothendieck biextensions, mainly of topological type, which arise in connection with the asymptotic height pairing. In §16, we use these results, to generalize the asymptotic height pairing from the case of variations with unipotent monodromy in the normal crossing case to arbitrary pure variations on S in a neighborhood of some $s \in \bar{S}$ (where S is a Zariski open subset of a complex manifold $\bar{S}$).

In this paper, we translate the term *extensions panachées* as "mixed extensions." The (probably better) term "blended extensions" is used by D. Bertrand in the abstract to [2].

1.4. **Related Work.** The thesis [11] of D. Lear contains the initial results of the asymptotics of the height over a punctured disk. An explicit example of the non-linearity of μ appears in Example (5.44) of [34]. The asymptotics of the biextension bundle also appear in the recent work of J. Burgos Gill, D. Holmes and R. de Jong [7] and [6]. Another approach to limits of heights having to do with Feynman amplitudes is given in [1]. A preliminary discussion of heights from the viewpoint of log geometry and compactification of mixed period domains may be found at the end of [30], with additional results appearing in [29].

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The authors have also had extensive discussions of the asymptotics of the height pairing with C. Schnell, M. Kerr and J. Lewis. We thank B. Moonen for informing us about the sections in [41] containing the *extensions panachées*. We also thank H. Boas and S. Biard for helpful discussions concerning extensions of plurisubharmonic functions.

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2. Variations of Mixed Hodge Structure

Weights. Suppose M is any object with a weight filtration W. For example, M could be a mixed Hodge structure, a variation of mixed Hodge structure or a mixed Hodge module. We say that M has *weights in an interval* $[a,b]$ if $\mathrm{Gr}^W_k M = 0$ for $k \notin [a,b]$.

Mixed Hodge Structures. In this paper, all mixed Hodge structure are defined over $A = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. We let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$. For an integer n, we let $A(n)$ denote the pure A-Hodge structure of weight $-2n$ with underlying A-module A [not $(2\pi i)^n A$]. If H and K are mixed Hodge structures and $n \in \mathbb{Z}$, an element $f \in \mathrm{Hom}_{\mathrm{MHS}}(H, K(-n))$ is called a (n,n) morphism of Hodge structures. A morphism of mixed Hodge structures is a $(0,0)$ morphism.

A mixed Hodge structure (F,W) induces a unique, functorial bigrading [Theorem (2.13), [9]]

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \tag{28}$$

on the underlying complex vector space $V_{\mathbb{C}}$ such that

(a) $F^p = \oplus_{a\geq p} I^{a,b}$;
(b) $W_k = \oplus_{a+b\leq k} I^{a,b}$;
(c) $I^{\bar{p},q} = I^{q,p} \mod \oplus_{a<q,b<p} I^{a,b}$.

A mixed Hodge structure is said to be split over $\mathbb{R}$ if $\bar{I}^{p,q} = I^{q,p}$. In this case

$$I^{p,q} = F^p \cap \bar{F}^q \cap W_{p+q} \tag{29}$$

If K is a field of characteristic zero and W is an increasing filtration of a finite dimensional K-vector space V then a grading of W is a semisimple endomorphism Y of V with integral eigenvalues such that

$$W_k = E_k(Y) \bigoplus W_{k-1}$$

for each index k, where $E_k(Y)$ denotes the k-eigenspace of Y. In particular, a mixed Hodge structure (F,W) determines a $\mathbb{C}$-grading $Y = Y_{(F,W)}$ of W by the rule

$$E_k(Y) = \bigoplus_{p+q=k} I^{p,q} \tag{30}$$

In what follows, the grading $Y_{(F,W)}$ will be called the Deligne grading of (F,W).

The set of all gradings of W will be denoted $\mathcal{Y}(W)$. The subalgebra of $gl(V)$ consisting of all elements α such that

$$\alpha(W_k) \subseteq W_{k-\ell}$$

for all k will be denoted $W_{-\ell}gl(V)$.

Proposition 31 (Prop. (2.2) [9]). *$\exp(W_{-1}gl(V))$ acts simply transitively on $\mathcal{Y}(W)$ by the adjoint action.*

Proposition 32. *A mixed Hodge structure (F, W) is split over $\mathbb{R}$ if and only if $Y = \bar{Y}$ where Y is the Deligne grading of (F, W).*

Proof. If (F, W) is $\mathbb{R}$-split then $Y = \bar{Y}$ since $\bar{I}^{p,q} = I^{q,p}$. Conversely, suppose that $Y = \bar{Y}$ and let $\oplus_{p+q=k} H^{p,q}$ be the Hodge decomposition of Gr^W_k. Define

$$J^{p,q} = \{v \in E_{p+q}(Y) \mid [v] \in H^{p,q} \subseteq Gr^W_{p+q}\}$$

Then, $V_{\mathbb{C}} = \oplus_{p,q} J^{p,q}$ satisfies conditions (a)–(c) of (28): Condition (a) follows from the fact that Y is a grading of W which preserves F. Condition (b) follows from the fact that Y is a grading of W. Finally, condition (c) follows from the fact that $\bar{J}^{p,q} = J^{q,p}$. Thus, $I^{p,q} = J^{p,q}$ by the uniqueness of (28), and hence (F, W) is split over $\mathbb{R}$. □

If (F, W) is a mixed Hodge structure with underlying complex vector space $V_{\mathbb{C}}$ then

$$\Lambda^{-1,-1}_{(F,W)} = \bigoplus_{p,q<0} gl(V_{\mathbb{C}})^{p,q} \tag{33}$$

Note that by property (c) of (28), $\Lambda^{-1,-1}$ is closed under complex conjugation. It follows from the defining properties (a)–(c) of (28) that

$$e^{\lambda} I^{p,q}_{(F,W)} = I^{p,q}_{(e^{\lambda}F,W)} \tag{34}$$

Theorem 35 (Theorem (2.20) [9]). *Given a mixed Hodge structure (F, W) with underlying real vector space $V_{\mathbb{R}}$ there exists a unique real element*

$$\delta \in \Lambda^{-1,-1}_{(F,W)} \cap gl(V_{\mathbb{R}}) \tag{36}$$

such that $(e^{-i\delta}F, W)$ is a mixed Hodge structure which is split over $\mathbb{R}$. Moreover, δ commutes with all (r, r)-morphisms of (F, W).

The splitting (36) is henceforth called the Deligne splitting or δ-splitting of (F, W). The defining equation (ibid) for δ is

$$\overline{Y_{(F,W)}} = e^{-2i\,\mathrm{ad}\,\delta} Y_{(F,W)} \tag{37}$$

Definition 38. An unoriented biextension in the category of A-mixed Hodge structures is an A-mixed Hodge structure (F, W) with weights in $[-2, 0]$ such that Gr^W_0, Gr^W_{-2} have rank 1. A biextension mixed Hodge structure is an unoriented biextension equipped with a choice of isomorphisms $Gr^W_0 \cong A(0)$ and $Gr^W_{-2} \cong A(1)$. The positive generators of $A(0)$ and $A(1)$ will be denoted 1 and $1^{\vee}$ respectively.

If (F, W) is a biextension mixed Hodge structure then η is the unique $(-1, -1)$-morphism of (F, W) such that the induced map $Gr^W_0 \to Gr^W_{-2}$ sends 1 to $1^{\vee}$. The group $(\mathbb{C}, +)$ acts additively on the set of biextension mixed Hodge structures by the rule

$$t + (F, W) = (e^{t\eta}F, W) \tag{39}$$

Since η is a $(-1, -1)$-morphism, $t + (F, W) = (F, W)$ if and only if $t = 0$.

Definition 40. The height of a biextension mixed Hodge structure (F, W) is the unique real number $h(F, W)$ such that

$$2\pi\delta_{(F,W)} = h(F, W)\eta \tag{41}$$

Lemma 42. *Let (F, W) be a biextension mixed Hodge structure. Then,*

(a) *For any complex number t,*

$$h(e^{t\eta}F, W) = h(F, W) + 2\pi\mathrm{Im}(t)\eta \tag{43}$$

(b) *$h(F, W)$ depends only the underlying isomorphism class of (F, W) as an $\mathbb{R}$-biextension, i.e. $h(F, W)$ is invariant under isomorphisms of $\mathbb{R}$-mixed Hodge structures such that the induced maps*

$$Gr^W_0 \to Gr^W_0, \qquad Gr^W_{-2} \to Gr^W_{-2}$$

are the identity.

(c) *$h(F, W) = 0$ if and only if (F, W) is $\mathbb{R}$-split.*

Proof. Item (a) follows by direct computation from equations (37) and (39). Item (b) follows from the fact that δ is an isomorphism invariant of $\mathbb{R}$-mixed Hodge structures, i.e. an isomorphism of $\mathbb{R}$-mixed Hodge structures induces an isomorphism of Deligne bigradings, and hence intertwines δ-invariants. Regarding item (c), observe that by construction $\delta_{(F,W)}$ vanishes if and only if (F, W) is split over $\mathbb{R}$. □

In the case where X is a smooth complex projective variety of dimension n and Z and W are null homologous cycles of dimensions d and e such that $|Z| \cap |W|$ is empty and $d + e = n - 1$ there exists a canonical subquotient (F, W) of the mixed Hodge structure on $H_{2d+1}(X - |W|, |Z|)$ such that $h(F, W)$ is the archimedean height of the pair (Z, W). See [19] and equation equation (5.4) of [34] for details.

If (F, W) is a biextension mixed Hodge structure let $\tilde{B} = \tilde{B}(F, W)$ denote the $(\mathbb{C}, +)$-orbit of (F, W) under (39). Since η is a $(-1, -1)$-morphism of (F, W), any two elements of $\tilde{B}(F, W)$ induce the same mixed Hodge structures on Gr^W_k as well as W_0/W_{-2} and W_{-1}. Let B be the quotient of $\tilde{B}$ by $(A, +) \subset (\mathbb{C}, +)$. By (43), the height function (41) descends to a function $h : B \to \mathbb{R}$. For $b \in B$ we define

$$|b| = \exp(-h(b)) \tag{44}$$

In the case $A = \mathbb{Z}$, the set B has a simply transitive $\mathbb{C}^*$ action given by

$$t.[(\tilde{F}, W)] = [(e^{\frac{1}{2\pi i}\log(t)\eta}\tilde{F}, W)] \tag{45}$$

since $\frac{1}{2\pi i}\log(t)$ is well defined modulo $\mathbb{Z}$.

Proposition 46. *Let (F, W) be a $\mathbb{Z}$-biextension mixed Hodge structure and $t \in \mathbb{C}^*$. Then, $|t.b| = |t||b|$.*

Proof. By (43), $h(t.b) = h(b) - \log|t|\eta$. □

Classifying Spaces. A polarization of a pure A-Hodge structure H of weight k is a $(-1)^k$-symmetric, non-degenerate bilinear form

$$Q : H \otimes H \to A(-k)$$

which is a morphism of Hodge structure such that if C is the Weil operator which acts on $H^{p,q}$ as i^{p-q} then $< u, v >= Q(Cu, \bar{v})$ is a positive definite hermitian inner product on $H_{\mathbb{C}}$.

A graded-polarization of an A-mixed Hodge structure (F, W) is a collection of non-degenerate bilinear forms

$$Q_k : Gr^W_k \otimes Gr^W_k \to A(-k)$$

which polarize the pure Hodge structure FGr^W_k for each index k.

Given a graded-polarized mixed Hodge structure $\{(F,W),Q_\bullet\}$ with underlying A-module V_A let X denote the generalized flag variety consisting of all decreasing filtrations $\tilde{F}$ of $V_{\mathbb{C}}$ such that $\dim \tilde{F}^p = \dim F^p$ for each index p. Let $\mathcal{M} \subset X$ denote the classifying space of all $\tilde{F} \in X$ such that

— $(\tilde{F},W)$ is a mixed Hodge structure with the same graded Hodge numbers as (F,W);
— $Q_\bullet$ is a graded-polarization of $(\tilde{F},W)$.

Let $GL(V_{\mathbb{C}})^W$ denote the subgroup of $GL(V_{\mathbb{C}})$ consisting of elements which preserve W. Define $G_{\mathbb{C}}$ to be the complex subgroup of $GL(V_{\mathbb{C}})^W$ consisting of elements which act by isometries of $Q_\bullet$ on Gr^W. Let $G_{\mathbb{R}} = G_{\mathbb{C}} \cap GL(V_{\mathbb{R}})$ and $G \subset G_{\mathbb{C}}$ be the set consisting of elements of $G_{\mathbb{C}}$ which act by real isometries of $Q_\bullet$ on Gr^W. Then, G is a real Lie group which acts transitively on $\mathcal{M}$ by biholomorphisms [35]. The "compact dual" $\check{\mathcal{M}}$ is defined to be the $G_{\mathbb{C}}$ orbit of F in X. In general, $\check{\mathcal{M}}$ is not compact due to the fact that $G_{\mathbb{C}}$ contains the complex subgroup G_{-1} consisting of element of $GL(V_{\mathbb{C}})^W$ which act trivially on Gr^W. Since G also contains G_{-1}, viewed as a real Lie group, it follows that $G_{\mathbb{C}}$ is not the complexification of G unless $G_{-1} = 1$.

See [35] and the references therein for further details on classifying spaces of mixed Hodge structures. In general, variations of mixed Hodge structure do not have good asymptotic behavior in the absence of the existence of a graded-polarization.

Period Maps. The axioms of an admissible variation of mixed Hodge structure are given in [44, 27]. As described in [35], given a variation of (graded-polarized) mixed Hodge structure $\mathcal{V} \to S$ we let

$$\varphi : S \to \Gamma\backslash\mathcal{M} \tag{47}$$

denote the corresponding period map, determined by a choice of point $s_0 \in S$ and Γ is the monodromy group of $\mathcal{V}$.

For the remainder of this section, we fix a polydisk Δ^r with local coordinates $(s_1,\dots,s_r)$ and let Δ^{*r} denote the complement of $s_1\cdots s_r = 0$. Let $(z_1,\dots,z_r)$ denote the standard Euclidean coordinates on $\mathbb{C}^r$ and $\mathfrak{h}^r$ be the product of upper half-planes defined by $\mathrm{Im}(z_1),\dots,\mathrm{Im}(z_r) > 0$. Let $\mathfrak{h}^r \to \Delta^{*r}$ be the universal covering defined by $s_j = e^{2\pi i z_j}$.

An admissible period map [44, 27] $\varphi : \Delta^{*r} \to \Gamma\backslash\mathcal{M}$ with unipotent monodromy then has a local normal form [35]

$$F(z_1,\dots,z_r) = e^{\sum_j z_j N_j} e^{\Gamma(s)} F_\infty \tag{48}$$

where $T_j = e^{N_j}$, F_∞ is the limit Hodge filtration and $\Gamma(s)$ is a holomorphic function on Δ^r which vanishes at the origin and takes values in a vector space complement to the stabilizer of F_∞ in $Lie(G_{\mathbb{C}})$. See [35] for details.

The local normal form (48) for variations of pure Hodge structure appears in [8], and works in the setting of admissible $\mathbb{R}$-VMHS with unipotent monodromy. Indeed, the special case $\Gamma(s) = 0$ is an admissible nilpotent orbit (or an infinitesimal mixed Hodge module).

Passage from the period map $\varphi : \Delta^{*r} \to \Gamma\backslash\mathcal{M}$ to the local normal form (48) may involve replacing Δ^{*r} by a polydisk of smaller radius. Since we are only interested in the asymptotic behavior of the period map in our discussions involving the local normal form, we generally omit this step. However, when it becomes necessary to explicitly restrict to a smaller disk we write

$$\Delta^r_a = \{\, s \in \Delta^r \mid |s_1|,\dots,|s_r| < a \,\}. \tag{49}$$

Remark 50. Variations of pure, polarized Hodge structure with unipotent monodromy are admissible by the results of Schmid [38] and Cattani, Kaplan and Schmid [9]. If $\mathcal{V}$ is a variation of pure Hodge structure, we write $\check{\mathcal{D}}$ and $\mathcal{D}$ in place of $\check{\mathcal{M}}$ and $\mathcal{M}$.

2.1. **Nilpotent Orbits.** The nilpotent orbit attached to an admissible variation $\mathcal{V} \to \Delta^{*r}$ is the map $\theta(z) = \exp(\sum_j z_j N_j)F_\infty$ from $\mathbb{C}^r \to \check{\mathcal{M}}$ obtained by setting $\Gamma = 0$ in (48). In language of [27], the data $(N_1, \dots, N_r; F, W)$ determines an infinitesimal mixed Hodge module (IMHM) We shall often use the alternative term admissible nilpotent orbit instead of IMHM. In the case where $\mathcal{M}$ is a classifying space of pure Hodge structures, an IMHM is just a nilpotent orbit in the sense of Schmid [38]. If $\mathcal{V} \to \Delta^{*r}$ is an admissible variation of mixed Hodge structure, we also use the notation $\mathcal{V}_{\mathrm{nilp}}$ to denote the associated nilpotent orbit.

Definition 51. Let $\mathcal{D}$ be a classifying space of pure Hodge structures of weight k polarized by Q and $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebra consisting of infinitesimal, real automorphisms of Q. Then, an r-variable nilpotent orbit θ with values in $\mathcal{D}$ is a map $\theta : \mathbb{C}^r \to \check{\mathcal{D}}$ of the form

$$\theta(z) = \exp(\sum_j z_j N_j)F \tag{52}$$

where $N_1, \dots, N_r \in \mathfrak{g}_{\mathbb{R}}$ are commuting nilpotent endomorphisms and $F \in \check{\mathcal{D}}$ such that

(i) $N_j(F^p) \subseteq F^{p-1}$;
(ii) There exists a constant a such that if $\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_r) > a$ then $\theta(z) \in \mathcal{D}$.

In particular, since $\mathcal{D}$ encodes the weight of the Hodge structures it parametrizes, the data of a nilpotent orbit with values in $\mathcal{D}$ reduces to $(N_1, \dots, N_r; F)$. If the data of $\mathcal{D}$ is defined over a subfield K of $\mathbb{R}$, one can define the notion of an K-nilpotent orbit by requiring that $N_1, \dots, N_r \in \mathfrak{g}_K = Lie(\mathrm{Aut}_K(Q))$.

Lemma 53. *If $(N_1, \dots, N_r; F)$ generate a K-nilpotent orbit of pure Hodge structure then so does*

$$(t_1 N_1, \dots, t_r N_r; F)$$

for any choice of positive scalars $t_1, \dots, t_r \in K_+$.

Proof. Conditions (i) and (ii) remain true upon replacing N_j by $t_j N_j$. □

To continue, we recall that if N is a nilpotent endomorphism of a finite dimensional vector space V defined over a field of characteristic zero then there exists a unique, increasing filtration $W = W(N)$ of V such that [cf. [9]]

(a) $N(W_j) \subseteq W_{j-2}$;
(b) $N^\ell : Gr^W_\ell \to Gr^W_{-\ell}$ is an isomorphism;

for all j and ℓ.

Theorem 54. *(See [9]) Let θ be a nilpotent orbit of pure Hodge structure of weight k and*

$$\mathcal{C} = \{\sum_j a_j N_j \mid a_1, \dots, a_r > 0\}$$

Then, the map $N \mapsto W(N)$ is constant on $\mathcal{C}$. Pick $N \in \mathcal{C}$ and define $W(N)[-k]_j = W_{j-k}(N)$. Then,

$$(F, W(N)[-k]) \tag{55}$$

is a mixed Hodge structure with respect to which each N_j is a $(-1,-1)$-morphism.

The pair (55) is called the limit mixed Hodge structure of θ. Analysis of the possible nilpotent orbits which can arise can then be reduced to the study of nilpotent orbits with limit mixed Hodge structure which are split over $\mathbb{R}$ by the following results of Cattani, Kaplan and Schmid [9].

Theorem 56. [9] *If $\theta(z_1, \dots, z_r)$ is a nilpotent orbit of pure Hodge structure such that (55) is split over $\mathbb{R}$ then $\theta(z)$ takes values in the appropriate classifying space of pure Hodge structure as soon as $\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_r) > 0$.*

Theorem 57. [9] *If $\theta(z_1,\dots,z_r)$ is a nilpotent orbit of pure Hodge structure and $(\tilde{F},W)=(e^{-i\delta}F,W)$ is the δ-splitting of the limit mixed Hodge structure* (55) *then $\tilde{\theta}(z_1,\dots,z_r)=e^{\sum_j z_jN_j}\tilde{F}$ is a nilpotent orbit of pure Hodge structure.*

The SL_2-orbit theorem of [38, 9] involves the sl_2 or canonical splitting

$$(F,W)\mapsto(e^{-\xi}F,W) \tag{58}$$

where ξ is given by universal Lie polynomials in the Hodge components of δ. In particular, the mixed Hodge structures (F,W), $(e^{-i\delta}F,W)$ and $(e^{-\xi}F,W)$ all induce the same pure Hodge structures on Gr^W since $\delta,\xi\in\Lambda^{-1,-1}_{(F,W)}$.

Definition 59. Let $\theta(z)=e^{\sum_j z_jN_j}F$ be a nilpotent orbit with values in $\mathcal{D}$ and limit mixed Hodge structure (F,W). Let $(\tilde{F},W)$ be the Deligne δ-splitting of (F,W). Then, $\tilde{\theta}(z)=e^{\sum_j z_jN_j}\tilde{F}$ is called the δ-splitting of θ. Likewise, if $(\hat{F},W)$ is the sl_2-splitting of (F,W) then $\hat{\theta}(z)=e^{\sum_j z_jN_j}\hat{F}$ is called the sl_2-splitting of θ.

Remark 60. Theorem (57) holds mutatis mutandis for $\hat{\theta}$. Since the property of being a K-nilpotent orbit does not involve the limit Hodge filtration, passage to the δ or sl_2-splitting does not effect the property of being a K-nilpotent orbit.

For later use, we record that if (F,W) is a biextension mixed Hodge structure then

$$W_{-2}gl(V)=gl(V)^{-1,-1}_{(F,W)} \tag{61}$$

for every point $F\in\mathcal{M}$. Moreover, in the graded-polarized case,

$$\eta\in Z(\mathfrak{g}_{\mathbb{C}}) \tag{62}$$

i.e. η commutes with every element of $\mathfrak{g}_{\mathbb{C}}$. Indeed, any element of $\mathfrak{g}_{\mathbb{C}}$ must act trivially on Gr^W_0 and Gr^W_{-2} and η maps Gr^W_0 to Gr^W_{-2} and annihilates $W_{-1}(V)$.

2.2. **Heights of Nilpotent Orbits.** Let V be a finite dimensional vector space over a field of characteristic zero equipped with a nilpotent endomorphism N and increasing filtration W such that $N(W_k)\subseteq W_k$ for each index k. Then, there exists at most one increasing filtration $M=M(N,W)$, called the relative weight filtration of W with respect to N (see [44]), such that

(a) $N(M_k)\subseteq M_{k-2}$ for all k;
(b) For any k and ℓ, the induced map $N^\ell: Gr^M_{\ell+k}Gr^W_k\to Gr^M_{k-\ell}Gr^W_k$ is an isomorphism.

Definition 63. [44] The data (N,F,W) defines an admissible nilpotent orbit $\theta(z)=(e^{zN}F,W)$ with values in the classifying space $\mathcal{M}$ of graded-polarized mixed Hodge structure provided that

— N is nilpotent, preserves W and acts by infinitesimal isometries on Gr^W;
— $M=M(N,W)$ exists;
— $F\in\check{\mathcal{M}}$ and $N(F^p)\subseteq F^{p-1}$;
— $e^{zN}F$ induces nilpotent orbits of pure, polarized Hodge structure on Gr^W.

Theorem 64. [44] *If (F,N,W) defines an admissible nilpotent orbit and $M=M(N,W)$ then (F,M) is a mixed Hodge structure for which N is a $(-1,-1)$-morphism.*

As in the pure case, (F,M) is called the limit mixed Hodge structure of the nilpotent orbit generated by (N,F,W). An admissible nilpotent orbit θ induces nilpotent orbits of pure Hodge structure on Gr^W such that the limit mixed Hodge structure of θ induces the limit mixed Hodge structure of the graded orbits. The analogs of Theorems (56) and (57) hold mutatis mutandis in the mixed case.

As in Definition (59), one defines the δ-splitting and the sl_2-splitting of an admissible nilpotent orbit θ by replacing by replacing the limit mixed Hodge structure of θ with the corresponding Deligne or sl_2-splitting.

Lemma 65 (Deligne [14, 4])**.** *Let (N, F, W) generate an admissible nilpotent orbit θ with δ-splitting generated by $(N, \tilde{F}, W)$. Define,*

$$Y(N, F, W) = \mathrm{Ad}(e^{-iN})Y_{(e^{iN}\tilde{F}, W)}, \qquad \tilde{Y}(N, F, W) = Y_{(\tilde{F}, M)} \tag{66}$$

Then, writing $Y = Y(N, F, W)$ and $\tilde{Y} = \tilde{Y}(N, F, W)$,

(i) $Y = \bar{Y}$;
(ii) *Y preserves $\tilde{F}$;*
(ii) *Y commutes with $\tilde{Y}$ ($\implies$ Y preserves M);*
(iii) *If $N = \sum_{j\geq 0} N_{-j}$ relative to $\mathrm{ad}\, Y$ then N_0 and $H = \tilde{Y} - Y$ form an sl_2-pair, with associated sl_2-triple (N_0, H, N_0^+).*
(iv) $[N - N_0, N_0^+] = 0$.

Remark 67.

— In the language of Deligne systems, Y is the grading attached to N and $\tilde{Y}$ (cf. [4]).
— For $j > 0$ it follows that N_{-j} is highest weight $j - 2$ for (N_0, H, N_0^+). This forces $N_{-1} = 0$ and $[N_0, N_{-2}] = 0$.
— In the case where $(e^{zN}\tilde{F}, W)$ is a biextension mixed Hodge structure then $N = N_0 + N_{-2}$.

Theorem 68 (Theorem (5.19) [34])**.** *For an admissible biextension variation $\mathcal{V} \to \Delta^*$ with nilpotent orbit generated by (N, F, W),*

$$\mu(\mathcal{V}) = N_{-2}\eta \tag{69}$$

where $N = N_0 + N_{-2}$ defined as in Lemma (65)*.*

Proof. Theorem (5.19) of [34] asserts that any grading Y of W such that $[Y, N] \in W_{-2}gl(V)$ can be used to compute $\mu(\mathcal{V})$ using (69). □

Definition 70. Let (N, F, W) generate an admissible nilpotent orbit of biextension type with Deligne splitting δ of the limit mixed Hodge structure and $Y = Y(N, F, W)$ defined as in (66). Let $\delta = \sum_{k\leq 0} \delta_k$ with $[Y, \delta_k] = k\delta_k$. Then, the limit height $ht(N, F, W)$ of the nilpotent orbit $(e^{zN}F, W)$ is given by the formula

$$2\pi\delta_{-2} = ht(N, F, W)\eta \tag{71}$$

Remark 72. A biextension mixed Hodge structure (F, W) on V determines an admissible nilpotent orbit by setting $N = 0$. In this case $M = W$, δ is the Deligne splitting of (F, W) and $Y(N, F, W)$ is a grading of W. Since $\delta \in W_{-2}gl(V)$ it then follows from the short length of W that $\delta = \delta_{-2}$ relative to $Y(N, F, W)$ and hence $ht(N, F, W) = h(F, W)$.

3. Regularity Results

The following result justifies the name limit height (Theorem (76)): Let $\mathcal{V} \to \Delta^*$ be an admissible biextension VMHS with nilpotent orbit $(e^{zN}F_\infty, W)$ and asymptotic height $h(\mathcal{V}) \cong -\mu \log|s|$. Then,

$$\bar{h}(s) = h(\mathcal{V}) + \mu \log|s|$$

has a continuous extension to Δ with $\bar{h}(0) = ht(N, F_\infty, W)$.

SL_2**-orbits.** In this section we review the theory of SL_2 orbits for nilpotent orbits of biextension type following [34]. Let $\mathcal{M}$ be the ambient classifying space with associated Lie group G and $\tilde{G}$ be the subgroup of G which acts as real transformations on W_k/W_{k-2} for all k.

Let $(e^{zN}F, W)$ be an admissible nilpotent orbit of biextension type with limit mixed Hodge structure (F, M), let $(\tilde{F}, M) = (e^{-i\delta}F, M)$ denote the Deligne splitting of (F, M) and $\tilde{Y} = Y(N, F, W)$, $Y = Y(N, F, W)$, and $N = N_0 + N_{-2}$, N_0^+ etc. as in Lemma (65).

To continue, we recall the SL_2-orbit Theorem of [34]:

Theorem 73 (Theorem 4.2 [34]). *Let $(e^{zN}F, W)$ be an admissible nilpotent orbit of biextension type. Then, there exists*

$$\chi \in Lie(\tilde{G}) \cap \ker(N) \cap \Lambda^{-1,-1}_{(\tilde{F},M)}$$

and distinguished real analytic function $g : (a, \infty) \to \tilde{G}$ such that

(a) $e^{iyN}.F = g(y)e^{iyN}.\tilde{F}$;
(b) *$g(y)$ and $g^{-1}(y)$ have convergent series expansions about ∞ of the form*

$$g(y) = e^{\chi}(1 + g_1 y^{-1} + g_2 y^{-2} + \cdots)$$
$$g^{-1}(y) = (1 + f_1 y^{-1} + f_2 y^{-2} + \cdots)e^{-\chi}$$

with g_k, $f_k \in \ker(\mathrm{ad}\,\mathrm{N_0})^{\mathrm{k}+1} \cap \ker(\mathrm{ad}\,\mathrm{N_{-2}})$;
(c) *δ, χ and the coefficients g_k are related by the formula*

$$e^{i\delta} = e^{\chi}\left(1 + \sum_{k>0} \frac{1}{k!}(-i)^k (\mathrm{ad}\ \mathrm{N_0})^{\mathrm{k}}\, \mathrm{g_k}\right)$$

Moreover, χ can expressed as a universal Lie polynomial over $\mathbb{Q}(\sqrt{-1})$ in the Hodge components $\delta^{r,s}$ of δ with respect to $(\tilde{F}, W)$. Likewise, the coefficients g_k and f_k can be expressed as universal, non-commuting polynomials over $\mathbb{Q}(\sqrt{-1})$ in $\delta^{r,s}$ and $\mathrm{ad}\ \mathrm{N}_0^+$.

Corollary 74 (Corollary (4.3) [34]). *Let $\mathcal{V} \to \Delta^*$ be an admissible biextension variation, with period map $F(z) : \mathfrak{h} \to \mathcal{M}$ and nilpotent orbit $e^{zN}F$. Let $F_0 = e^{iN_0}\tilde{F}$. Then, (cf. Theorem (73)), there exists a distinguished, real–analytic function $\gamma(z)$ with values in $Lie(\tilde{G})$ such that, for* $\mathrm{Im(z)}$ *sufficiently large,*

(i) $F(z) = e^{xN}g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.F_0$;
(ii) *$|\gamma(z)| = O(\mathrm{Im}(z)^{\beta}e^{-2\pi \mathrm{Im}(z)})$ as $y \to \infty$ and x restricted to a finite subinterval of $\mathbb{R}$, for some constant $\beta \in \mathbb{R}$.*

Punctured Disk. Let $\mathcal{V} \to \Delta^*$ be an admissible biextension with nilpotent orbit $(e^{zN}F, W)$ and define δ, Y, $\tilde{Y}$ etc. as in the previous subsection. Observe that since $\tilde{Y}$ and Y commute, we can write

$$\delta = \sum_{a,b} \delta_{a,b}, \qquad [\tilde{Y}, \delta_{a,b}] = a\delta_{a,b}, \quad [Y, \delta_{a,b}] = b\delta_{a,b}$$

Define $\chi = \sum_{a,b} \chi_{a,b}$ similarly. Let

$$\delta(j) = \sum_{a-b=j} \delta_{a,b}, \qquad \chi(j) = \sum_{a-b=j} \chi_{a,b}$$

and note that $\delta(j)$ and $\chi(j)$ are the respective projections of δ and χ onto the j-eignespace of H. In particular, since $N = N_0 + N_{-2}$ and N_{-2} is central in $\mathfrak{g}_{\mathbb{C}}$ it follows that $[N_0, \delta(j)] = 0$ and $[N_0, \chi(j)] = 0$. Consequently, $\delta(j)$ and $\chi(j)$ vanish for $j > 0$.

Lemma 75. *$g^{\ddagger}(y) = \mathrm{Ad}(y^{H/2})g(y)$ is given by a convergent power series in powers of $y^{-1/2}$ with*

$$\lim_{y\to\infty} g^{\ddagger}(y) = e^{i\delta_{-2}}$$

where $\delta = \delta_0 + \delta_{-1} + \delta_{-2}$ *relative to* $\operatorname{ad} Y$.

Proof. To compute the limiting value, it is sufficient to show that

(a) $\lim_{y\to\infty} \operatorname{Ad}(y^{H/2})(e^{-\chi}g(y)) = 1$;
(b) $\lim_{y\to\infty} \operatorname{Ad}(y^{H/2})e^{\chi} = e^{i\delta_{-2}}$;

The intermediate computations will show that $g^{\ddagger}(y)$ has a power series expansion of the stated form.

To verify (a), recall that

$$g(y) = e^{\chi}(1 + g_1 y^{-1} + g_2 y^{-2} + \cdots)$$

with $g_k \in \ker(\operatorname{ad}(N_0)^{k+1})$. Let $g(y)(j)$ denote the projection of $e^{-\chi}g(y)$ to the j-eigenspace of $\operatorname{ad}(H)$. (Note the slight conflict with the notation used above for δ and χ.) Define

$$g_-(y) = \sum_{j<0} g(y)(j), \qquad g_0(y) = g(y)(0), \qquad g_+(y) = \sum_{j>0} g(y)(j)$$

Then,

$$e^{-\chi}g(y) = g_-(y) + g_0(y) + g_+(y).$$

By construction, $\operatorname{Ad}(y^{H/2})g_-(y)$ is a convergent power series in $y^{-1/2}$ with constant term zero. Likewise, $\operatorname{Ad}(y^{H/2})g_0(y) = g(y)(0)$ is a convergent series in $y^{-1/2}$ with constant term 1.

To continue, observe that $g_k(j) = 0$ unless $k \geq j$ since $\operatorname{ad}(N_0)^{k+1}g_k = 0$. Consequently, for $j > 0$ we have

$$g(y)(j) = \sum_{k\geq j} g_k(j) y^{-k} = y^{-j}\sum_{k\geq j} g_k(j) y^{j-k}$$

As such, for $j > 0$, $\operatorname{Ad}(y^{H/2})g(y)(j)$ is a convergent power series in $y^{-1/2}$ with constant term zero. Combining this with the previously computations, this shows that $g^{\ddagger}(y)$ has a convergent series expansion in $y^{-1/2}$ with constant term 1. This establishes (a).

To verify (b), we note that

$$e^{iyN}e^{i\delta}\tilde{F} = e^{\chi}e^{-\chi}g(y)e^{iyN}\tilde{F}.$$

Since N_{-2} is central in $\mathfrak{g}_{\mathbb{C}}$, it follows that

$$e^{iyN_0}e^{i\delta}\tilde{F} = e^{\chi}e^{-\chi}g(y)e^{iyN_0}\tilde{F}.$$

Applying $y^{H/2}$ to each side of this equation, and using the fact that H preserves $\tilde{F}$ it follows that

$$e^{iN_0}e^{i\sum_{j\leq 0} y^{j/2}\delta(j)}\tilde{F} = e^{\sum_{j\leq 0} y^{j/2}\chi(j)}\operatorname{Ad}(y^{H/2})(e^{-\chi}g(y))e^{iN_0}\tilde{F}$$

Using part (a), and taking the limit at $y \to \infty$ yields

$$e^{iN_0}e^{i\delta(0)}\tilde{F} = e^{\chi(0)}e^{iN_0}\tilde{F}$$

In particular, since $[N_0, \chi] = 0$ it follows that $[N_0, \chi(j)] = 0$ for each j. Applying e^{-iN_0} to the previous equation therefore implies

$$e^{i\delta(0)}\tilde{F} = e^{\chi(0)}\tilde{F}$$

Finally, $\delta(0)$ and $\chi(0)$ take values in $\Lambda^{-1,-1}_{(\tilde{F},M)}$ and the map $\lambda \in \Lambda^{-1,-1}_{(\tilde{F},M)} \to e^{\lambda}\tilde{F}$ is injective. Thus, $i\delta(0) = \chi(0)$.

To finish the proof of part (b), observe that δ is real and χ is required to act by real transformations on W_k/W_{k-2}. Therefore $i\delta(0) = \chi(0)$ implies that $\chi(0) = \chi_{-2,-2} = i\delta_{-2,-2}$. Indeed,

$$i\delta(0) = \chi(0) \iff i\delta_{p,p} = \chi_{p,p}$$

for each p. In order for χ to act by real transformations on W_k/W_{k-2} we must have $p = -2$. Since $W_{-2}Lie(\mathbb{G}_{\mathbb{C}})$ is central,

$$[\tilde{Y}, \delta_{-2}] = [H + Y, \delta_{-2}] = [Y, \delta_{-2}] = -2\delta_{-2}$$

Consequently, $\chi(0) = i\delta_{-2,-2} = i\delta_{-2}$. Using the fact that $\chi(j) = 0$ for $j > 0$, the computation of the limit in (b) then follows directly. □

Theorem 76. *Let $\mathcal{V} \to \Delta^*$ be an admissible biextension with unipotent monodromy $T = e^N$ and limit mixed Hodge structure (F, M). Then*

$$\lim_{s\to 0} h(s) = \lim_{s\to 0} h(\mathcal{V}) + \mu \log|s| = ht(N, F, W).$$

Proof. Let $(\tilde{F}, M) = (e^{-i\delta}F, M)$ be Deligne's δ-splitting of the limit mixed Hodge structure (F, M) of $\mathcal{V}$. Let $F(z)$ be the lifting of the period map of $\mathcal{V}$ to the upper half-plane. By Corollary (74), we have

$$\delta_{(F(z),W)} = \delta_{(e^{xN}g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.F_0,W)}$$

where $F_0 = e^{iN_0}\tilde{F}$, and we use $g.F_0$ instead of gF to break up long computations. Moreover, $|\gamma(z)| = O(\mathrm{Im}(z)^{\beta}e^{-2\pi\mathrm{Im}(z)})$ as $y \to \infty$ and x restricted to a finite subinterval of $\mathbb{R}$. Since xN is real, it follows from that

$$\begin{aligned}\delta_{(F(z),W)} &= \mathrm{Ad}(e^{xN})\delta_{(g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.F_0,W)}\\ &= \delta_{(g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.F_0,W)}.\end{aligned}$$

Likewise, since $N_{-2} \in W_{-2}\mathfrak{g}_{\mathbb{C}}$ is central, it follows (using (41) and (43)) that

$$\begin{aligned}\delta_{(F(z),W)} &= \delta_{(g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.F_0,W)}\\ &= \delta_{(g(y)y^{-H/2}e^{\gamma(z)}.F_0,W)} + yN_{-2}.\end{aligned}$$

Consequently,

$$\begin{aligned}\delta_{(F(z),W)} - yN_{-2} &= \delta_{(g(y)y^{-H/2}e^{\gamma(z)}.F_0,W)}\\ &= \delta_{(y^{-H/2}y^{H/2}g(y)y^{-H/2}e^{\gamma(z)}.F_0,W)}\\ &= \mathrm{Ad}(y^{-H/2})\delta_{(y^{H/2}g(y)y^{-H/2}e^{\gamma(z)}.F_0,W)}\\ &= \delta_{(y^{H/2}g(y)y^{-H/2}e^{\gamma(z)}.F_0,W)}\end{aligned}$$

since $W_{-2}gl(V) \subset Z(\mathfrak{g}_{\mathbb{C}})$ and hence $\mathrm{Ad}(y^{-H/2})$ acts trivially on $W_{-2}gl(V)$. Therefore,

$$\lim_{\mathrm{Im}(z)\to\infty} \delta_{(F(z),W)} - yN_{-2} = \delta_{(e^{i\delta_{-2}}e^{iN_0}.\tilde{F},W)} \tag{77}$$

for $\mathrm{Re}(z)$ restricted to an interval of finite length.

To finish the proof, we note that $(e^{iN_0}.\tilde{F}, W)$ is a mixed Hodge structure which is split over $\mathbb{R}$ with $Y_{(e^{iN_0}.\tilde{F},W)} = Y$. Consequently,

$$\delta_{(e^{i\delta_{-2}}e^{iN_0}.\tilde{F},W)} = \delta_{-2} + \delta_{(e^{iN_0}.\tilde{F},W)} = \delta_{-2}.$$

Therefore, using equations (71), (69) and (41), equation (77) becomes

$$\lim_{\mathrm{Im}(z)\to\infty} h(F(z), W) + \mu\log|s| = ht(N, F, W).$$

□

Smooth Divisor. Consider now the case where $\mathcal{V}$ is a biextension variation over the complement of $s_1 = 0$ in a polydisk Δ^r with coordinates $(s_1, \ldots, s_r)$. Let

$$F(z_1, s_2, \ldots, s_r) = e^{z_1N_1}e^{\Gamma(s)}.F_\infty \tag{78}$$

denote the local normal form of the period map of $\mathcal{V}$ after lifting to $U \times \Delta^{r-1}$ where U is the upper half-plane with coordinate z_1 and $s_1 = e^{2\pi i z_1}$. To simplify notation, we shall write N in place of N_1 and $z = x + iy$ in place of z_1 where convenient.

To continue, we recall [25] that

$$[N_1, \Gamma(0, s_2, \ldots, s_r)] = 0$$

Accordingly, we let

$$\Gamma_1 = \Gamma(0, s_2, \ldots, s_r), \qquad e^{\Gamma(s)} = e^{\Gamma^1} e^{\Gamma_1}$$

and note that $\Gamma^1(0, s_2, \ldots, s_r) = 0$ by construction, and hence $s_1 | \Gamma^1$.

Let $(\tilde{F}_\infty, M) = (e^{-i\delta}.F_\infty, M)$ be the Deligne splitting of the limit mixed Hodge structure of $\mathcal{V}$. Define $N = N_0 + N_{-2}$ and $H = \tilde{Y} - Y$ etc. as in Theorem (73) for the nilpotent orbit $(e^{zN}.F_\infty, W)$. Then, since N_{-2} is central in $\mathfrak{g}_\mathbb{C}$ and $[N, \Gamma_1] = 0$ it follows that $[N_0, \Gamma_1] = 0$. This forces

$$\Gamma_1 = \sum_{k \le 0} \Gamma_{1,k}, \qquad [H, \Gamma_{1,k}] = k\Gamma_{1,k}.$$

Let $\Gamma^1 = \sum_k \Gamma^1_k$ where $[H, \Gamma^1_k] = k\Gamma^1_k$.

Theorem 79. *Let $\mathcal{V}$ be a biextension variation with unipotent monodromy defined on the complement of the divisor $s_1 = 0$ in Δ^r. Then,*

$$\bar{h}(\mathcal{V}) = h(\mathcal{V}) + \mu \log |s_1|$$

extends continuously to Δ^r.

Proof. By the above remarks, we can write

$$\begin{aligned}
F(z, s_2, \ldots, s_r) &= e^{xN} e^{iyN} e^{\Gamma(s)} e^{-iyN} e^{iyN}.F_\infty \\
&= e^{xN} e^{iyN} e^{\Gamma^1(s)} e^{\Gamma_1(s)} e^{-iyN} e^{iyN}.F_\infty \\
&= e^{xN} e^{iyN} e^{\Gamma^1(s)} e^{-iyN} e^{\Gamma_1(s)} e^{iyN}.F_\infty \\
&= e^{xN} e^{iyN_0} e^{iyN_{-2}} e^{\Gamma^1(s)} e^{-iyN_{-2}} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN}.F_\infty \\
&= e^{xN} e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN}.F_\infty.
\end{aligned}$$

By theorem (73), we can then write

$$\begin{aligned}
e^{iyN} F_\infty &= g(y) e^{iyN}.\tilde{F}_\infty \\
&= g(y) e^{iyN_{-2}} e^{iyN_0} \tilde{F}_\infty \\
&= e^{iyN_{-2}} g(y) e^{iyN_0} \tilde{F}_\infty \\
&= e^{iyN_{-2}} g(y) y^{-H/2} e^{iN_0} \tilde{F}_\infty \\
&= e^{iyN_{-2}} y^{-H/2} g^\ddagger(y) e^{iN_0} \tilde{F}_\infty
\end{aligned}$$

where $g^\ddagger(y) = \mathrm{Ad}(y^{H/2}) g(y)$ as in Lemma (75).

Combining the previous two paragraphs yields

$$\begin{aligned}
\delta_{(F(z, s_2, \ldots, s_r), W)} &= \delta_{(e^{xN} e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN}.F_\infty, W)} \\
&= \delta_{(e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN}.F_\infty, W)} \\
&= \delta_{(e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN_{-2}} y^{-H/2} g^\ddagger(y) e^{iN_0} \tilde{F}_\infty, W)} \\
&= \delta_{(e^{iyN_{-2}} e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} y^{-H/2} g^\ddagger(y) e^{iN_0} \tilde{F}_\infty, W)} \\
&= \delta_{(e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} y^{-H/2} g^\ddagger(y) e^{iN_0} \tilde{F}_\infty, W)} + yN_{-2}.
\end{aligned}$$

To complete the proof, let $\Gamma^1_\ddagger = \mathrm{Ad}(y^{H/2})\Gamma^1$, and observe that $\Gamma^1_\ddagger$ is a finite sum of functions $(\log|s_1|)^{k/2} s_1 f$ where f is a holomorphic function on Δ^r with values in $\mathfrak{q}$, and hence has a continuous extension to Δ^r. Likewise,

$$\Gamma_1^\ddagger = \mathrm{Ad}(y^{H/2})\Gamma_1$$

is a finite sum of functions of the form $(\log|s_1|)^{-k/2} f$ for $k \geq 0$ and f a holomorphic function on Δ^r with values in $\mathfrak{q}$ which is independent of s_1, and hence also has a continuous extension to Δ^r. Finally, $g^\ddagger(y)$ is independent of $(s_2, \dots, s_r)$ and has a convergent series expansion in powers of $(\log|s_1|)^{-1/2}$ by Lemma (75).

Combining the above, it follows that

$$\begin{aligned}
\delta_{(F(z,s_2,\dots,s_r),W)} - yN_{-2} &= \delta_{(e^{iyN_0}e^{\Gamma^1(s)}e^{-iyN_0}e^{\Gamma_1(s)}y^{-H/2}g^\ddagger(y)e^{iN_0}\tilde{F}_\infty,W)} \\
&= \delta_{(y^{-H/2}e^{iN_0}e^{\Gamma^1_\ddagger(s)}e^{-iN_0}e^{\Gamma_1^\ddagger(s)}g^\ddagger(y)e^{iN_0}.\tilde{F}_\infty,W)} \\
&= \mathrm{Ad}(y^{-H/2})\delta_{(e^{iN_0}e^{\Gamma^1_\ddagger(s)}e^{-iN_0}e^{\Gamma_1^\ddagger(s)}g^\ddagger(y)e^{iN_0}.\tilde{F}_\infty,W)} \\
&= \delta_{(e^{iN_0}e^{\Gamma^1_\ddagger(s)}e^{-iN_0}e^{\Gamma_1^\ddagger(s)}g^\ddagger(y)e^{iN_0}.\tilde{F}_\infty,W)}
\end{aligned}$$

since $\mathrm{Ad}(y^{-H/2})$ acts trivially on $W_{-2}\mathfrak{g}_\mathbb{C}$. By the remarks of the previous paragraph

$$e^{iN_0}e^{\Gamma^1_\ddagger(s)}e^{-iN_0}e^{\Gamma_1^\ddagger(s)}g^\ddagger(y)e^{iN_0}$$

has a continuous extension to Δ^r, and hence so does $\delta_{(F(z,s_2,\dots,s_r),W)} - yN_{-2}$. Using equations (71), (69) and (41), it follows as in the last paragraph of the proof of Theorem (76) that $\bar{h}(\mathcal{V})$ has a continuous extension to Δ^r. □

Remark 80. Since $\bar{h}(\mathcal{V})$ extends continuously to Δ^r its value at any point $(0, s_2, \dots, s_r)$ can be computed by taking the limit along $s \to (s, s_2, \dots, s_r)$. Since this corresponds to a 1 parameter degeneration, it follows from Theorem (76) that $\bar{h}(\mathcal{V})(0, s_2, \dots, s_r)$ depends only on the limit mixed Hodge structure of $\mathcal{V}$ at $(0, s_2, \dots, s_r)$.

4. Existence of Biextensions

In this section we prove the following result, which shows that over Δ^{*r} a pair of admissible normal functions can be glued together to give an admissible biextension. For the remainder of this section, A is either $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, and

- — Δ^r is a polydisk with coordinates $(s_1, \dots, s_r)$;
- — D is the divisor defined by $s_1 \cdots s_k = 0$;
- — $\mathcal{H}$ is a torsion free, polarized variation of pure A-Hodge structure of weight -1 on $\Delta^r - D$ with quasi-unipotent monodromy;
- — ν is an admissible extension of $A -$ VMHS

  $$0 \to \mathcal{H} \to \mathcal{V}_\nu \to A(0) \to 0$$

  with respect to inclusion of $\Delta^r - D$ into Δ^r;
- — $\omega$is an admissible extension of $A -$ VMHS

  $$0 \to A(1) \to \mathcal{V}^\vee_\omega \to \mathcal{H} \to 0$$

  with respect to inclusion of $\Delta^r - D$ into Δ^r.

Theorem 81. *The set $B(\nu, \omega)$ of admissible biextension variations of type (ν, ω) over $\Delta^r - D \hookrightarrow \Delta^r$ is non-empty.*

We prove Theorem (81) in the case of unipotent monodromy first, and hence assume that all VMHS have unipotent monodromy unless otherwise noted. By formally allowing the monodromy about $s_j = 0$ to be trivial, we can reduce the proof of Theorem (81) to the case where D is the divisor $s_1 \dots s_r = 0$. The outline of the proof is as follows: The normal functions ν and ω have local normal forms (48). Likewise, as a consequence of the SL_2-orbit theorem if $F : \mathfrak{h} \to \mathcal{M}$ is the period map of an admissible nilpotent orbit then

$$\lim_{\mathrm{Im}(z)\to\infty} Y_{(F(z),W)}$$

exists as a grading of W. Moreover, this grading will preserve the $I^{p,q}$'s of the sl_2-splitting of the limit mixed Hodge structure of $(F(z), W)$. This produces a trigrading of the underlying vector space. For ν and ω, we must then glue these two trigradings together, and use the resulting new vector space as the model for the biextension variation. It remains then to add an integral structure, local monodromy logarithms et. cetera.

A sketch of the proof of Theorem (81) in the quasi-unipotent case is a follows: A quasi-unipotent VMHS can be viewed as a VMHS on the pullback to the unipotent setting which has the extra property that when the i'th coordinate is multiplied by a suitable root of unity, the Hodge filtration changes by the appropriate semisimple factor. Gluing ν and ω together in the quasi-unipotent case therefore reduces to the unipotent case, together with the fact that the semisimple parts of the local monodromy are morphisms of Hodge structure in the limit, and are therefore compatible with the gluing procedure.

Trigraded Vector Spaces. The *weight filtration* of a trigraded vector space

$$L = \bigoplus_{p,q,i} L_i^{p,q} \tag{82}$$

is the increasing filtration $W_k(L) = \bigoplus_{i\leq k,p,q} L_i^{p,q}$. The subspaces $L_i = \bigoplus_{p,q} L_i^{p,q}$ define a grading of $W_\bullet(L)$. Likewise, we set $L^{p,q} = \oplus_i L_i^{p,q}$ and call

$$L = \bigoplus_{p,q} L^{p,q}$$

is the associated *bigrading* of L. We define the *relative weight filtration* of L to be the increasing filtration $M_k(L) = \oplus_{p+q\leq k,i} L_i^{p,q}$. If L is a complex vector space with real form $L_{\mathbb{R}}$ then a trigrading of L is said to split over $\mathbb{R}$ if $\bar{L}_i^{p,q} = L_i^{q,p}$. In this case, the decreasing filtration $F^p(L) = \oplus_{a\geq p,b} L^{a,b}$ pairs with M to define an $\mathbb{R}$-split mixed Hodge structure (F, M) which induces pure Hodge structures on Gr^W and has Deligne bigrading $I^{p,q} = L^{p,q}$.

If U and V are vector spaces equipped with linear maps $f : U \to Z$ and $g : V \to Z$ then the fiber product

$$U \times_Z V = \{(u,v) \in U \times V \mid f(u) = g(v)\}$$

is called the *gluing* of U and V along Z. We let $\pi_U : U \times_Z V \to U$ and $\pi_V : U \times_Z V \to V$ denote the natural projection maps.

Let U and V be trigraded vector spaces. Then U *abuts* V if there exists an index ℓ such that

(a) $U_i = 0$ for $i > \ell$ and $V_i = 0$ for $i < \ell$;
(b) There exists an isomorphism $Gr_\ell^W(U) \cong Gr_\ell^W(V)$ for which the composite map

$$\sigma : U_\ell \to Gr_\ell^W(U) \cong Gr_\ell^W(V) \to V_\ell$$

induces an isomorphism $U_\ell^{p,q} \to V_\ell^{p,q}$ for each bi-index (p,q).

If U abuts V (at ℓ) we define $Gr^W_\ell \subset Gr^W_\ell(U) \times Gr^W_\ell(V)$ to be the subspace consisting of points $([u],[v])$ which are identified under the isomorphism (b). We then define

$$U \star V = U \times_{Gr^W_\ell} V$$

relative to the linear maps $f : U \to U_\ell \to Gr^W_\ell$ and $g : V \to V_\ell \to Gr^W_\ell$.

Let ι_U from $U \to U \star V$ denote the linear map which sends $u \in U_i$ to $(u,0) \in U \star V$ for $i < \ell$ and maps $u \in U_\ell$ to $(u, \sigma(u)) \in U \star V$. Let ι_V from $V \to U \star V$ be the map that send $v \in V_i$ to $(0,v) \in U \star V$ for $i > \ell$ and maps $v \in V_\ell$ to $(\sigma^{-1}(v), v) \in U \star V$. Accordingly, we can introduce a trigrading on $U \star V$ by setting

$$(U \star V)^{p,q}_i = \begin{cases} \iota_U(U^{p,q}_i), & i \leq \ell \\ \iota_V(V^{p,q}_i), & i > \ell \end{cases} \tag{83}$$

If the trigradings of U and V are split over $\mathbb{R}$ and the isomorphism $Gr^W_\ell(U) \to Gr^W_\ell(V)$ intertwines complex conjugation then the trigrading (83) is split over $\mathbb{R}$.

Endomorphisms. Let U and V be trigraded vector spaces which abut (at ℓ). Let α_U and α_V be endomorphisms of U and V respectively which preserve the corresponding weight filtrations $W(U)$ and $W(V)$. Assume that α_U and α_V induce the same map on Gr^W_ℓ. For $\eta \in (U \star V)_i$ define

$$(\alpha_U \star \alpha_V)(\eta) = \begin{cases} \iota_U \circ \alpha_U \circ \pi_U(\eta), & i \leq \ell \\ \iota_V \circ \alpha_V \circ \pi_V(\eta), & i > \ell. \end{cases} \tag{84}$$

Then, $\alpha = \alpha_U \star \alpha_V$ is an endomorphism of $U \star V$ which preserves $W(U \star V)$ such that

$$\pi_U \circ \alpha \circ \iota_U = \alpha_U, \qquad \pi_V \circ \alpha \circ \iota_V = \alpha_V.$$

.

Lemma 85. *The construction* (84) *is compatible with the bigrading* (83)*, i.e. if* $\alpha_U(U^{a,b}) \subseteq U^{p+a,q+b}$ *and* $\alpha_V(V^{a,b}) \subseteq V^{p+a,q+b}$ *then*

$$\alpha(U \star V)^{a,b} \subseteq (U \star V)^{p+a,q+b} \tag{86}$$

Proof. This follows from that fact that $\pi_U \circ \iota_U$ is the identity on U, $\pi_V \circ \iota_V$ is the identity on V and both the maps π_U, π_V, ι_U, ι_V all preserve the relevant bigradings by construction. □

Limiting Splitting. Let $\psi : \Delta^* \to \Gamma\backslash\mathcal{M}$ be the period map of an admissible extension of $\mathbb{R}(0)$ by a variation of pure, polarized $\mathbb{R}$-Hodge structure of weight -1. Let V denote the underlying vector space of ψ, let W denote the weight filtration, and assume the monodromy is unipotent and given by $T = e^N$. Let

$$F(z) : \mathfrak{h} \to \mathcal{M}, \qquad F(z+1) = TF(z)$$

be a lifting of ψ to a map from the upper half-plane $\mathfrak{h}$ into $\mathcal{M}$. Then, by [5, Theorem 3.9], it follows that

$$Y^\ddagger = \lim_{\mathrm{Im}(z) \to \infty} Y_{(F(z),W)} \tag{87}$$

exists, where the limit is taken with $\mathrm{Re}(z)$ restricted to a finite interval. More precisely,

$$Y^\ddagger = \mathrm{Ad}(\mathrm{e}^{-\mathrm{i}\mathrm{N}})\mathrm{Y}_{(\mathrm{e}^{\mathrm{i}\mathrm{N}}\hat{\mathrm{F}},\mathrm{W})}$$

where $(\hat{F}, M) = (e^{-\xi}F, M)$ is sl_2-splitting of the limit mixed Hodge structure (F, M) of $F : \mathfrak{h} \to \mathcal{M}$.

Corollary 88. *The subspaces*

$$L^{p,q}_k = \{\, v \in I^{p,q}_{(\hat{F},M)} \mid Y^\ddagger(v) = kv \,\}$$

form a trigrading of V which is split over $\mathbb{R}$ such that $W(L) = W$, $F(L) = \hat{F}$ and $M(L) = M$ as in (82).

For future use, we record that in this setting, the element ξ appearing (58) in the sl_2-splitting of (F, M) also preserves the weight filtration W [4]. The next result is only used in the quasi-unipotent case:

Proposition 89. *If γ is a morphism of (F, M) which preserves W such that $[\gamma, N] = 0$ then $[\gamma, Y^{\ddagger}] = 0$.*

Proof. Since γ is a morphism of (F, M) it follows that γ commutes with the Hodge components $\{\delta^{p,q}\}$ of the Deligne splitting of (F, M). As ξ is given by universal Lie polynomials in the Hodge components of δ, it follows that γ commutes with ξ. Therefore, γ is a morphism of $(\hat{F}, M) = (e^{-\xi}F, M)$. By [5],

$$Y^{\ddagger} = \mathrm{Ad}(e^{-iN})Y_{(e^{iN}\hat{F},W)}.$$

As γ commutes with N and preserves W, it follows that γ is of type $(0,0)$ with respect to $(e^{iN}\hat{F}, W)$ as well. Thus, $[\gamma, Y^{\ddagger}] = 0$. □

Limit Mixed Hodge Structure. We now return to the notation from the beginning of this section. In particular, $\mathcal{V}_\nu$ and $\mathcal{V}_\omega$ are as in the paragraph before Theorem 81.

Let $\Delta^* \subset \Delta^{*r}$ be the punctured disk defined by the equation $s_1 = \cdots = s_r$, and fix a point $s_0 \in \Delta^*$. Let V be the fiber of $\mathcal{V}_\nu$ over s_0. Then, applying Corollary 88 to the restriction of $\mathcal{V}_\nu$ to $\Delta^* \subset \Delta^{*r}$ produces a trigrading $V_k^{p,q}$ of V which is split over $\mathbb{R}$. Likewise, let U be the fiber of $\mathcal{V}_\omega^\vee$ over s_0. Then, using duality, restriction of $\mathcal{V}_\omega^\vee$ to $\Delta^* \subset \Delta^{*r}$ produces a trigrading $U_k^{p,q}$ of U which is also split over $\mathbb{R}$.

Since U and V abut at $\ell = -1$, we can form the trigraded vector space

$$B = U \star V$$

with associated trigrading which is split over $\mathbb{R}$. The associated bigrading determines an $\mathbb{R}$-split mixed Hodge structure $(\hat{F}_\infty, M) = (\hat{F}_\infty(B), M(B))$ as in (82).

Let (F_ν, M^ν) be the limit mixed Hodge structure of $\mathcal{V}_\nu$ and (F_ω, M^ω) be the limit mixed Hodge structure $\mathcal{V}_\omega^\vee$. Let $\xi_\nu \in gl(V)$ and $\xi_\omega \in gl(U)$ be the corresponding endomorphisms (58) of V and U. Recall that ξ_ν and ξ_ω preserve the weight filtrations $W(V)$ and $W(U)$ respectively. Moreover, they induce the same action on Gr_{-1}^W since $\mathcal{V}_\nu$ and $\mathcal{V}_\omega^\vee$ induce the same limit mixed Hodge structure on Gr_{-1}^W. Define

$$F_\infty = e^{\xi}\hat{F}_\infty \tag{90}$$

where $\xi = \xi_U \star \xi_V$. Since ξ_ν and ξ_ω act trivially on $Gr^M(V)$ and $Gr^M(U)$ it follows that ξ acts trivially on Gr^M by (86).

Remark 91. The mixed Hodge structure $(\hat{F}_\infty, M)$ projects to $(\hat{F}_\nu, M^\nu)$ and $(\hat{F}_\omega, M^\omega)$ on V and U respectively. The monodromy logarithms N_j^V and N_j^U of $\mathcal{V}_\nu$ and $\mathcal{V}_\omega^\vee$ are $(-1,-1)$-morphisms of $(\hat{F}_\nu, M^\nu)$ and $(\hat{F}_\omega, M^\omega)$. For future use, we also note that $W_{-2}\,\mathrm{End}(B)$ consists of $(-1,-1)$-morphisms of $(\hat{F}_\infty, M)$ and (F_∞, M). This follows from the fact that $B_0 = B_0^{0,0}$ while $B_{-2} = B_{-2}^{-1,-1}$.

Local System. The monodromy logarithms N_j^V and N_j^U induce the same action on Gr_{-1}^W and hence glue together to define an endomorphism $\tilde{N}_j$ of $B_{\mathbb{R}}$. By the previous remark, $\tilde{N}_j$ is a $(-1,-1)$-morphism of $(\hat{F}_\infty, M)$. To show that the endomorphisms $\{\tilde{N}_j\}$ define a local system over Δ^{*r} with fiber $B_{\mathbb{R}}$, it is sufficient to show that $[\tilde{N}_j, \tilde{N}_k] = 0$ for all j and k.

Let $\{V_i^{p,q}\}$ be the trigrading of V of Corollary (88), and let $v_0 \in V_0$ be a lifting of $1 \in Gr_0^W$. Set $b_0 = \iota_V(v_0)$. Then, since v_0 is type $(0,0)$ with respect to the bigrading of V, the same is true of b_0 with respect to the bigrading of B. Moreover, tracing through the above definitions, one sees that

$$\pi_V[\tilde{N}_j, \tilde{N}_k]b_0 = [N_j^V, N_k^V]v_0 = 0$$

Therefore, $[\tilde{N}_j, \tilde{N}_k]b_0$ is an element of $W_{-2}(B)$ which is of type $(-2,-2)$ with respect to the bigrading of B. But $W_{-2}(B)$ is pure of type $(-1,-1)$, and hence $[\tilde{N}_j, \tilde{N}_k]b_0 = 0$. Likewise, $\pi_U[\tilde{N}_j, \tilde{N}_k]\iota_U(u) = [N^U_j, N^U_k]u = 0$. Consequently, $[\tilde{N}_j, \tilde{N}_k] = 0$.

A-**Structure.** Pick an element $v_A \in V_A$ which projects to $1 \in Gr^W_0(V)$ and let $b_A = \iota_V(v_A)$. Define

$$B_A = Ab_A \oplus \iota_U(U_A)$$

Then, $B_A \otimes \mathbb{R} = B_{\mathbb{R}}$ since $v_A = v_0 + v_{-1}$ with v_0, v_{-1} real and v_0 as above in $V^{0,0}_0$.

Let u_{-2} be the generator of $W_{-2}(U_A) \cong A(1)$ corresponding to $1^\vee$ (with $1^\vee$ a generator of $A(1)$ as in Definition 38). Set $b_{-2} = \iota_U(u_{-2})$. Note that $b_{-2} \in B^{-1,-1}_{-2}$. Let η be the endomorphism of $B_{\mathbb{R}}$ which annihilates $W_{-1}(B)$ and maps b_A to b_{-2} (or, equivalently, b_0 to b_{-2}). We are going to set

$$N_j = \tilde{N}_j + c_j\eta \tag{92}$$

for some scalar c_j to be determined below.

Since $W_{-2}(B)$ has rank 1 and $\tilde{N}_j$ is nilpotent and preserves $W(B)$, it follows that $\tilde{N}_j$ acts trivially on $W_{-2}(B)$. Consequently, $[\tilde{N}_j, \eta] = 0$ and hence $[N_j, N_k] = 0$. Likewise, $[N_j, \eta] = 0$ and hence $T_j = e^{N_j} = e^{\tilde{N}_j + c_j\eta} = (\tilde{T}_j)(1 + c_j\eta)$. Therefore,

$$T_j(b_A) = \tilde{T}_j(b_A) + c_j u_{-2} = b_A + (\tilde{T}_j - 1)b_A + c_j u_{-2}$$

In particular, because $(T^V_j - 1)v$ maps to an integral class in Gr^W_{-1}, it follows that for suitable choice of real scalar c_j, $T_j(b) \in B_A$. Likewise, T_j acts as T^U_j on U, and hence T_j preserves B_A.

Hodge filtration. By rescaling s if necessary, we can assume that $\mathcal{V}_\nu$ and $\mathcal{V}^\vee_\omega$ have local normal forms Γ^V and Γ^U on Δ^r as in (48). Then, Γ^V preserves $W(V)$, Γ^U preserves $W(U)$ and the induced actions on Gr^W_{-1} agree since the induced VHS on Gr^W_{-1} coincide. Therefore, we can glue Γ^V to Γ^U to define a function Γ on Δ^r which preserves $W(U \star V)$ and acts by infinitesimal isometries on $\mathrm{Gr}^W B$. Define

$$F(z) = e^{\sum_j z_j N_j} e^{\Gamma(s)} F_\infty \tag{93}$$

where F_∞ is the filtration (90), and N_j is the endomorphism (92). Then, $F(z)$ induces the corresponding Hodge filtrations of $\mathcal{V}_\nu$ on $Gr^W_j(V)$ for $j = 0$, $j = -1$, and $\mathcal{V}^\vee_\omega$ for $j = -1, -2$. Clearly, $F(z)$ is holomorphic and descends to a period map over Δ^{*r} with monodromy $\{T_j\}$.

Horizontality. We need to show that $\frac{\partial}{\partial z_j} F^p(z) \subseteq F^{p-1}(z)$. By manipulation of (93), this is equivalent to

$$\left(e^{-\mathrm{ad}\Gamma} N_j + 2\pi i s_j e^{-\Gamma} \frac{\partial}{\partial s_j} e^{\Gamma}\right)(F^p_\infty) \subseteq F^{p-1}_\infty$$

By construction, this holds modulo $W_{-2}\operatorname{End}(B)$, which is sufficient to prove horizontality for $p > 0$. For the case $p = 0$, observe that $W_{-2}\operatorname{End}(B)$ consists of $(-1,-1)$-morphisms of (F_∞, M). For $p < 0$, horizontality follows from horizontality modulo $W_{-2}B$ and the case $p = 0$.

Admissibility. To show that (93) defines an admissible normal function, it remains to show that (i) the Hodge bundles extend holomorphically over Δ^r and induces the limit filtration of Schmid on each Gr^W, and (ii) the required relative weight filtrations exist.

The first condition follows from that fact that $\exp(-\sum_j z_j N_j)F(z) = e^{\Gamma(s)}F_\infty$ which obviously extends to a holomorphic filtration on Δ^r, and F_∞ induces Schmid's filtration on Gr^W by construction.

To prove the existence of the required relative weight filtrations, we recall the following result from [44, Theorem 2.20]: Let ${}_{(i)}M$ denote the relative weight filtration of N_{W_i} and W. Suppose that ${}_{(k-1)}M$ exists. Then, ${}_{(k)}M$ exists if and only if for all $j > 0$,

$$(N^j W_j) \cap W_{k-1} \subset N^j W_{k-1} + {}_{(k-1)}M_{k-j-1}$$

Set $N = N_j$ and $W = W(B)$. Then, the relative weight filtration of N restricted to W_{-1} exists and equals the image of $M(N_j^U, W(U))$ under ι_U. Therefore, we need only check the above for $k = 0$. Since $N = \tilde{N}_j + c_j\eta$,

$$(N^j W_j) \cap W_{-1} = N^j W_{-1} + \mathrm{Im}(c_j\eta) \subset N^j W_{-1} + {}_{(-1)}M_{-2}$$

because $W_{-2} \subset {}_{(-1)}M_{-2}$ since W_{-2} is pure of type $(-1,-1)$ for the limit mixed Hodge structure.

This completes the proof of Theorem 81.

Quasi-Unipotent Monodromy. In this section, we extend Theorem 81 to the case of admissible normal functions with quasi-unipotent monodromy, and give an analog of the local normal form (48) for admissible variations with quasi-unipotent monodromy.

Let T be an automorphism of a finite dimensional complex vector space. Then, by the Jordan decomposition theorem, $T = T_s + T_n$ where T_s is semisimple, T_n is nilpotent and $[T_s, T_n] = 0$. Moreover, there exists polynomials p and q without constant term such that $T_s = p(T)$ and $T_n = q(T)$. In particular, since T is an automorphism, T_s is invertible. Let $T_u = 1 + T_s^{-1}T_n$. Then, T_u is unipotent and $T = T_sT_u = T_uT_s$ is called the multiplicative Jordan decomposition of T.

Remark 94. If T is a automorphism of a finite dimensional rational vector space, and T_s has finite order m, then both T_s and T_u are rational. To see this, note that $T_u = e^{\alpha}$ and hence $T^m = T_u^m = e^{m\alpha}$ is rational. Therefore, α is rational, and hence so is T_u and $T_s = TT_u^{-1}$.

Proposition 95. *Let T and T' be commuting automorphisms of a finite dimensional complex vector space, and let $T = T_s + T_n$ and $T' = T'_s + T'_n$ be the Jordan decomposition of T and T'. Then, $\{T_s, T_n, T'_s, T'_n\}$ is a set of mutually commuting endomorphisms.*

Proof. Each of the elements listed is a polynomial in T or T'. □

Corollary 96. *Let T and T' be commuting automorphisms of a finite dimensional complex vector space, and let $T = T_sT_u$ and $T' = T'_sT'_u$ be the multiplicative Jordan decompositions of T and T'. Then, $\{T_s, T_u, T'_s, T'_u\}$ is a set of mutually commuting automorphisms.*

Let (Δ^r, s) be a polydisk with holomorphic coordinates $(s_1, \dots, s_r)$ and $\mathcal{V} \to \Delta^{*r}$ be a variation of mixed Hodge structure defined on the complement of the divisor $s_1 \cdots s_r = 0$. Assume that $\mathcal{V}$ has quasi-unipotent monodromy γ_i about $s_i = 0$, and let $\gamma_i = \gamma_{i,s}\gamma_{i,u}$ be the multiplicative Jordan decomposition of γ_i. Then, by the previous Corollary, $\{\gamma_{j,s}, \gamma_{j,u}, \gamma_{k,s}, \gamma_{k,u}\}$ is a set of mutually commuting automorphisms of the reference fiber of $\mathcal{V}$ for any pair of indices j and k. Let $m_i > 0$ denote the order of $\gamma_{i,s}$ and $N_i = \log \gamma_{i,u}$.

Assume that $\mathcal{V} \to (\Delta^{*r}, s)$ is admissible. Let (Δ^{*r}, t) be a polydisk with coordinates $(t_1, \dots, t_r)$ and $f : (\Delta^{*r}, t) \to (\Delta^{*r}, s)$ denote the covering map $t_j = s_j^{m_j}$. Then, $f^*(\mathcal{V})$ is an admissible variation of mixed Hodge structure over (Δ^{*r}, t) with unipotent monodromy $e^{m_jN_j}$ about $t_j = 0$. Let

$$\tilde{F} : (\mathfrak{h}^r, w) \to \mathcal{M}$$

denote the lifting of the period map of $f^*(\mathcal{V})$ to the product of upper half-planes $\mathfrak{h}^r \subset \mathbb{C}^r$ with Cartesian coordinates $(w_1, \dots, w_r)$ relative to the covering map $t_j = e^{2\pi i w_j}$.

For $w \in \mathbb{C}^r$ let $N(w) = \sum_k w_k m_k N_k$, and

$$\tilde{\psi}(w) = e^{-N(w)}\tilde{F}(w). \tag{97}$$

Let $\{e_1, \dots, e_r\}$ denote the standard Cartesian basis of $\mathbb{C}^r$. Then, $\tilde{\psi}(w + e_j) = \tilde{\psi}(w)$, and hence $\tilde{\psi}$ descends to a map

$$\psi : \Delta^{*r} \to \check{\mathcal{M}}.$$

By admissibility, ψ extends to a holomorphic map $\Delta^r \to \check{\mathcal{M}}$, such that

$$(m_1 N_1, \dots, m_r N_r; F_\infty, W) \tag{98}$$

is an admissible nilpotent orbit, where $F_\infty = \psi(0)$.

Let $\mathfrak{g}_{\mathbb{C}} = \oplus_{p,q<0}\, \mathfrak{g}^{p,q}$ relative to the limit mixed Hodge structure (F_∞, M) of (98). Then,

$$\mathfrak{q} = \oplus_{a<0,b}\mathfrak{g}^{a,b}$$

is a vector space complement to the stabilizer of F_∞ in $\mathfrak{g}_{\mathbb{C}}$. Therefore, there exists a unique holomorphic function

$$\Gamma : \Delta^r \to \mathfrak{q}$$

such that $e^{\Gamma(t)}F_\infty = \psi(t)$, after shrinking Δ^r as necessary. This gives the local normal form

$$\tilde{F}(w) = e^{N(w)}e^{\Gamma(t)}F_\infty. \tag{99}$$

The next result asserts that each $\gamma_{j,s}$ is a morphism of the limit mixed Hodge structure. In the geometric case this is due to J. Steenbrink [43].

Proposition 100. *Each $\gamma_{j,s}$ is morphism of (F_∞, M). Moreover,*

$$\Gamma \circ \rho_j(t) = \mathrm{Ad}(\gamma_{j,s})\Gamma(t) \tag{101}$$

where $\rho_j(t_1, \dots, t_r) = (t_1, \dots, e^{2\pi i/m_j}t_j, \dots, t_r)$.

Proof. To prove that $\gamma_{j,s}$ is a morphism of the limit mixed Hodge structure we begin with the observation that

$$\psi \circ \rho_j(t) = \gamma_{j,s}\psi(t) \tag{102}$$

To see this we note that we have a commutative diagram

$$\begin{array}{ccccc}
(\mathfrak{h}^r, w) & \xrightarrow{z_k = m_k w_k} & (\mathfrak{h}^r, z) & \xrightarrow{F(z)} & \mathcal{M} \\
{\scriptstyle t_k = e^{2\pi i w_k}}\downarrow & & \downarrow{\scriptstyle s_k = e^{2\pi i z_k}} & & \downarrow \\
(\Delta^{*r}, t) & \xrightarrow{s_k = t_j^{m_k}} & (\Delta^{*r}, s) & \xrightarrow{\varphi} & \Gamma\backslash\mathcal{M}
\end{array}$$

where $F : (\mathfrak{h}^r, z) \to \mathcal{M}$ is the local lifting of the period map of $\mathcal{V} \to (\Delta^{*r}, s)$ relative to the covering map $s_k = e^{2\pi i z_k}$ where $(z_1, \dots, z_r)$ are Cartesian coordinates on the product of upper half-planes $\mathfrak{h}^r \subset \mathbb{C}^r$. In particular, since $F(z + e_j) = \gamma_j F(z)$ it follows that $\tilde{F}(w + (1/m_j)e_j) = \gamma_j \tilde{F}(w)$. Therefore,

$$\psi \circ \rho_j(t) = e^{-N(w+(1/m_j)e_j)}\tilde{F}(w + (1/m_j)e_j) = e^{-N(w+(1/m_j)e_j)}\gamma_j\tilde{F}(w) = \gamma_{j,s}\psi(t).$$

Taking the limit as $t \to 0$ yields $F_\infty = \gamma_{j,s}F_\infty$.

By the remark at the beginning of this section, in the case where $A = \mathbb{Z}$ or $A = \mathbb{Q}$, $\gamma_{j,s}$ is rational since $\gamma_{j,s}$ is finite order and γ_j is rational. To prove that $\gamma_{j,s}$ is a morphism of (F, M) it remains to check that $\gamma_{j,s}$ preserves M. This follows from the defining properties of the relative weight filtration and the fact that:

— Since γ_j preserves W so does $\gamma_{j,s} = p(\gamma_j)$;
— $\gamma_{j,s}$ commutes with $N_1, \dots, N_r$ and $M = M(N_1 + \cdots + N_r, W)$.

To verify (101), note that by the derivation of (99) and equation (102), we have

$$\psi \circ \rho_j(t) = e^{\Gamma \circ \rho_j(t)} F_\infty = \gamma_{j,s} \psi(t) = \gamma_{j,s} e^{\Gamma(t)} F_\infty.$$

Since $\gamma_{j,s}$ preserves F_∞ by the previous paragraphs, it follows that

$$e^{\Gamma \circ \rho_j(t)} F_\infty = \gamma_{j,s} e^{\Gamma(t)} \gamma_{j,s}^{-1} F_\infty.$$

In particular, since $\gamma_{j,s}$ is a morphism of (F_∞, M) the adjoint action of $\gamma_{j,s}$ preserves $\mathfrak{q}$. Because $\mathfrak{q}$ is a vector space complement to the stabilizer of F_∞ in $\mathfrak{g}_\mathbb{C}$, it then follows that $\Gamma \circ \rho_j(t) = \mathrm{Ad}(\gamma_{j,s})\Gamma(t)$. □

To continue, we note that by commutative diagram of Proposition (100), we have $\tilde{F}(w_1, \dots, w_r) = F(m_1 w_1, \dots, m_r w_r)$. Setting $w_j = z_j / m_j$ it then follows from (99) that

$$F(z_1, \dots, z_r) = \tilde{F}(z_1/m_1, \dots, z_r/m_r) = e^{\sum_k z_k N_k} e^{\Gamma(v_1, \dots, v_r)} F_\infty \tag{103}$$

where $v_j = e^{2\pi i z_j / m_j}$.

Theorem 104. *Let*

(1) $\theta(z_1, \dots, z_r) = (e^{\sum_k z_k N_k} F_\infty, W)$ *be an admissible nilpotent orbit with limit mixed Hodge structure* (F_∞, M) *and values in the a classifying space* $\mathcal{M}$ *of graded-polarized mixed Hodge structure for* $\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_r) \gg 0$;

(2) $\{\gamma_{1,s}, \dots, \gamma_{r,s}\}$ *be a set of semisimple automorphisms of* (F_∞, M) *which act by finite order isometries on* Gr^W. *Let* m_j *be the order of* $\gamma_{j,s}$ *and* $\gamma_{j,u} = e^{N_j}$. *Assume that* $\{\gamma_{1,s}, \gamma_{1,u}, \dots, \gamma_{r,s}, \gamma_{r,u}\}$ *is a set of mutually commuting automorphisms;*

(3) $\Gamma : \Delta^r \to \mathfrak{q}$ *be a holomorphic function which vanishes at the origin and satisfies* (101).

Suppose that

$$\tilde{F}(w_1, \dots, w_r) = e^{\sum_k w_k m_k N_k} e^{\Gamma(t)} F_\infty \tag{105}$$

satisfies Griffiths infinitesimal period relation on the upper half-plane $(\mathfrak{h}^r, w)$ *where* $t_j = e^{2\pi i w_j}$. *Define* $F(z)$ *from* $\tilde{F}(w)$ *via equation* (103). *Then,*

(i) $F(z_1, \dots, z_j + 1, \dots, z_r) = \gamma_i F(z_1, \dots, z_r)$ *where* $\gamma_i = \gamma_{i,s}\gamma_{i,u}$;

(ii) *There exists a positive real number A such that*

$$F(z_1, \dots, z_r) = e^{\sum_j z_j N_j} e^{\Gamma(v_1, \dots, v_r)} F_\infty$$

descends to an admissible period map on the set of points in (Δ^{*r}, s) *such that* $|s_j| = |e^{2\pi i z_j}| < A$.

Proof. Since $v_j = e^{2\pi i z_j / m_j}$ it follows that changing z_j to $z_j + 1$ changes v_j to $e^{2\pi i / m_j} v_j$. Therefore, by equation (101),

$$\begin{aligned} F(z_1, \dots, z_j + 1, \dots, z_r) &= e^{N_j} e^{\sum_k z_k N_k} \gamma_{j,s} e^{\Gamma(v_1, \dots, v_r)} \gamma_{,s}^{-1} F_\infty \\ &= \gamma_{j,u} \gamma_{j,s} e^{\sum_k z_k N_k} e^{\Gamma(v_1, \dots, v_r)} F_\infty \end{aligned}$$

using hypothesis (2) and (3). This proves part (i).

To prove part (ii), we need to show that

(a) $F(z)$ satisfies Griffiths infinitesimal period relation;
(b) $F(z)$ takes values in $\mathcal{M}$ for $\mathrm{Im}(z_1), \dots, \mathrm{Imz_r} \gg 0$;
(c) Verify the existence of the requisite relative weight filtrations.

Condition (c) follows from the fact that θ is an admissible nilpotent orbit. On the other hand, since $(\Delta^{*r}, t) \to (\Delta^{*r}, s)$ is a covering map, it is sufficient to verify conditions (a) and (b) for the map (105). Condition (a) holds for $\tilde{F}(w)$ by assumption. To verify condition (b), we need only show that $\tilde{F}(w)$ induces pure, polarized Hodge structures on Gr^W for $\mathrm{Im}(w_1), \dots, \mathrm{Imw_r} \gg 0$. This follows immediately from the fact that θ is an admissible nilpotent orbit, $\tilde{F}(w)$ is horizontal and Theorem (2.8) [8]. □

Remark 106. In order to produce a variation with an integral (or rational structure) via Theorem (104), each $\gamma_{j,s}$ and $\gamma_{j,u}$ must be integral (or rational).

Theorem 107. *Theorem* (81) *remains valid in the setting where $\mathcal{H}$ has quasi-unipotent monodromy.*

Proof. By Theorem (104), it is sufficient to work on the pullback to the unipotent case and ensure that the resulting function Γ satisfies (101). Each of the functions Γ^V and Γ^U attached to the normal functions satisfy an appropriate version of (101) relative to the semisimple automorphisms $\gamma_{j,s}^V$ and $\gamma_{j,s}^U$ respectively. It follows from Proposition (89) that $\gamma_{j,s}^V$ and $\gamma_{j,s}^U$ preserve the the trigradings of V and U. Since they agree on Gr_{-1}^W it follows that they glue together to an automorphism $\gamma_{j,s}$ of B which preserves the trigrading of B. Therefore, the function Γ produced by gluing Γ^U and Γ^V also satisfies (101).

In the case where $A = \mathbb{Z}$ or $\mathbb{Q}$, we add the underlying A structure as follows: The element η appearing (92) commutes with each $\gamma_{j,s}$ since $\gamma_{j,s} \in gl(B)_0^{0,0}$ with respect to the induced trigrading on $gl(B)$, and $Gr(\gamma_{j,s})$ acts trivially on the 1-dimensional factors Gr_0^W and Gr_{-2}^W by hypothesis (i.e. these factors are assumed to have trivial global monodromy). The construction of the A-structure therefore follows exactly as in the unipotent case. □

5. Singularities of Normal Functions

In this section, we review the notion of the singularities of normal functions, in the setting where $\mathcal{H}$ is an admissible variation of mixed Hodge structure such that $W_{-1}\mathcal{H} = \mathcal{H}$.

5.1. **Admissible Normal Functions.**

Suppose $\bar{S}$ is a complex manifold and S is a Zariski open subset. Suppose $\mathcal{H}$ is an object in $\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$ with $W_{-1}\mathcal{H} = \mathcal{H}$ and underlying $\mathbb{Z}$-local structure $\mathcal{H}_{\mathbb{Z}}$. Following M. Saito, we define

$$\mathrm{ANF}(S, \mathcal{H})_{\bar{S}} := \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathbb{Z}, \mathcal{H})$$

to be the group of admissible normal functions on S relative to $\bar{S}$ with underlying variation of mixed Hodge structure $\mathcal{H}$.

5.2. **Singularities of admissible normal functions.**

Let S, $\bar{S}$ and $\mathcal{H}$ be as in §5.1, and let $s \in \bar{S} \setminus S$. Write $j : S \to \bar{S}$ and $i : \{s\} \to \bar{S}$ for the inclusions. Suppose $\nu \in \mathrm{ANF}(S, \mathcal{H})_{\bar{S}}$ is an admissible normal function. Let

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z} \to 0 \tag{108}$$

by the extension class corresponding to ν. By applying the forgetful functor from $\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$ to the category $\mathrm{Loc}(S)$ of local systems on S, we wind up with an extension in the category of local systems on S. Thus we can associate to ν a class $\mathrm{cl}(\nu) \in \mathrm{H}^1(S, \mathcal{H})$. Tensoring with $\mathbb{Q}$, we get a class $\mathrm{cl}_{\mathbb{Q}} \in \mathrm{H}^1(S, \mathcal{H}_{\mathbb{Q}}) = \mathrm{H}^1(\bar{S}, Rj_*\mathcal{H}_{\mathbb{Q}})$. Finally, by pull-back via i, we get a class

$$\mathrm{sing}_s(\nu) \in \mathrm{H}^1_s(\mathcal{H}_{\mathbb{Q}}) := \mathrm{H}^1(i^* Rj_* \mathcal{H}_{\mathbb{Q}}). \tag{109}$$

This class is called the *singularity* of ν. We write $\mathrm{cl}_s(\nu)$ for the image of $\mathrm{cl}(\nu)$ in $\mathrm{H}^1(i^* Rj_* \mathcal{H})$.

5.3. **Singularities.** Suppose $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$. Let $\mathcal{L}$ denote a $\mathbb{F}$-local system on S and $d = \dim_{\mathbb{C}} S$. The intersection complex of $\mathcal{L}$ is

$$\mathrm{IC}(\mathcal{L}) := j_{!*}(\mathcal{L}[d])$$

The intersection and local intersection cohomology spaces are defined by

$$\mathrm{IH}^k(\bar{S}, \mathcal{L}) := \mathrm{H}^{k-d}(\bar{S}, \mathrm{IC}(\mathcal{L}))$$
$$\mathrm{IH}^k_s(\mathcal{L}) := \mathrm{H}^{k-d}(i^* \,\mathrm{IC}(\mathcal{L})).$$

There is a morphism $\mathrm{IC}(\mathcal{L}) \to Rj_*\mathcal{L}$ which induces maps $\mathrm{IH}^k(\bar{S}, \mathcal{L}) \to \mathrm{H}^k(S, \mathcal{L})$ and $\mathrm{IH}^k_s(\mathcal{L}) \to \mathrm{H}^k_s(\mathcal{L}) := \mathrm{H}^k(i^*Rj_*\mathcal{H}_{\mathbb{Q}})$. For $k = 0$, these maps are isomorphisms, and for $k = 1$ they are injections.

Note that, by M. Saito's theory of mixed Hodge modules, if $\mathcal{H}$ is a $\mathbb{Q}$ variation of mixed Hodge structure, then $\mathrm{H}^p_s(\mathcal{H})$ and $\mathrm{IH}^p_s(\mathcal{H})$ both carry canonical mixed Hodge structures.

Theorem 110. [3] *Suppose $\mathcal{H}$ is a weight -1 variation of Hodge structure on S. The map*

$$\mathrm{cl}_{\mathbb{Q}} : \mathrm{ANF}(S, \mathcal{H})_{\bar{S}} \to \mathrm{H}^1(S, \mathcal{H}_{\mathbb{Q}})$$

factors through $\mathrm{IH}^1(S, \mathcal{H}_{\mathbb{Q}})$. Similarly, the map

$$\mathrm{sing}_s : \mathrm{ANF}(S, \mathcal{H})_{\bar{S}} \to \mathrm{H}^1_s(\mathcal{H}_{\mathbb{Q}})$$

factors through $\mathrm{IH}^1_s(\mathcal{H}_{\mathbb{Q}})$.

5.4. **Duality in the Weight -1 case.** If $\mathcal{H}$ is variation of pure Hodge structure of weight -1 on S, as in the introduction, we write $\mathcal{H}^\vee := \mathcal{H}^*(1)$. This is another variation of pure Hodge structure of weight -1 on S. There is a canonical morphism $\mathcal{H} \to (\mathcal{H}^\vee)^\vee$ which is an isomorphism if $\mathcal{H}$ is torsion-free.

Proposition 111. *Suppose $\mathcal{H}$ is torsion free. Then duality gives a canonical isomorphism*

$$\mathrm{ANF}(S, \mathcal{H})_{\bar{S}} \cong \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathcal{H}^\vee, \mathbb{Z}(1))$$

Proof. We have

$$\mathrm{ANF}(S, \mathcal{H})_{\bar{S}} = \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathbb{Z}, \mathcal{H}) \xrightarrow{D} \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathcal{H}^*, \mathbb{Z}) = \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathcal{H}^\vee, \mathbb{Z}(1)).$$

The map D, which is given by duality, is an isomorphism because $\mathcal{H}$ is torsion-free. ☐

5.5. **Admissible classes.**

Definition 112. Let $j : S \to \bar{S}$ and $i : \{s\} \to \bar{S}$ be as in 5.1. We call a holomorphic map $\bar{\varphi} : \Delta \to \bar{S}$ a *test curve at s* if $\bar{\varphi}(0) = s$ and $\bar{\varphi}(\Delta^*) \subset S$. Write $\varphi : \Delta^* \to S$ for the restriction of $\bar{\varphi}$ to Δ^*. Suppose $\mathcal{L}$ is a $\mathbb{F}$-local system on S for $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$ (as in §5.2). We call a class $\alpha \in \mathrm{H}^1_s(\mathcal{L})$ *admissible* if the pull-back $\varphi^*\alpha$ in $\mathrm{H}^1_0(\varphi^*\mathcal{L})$ vanishes for every test curve through s. Similarly, we call a class $\alpha \in \mathrm{H}^1(S, \mathcal{L})$ admissible if $\varphi^*\alpha$ vanishes in $\mathrm{H}^1(\Delta^*, \varphi^*\mathcal{L})$ for all test curves through all points of $\bar{S}$. We write $\mathrm{H}^1_s(\mathcal{L})^{\mathrm{ad}}$ (resp. $\mathrm{H}^1(S, \mathcal{L})^{\mathrm{ad}}$) for the set of admissible classes in $\mathrm{H}^1_s(\mathcal{L})$ (resp. $\mathrm{H}^1(S, \mathcal{L})$). Similarly, we write $\mathrm{IH}^1_s(\mathcal{L})^{\mathrm{ad}} := \mathrm{IH}^1_s(\mathcal{L}) \cap \mathrm{H}^1_s(\mathcal{L})^{\mathrm{ad}}$ and $\mathrm{IH}^1(\bar{S}, \mathcal{L})^{\mathrm{ad}} := \mathrm{IH}^1(\bar{S}, \mathcal{L}) \cap \mathrm{H}^1(S, \mathcal{L})^{\mathrm{ad}}$.

Theorem 113. *Let $\mathcal{L}$ be an variation of pure Hodge structure on S. Then for $s \in \bar{S}$, we have $\mathrm{IH}^1_s(\mathcal{L}) = \mathrm{IH}^1_s(\mathcal{L})^{\mathrm{ad}}$. In other words, every class is admissible.*

We will prove the theorem after the following lemma.

Lemma 114. *Suppose $\mathcal{L}$ is a variation of pure Hodge structure of weight w on Δ^*. Then $\mathrm{Gr}^W_k \mathrm{H}^1_0(\mathcal{L}) = 0$ for $k < w + 2$. In other words, $\mathrm{H}^1_0\mathcal{L}$ has weights in the interval $[w + 2, \infty)$.*

Proof of Lemma 114. We have an exact sequence

$$0 \to \mathrm{H}^0_0(\mathcal{L}) \to \psi_{z,1}\mathcal{L} \xrightarrow{N} \psi_{z,1}\mathcal{L}(-1) \to \mathrm{H}^1_0(\mathcal{L}) \to 0 \tag{115}$$

where $\psi_{z,1}$ denotes the unipotent nearby cycles functor and N denotes the logarithm of unipotent part of the monodromy (of $\mathcal{L}$ around 0). The weight filtration on $\psi_{z,1}\mathcal{L}$ is induced from the monodromy filtration of N so that

$$N^k : \mathrm{Gr}^W_{w-k}(\psi_{z,1}\mathcal{L}(k)) \to \mathrm{Gr}^W_{w-k}\,\psi_{z,1}\mathcal{L}$$

is an isomorphism for $k \geq 0$. So $\operatorname{coker}(N : \psi_{z,1}\mathcal{L}(1) \to \psi_{z,1}\mathcal{L})$ has weights in the interval $[w, \infty)$. The result follows immediately. □

Proof of Theorem 113. Suppose $\bar{\varphi} : \Delta \to \bar{S}$ is a test curve through s. Write f for the composition

$$\mathrm{IH}^1_s(\mathcal{L}) \to \mathrm{H}^1_s(\mathcal{L}) \to \mathrm{H}^1_0(\varphi^*\mathcal{L}).$$

Then f is a morphism of mixed Hodge structures. By the purity theorem of Cattani-Kaplan-Schmid and Kashiwara-Kawai, the source of f has weights in the interval $(-\infty, w+1]$. But by Lemma 114, the target has weights in the interval $[w+2, \infty)$. It follows that $f = 0$. □

6. The asymptotic height pairing in the normal crossing case

The goal of this section is to define the asymptotic height pairing on local intersection cohomology in the normal crossing case using a complex B^* which computes the local intersection cohomology in this case. (See Theorem 135.)

6.1. **Normal crossing setup.** In this section, we write $\mathbb{F}$ for a field which is either $\mathbb{Q}$ or $\mathbb{R}$ and r for a fixed non-negative integer. Let Δ^r denote a polydisk with local coordinates $(s_1, \ldots, s_r)$ and Δ^{*r} be the complement of the divisor $s_1 \cdots s_r = 0$. Let $\mathcal{L}$ be a local system of $\mathbb{F}$-vector spaces on Δ^{*r} with unipotent monodromy, and we write L for a fixed fiber of $\mathcal{L}$. We let $T_1, \ldots, T_r \in \mathrm{Aut}_{\mathbb{F}} L$ denote the monodromy operators. Then, for each i, the monodromy logarithm $N_i = \log T_i$ is nilpotent.

Note that , if $\mathcal{L}^*$ denotes the local system dual to $\mathcal{L}$, then the monodromy operators on the fiber L^* are given by $(T_i^*)^{-1}$ and the logarithms are given by $-N_i^*$.

6.2. **Some linear algebra.** Suppose T is an endomorphism of a finite dimensional vector space V over a field F. Write $T^* \in \operatorname{End} V^*$ for the adjoint of T and let $(\ ,\ ) : V \otimes V^* \to F$ denote the canonical pairing. There is a natural perfect bilinear pairing

$$(\ ,\ )_T : TV \otimes T^*V^* \to F, \text{ given by}$$
$$Tv \otimes T^*\lambda \mapsto (v, T^*\lambda) = (Tv, \lambda).$$

To see that $(\ ,\ )_T$ is well-defined, note that $Tv = 0 \Rightarrow (v, T^*\lambda) = (Tv, \lambda) = 0$. Similarly, $T^*\lambda = 0 \Rightarrow (Tv, \lambda) = (v, T^*\lambda) = 0$. To see that the pairing is perfect, suppose $(Tv, T^*\lambda)_T = 0$ for all λ. Then $(Tv, \lambda) = 0$ for all λ. So $Tv = 0$. Similarly, $(Tv, T^*\lambda)_T = 0$ for all v implies that $T^*\lambda = 0$.

6.3. **Koszul and partial Koszul complexes.** The Koszul and partial Koszul complexes are complexes of vector spaces which compute the cohomology groups $\mathrm{H}^p(\mathcal{L}) = \mathrm{H}^p(\Delta^{*r}, \mathcal{L})$ and the intersection cohomology groups $\mathrm{IH}^p(\mathcal{L}) = \mathrm{IH}^p(\Delta^r, \mathcal{L})$ when $\mathcal{L}$ is as in §6.1. We follow the notation of Kashiwara and Kawai and Cattani, Kaplan and Schmid for these complexes. (See [28, §3.4] and [10].)

Let E denote an r dimensional $\mathbb{F}$-vector space with basis $\{e_1, \dots, e_r\}$. As a $\mathbb{F}$-vector space, the Koszul complex is $K(\mathcal{L}) = L \otimes \bigwedge^* E$. It is graded by the usual grading of $\bigwedge^* E$, and it has differential given by

$$d(l \otimes \omega) = \sum_{i=1}^{r} N_i l \otimes (e_i \wedge \omega). \tag{116}$$

There is a canonical isomorphism $\mathrm{H}^p(\mathcal{L}) = \mathrm{H}^p K(\mathcal{L})$.

Suppose $J = (j_1, \dots, j_p)$ is a multi-index in $\{1, \dots, r\}$. Let N_J denote the product $N_{j_1} \cdots N_{j_p}$, and write $e_J = e_{j_1} \wedge \cdots \wedge e_{j_p}$. The partial Koszul complex is the subcomplex $B(\mathcal{L})$ of $K(\mathcal{L})$ given by $\sum_J N_J(L) \otimes e_J$. By the results of Kashiwara, Kawai [28] and Cattani, Kaplan, Schmid [10], there is a canonical isomorphism $\mathrm{IH}^p(\mathcal{L}) = H^p(B(\mathcal{L}))$. In particular, the canonical map $\mathrm{IH}^p(\Delta^r, \mathcal{L}) \to \mathrm{H}^p(\Delta^{*r}, \mathcal{L})$ is precisely the map induced by the inclusion of complexes $B(\mathcal{L}) \to K(\mathcal{L})$.

Note that a morphism $f : \mathcal{L} \to \mathcal{M}$ of local systems on Δ^{*r} induces morphisms $f : K^*(\mathcal{L}) \to K^*(\mathcal{M})$ and $f : B^*(\mathcal{L}) \to B^*(\mathcal{M})$ in the obvious way: $l \otimes \omega \mapsto f(l) \otimes \omega$. These induce the corresponding homomorphisms $\mathrm{H}^*(\mathcal{L}) \to \mathrm{H}^*(\mathcal{M})$ and $\mathrm{IH}^*(\mathcal{L}) \to \mathrm{IH}^*(\mathcal{M})$.

Note also that the complexes $K(\mathcal{L})$ and $B(\mathcal{L})$ depend on only on the data $(L, N_1, \dots, N_r)$.

6.4. If (C, d) is a complex and p is a an integer, we write $Z^p(C) = \{\alpha \in C^p : d\alpha = 0\}$.

6.5. Suppose $t \in \mathbb{F}^r$. Write $N(t) := \sum_{i=1}^r t_i N_i$, and write $\mathcal{L}_t$ for the local system on Δ^* with monodromy logarithm $N(t) \in \operatorname{End} L$. Then, if $\mathbb{F}\{e\}$ denotes the free $\mathbb{F}$-vector space on one generator e, we have $K(\mathcal{L}_t) = L \otimes \bigwedge^* \mathbb{F}\{e\}$. As a complex, $K(\mathcal{L}_t)$ is just the map $N(t) : L \to L$ placed in degrees 0 and 1.

The map of vector spaces $E \to \mathbb{F}\{e\}$ given by $e_i \mapsto t_i e$ induces a map $\lambda_t : \wedge^* E \to \wedge^* \mathbb{C}\{e\}$. Write $\phi_t^\dagger$ for the map $\mathrm{id}_L \otimes \lambda_t : K(\mathcal{L}) \to K(\mathcal{L}_t)$.

If $\alpha = \sum_{i=1}^r \alpha_i \otimes e_i \in K^1(\mathcal{L})$, write $\alpha(t) := \sum_{i=1}^r t_i \alpha_i$. Then note that $\phi_t^\dagger \alpha = \alpha(t) \otimes e$.

Lemma 117. *The map $\phi_t^\dagger : K(\mathcal{L}) \to K(\mathcal{L}_t)$ is a morphism of complexes.*

Proof. Since $K(\mathcal{L}_t)$ is concentrated in degrees 0 and 1, we just need to check that $d\phi_t^\dagger(l) = \phi_t^\dagger(dl)$ for $l \in K^0(\mathcal{L})$. We compute $\phi_t^\dagger dl = \phi_t^\dagger \sum N_i l \otimes e_i = \sum t_i N_i l \otimes e = N(t) l \otimes e$ as desired. ☐

6.6. Suppose $t \in \mathbb{Z}_{\geq 0}^r$ and pick a $a \in \Delta^*$. Define a test curve $\phi_t : \Delta^* \to \Delta^{*r}$ by setting $\phi_t(s) = a(s^{t_1}, \dots, s^{t_r})$. Then ϕ_t induces a map

$$\phi_t^* : \mathrm{H}^p(\Delta^{*r}, \mathcal{L}) \to \mathrm{H}^p(\Delta^*, \mathcal{L}). \tag{118}$$

We leave it to the reader to check that, under the isomorphism of §6.3, $\phi_t^* = \phi_t^\dagger$.

6.7. In the rest of this section, $\mathcal{H}$ will denote a variation of pure, polarized $\mathbb{F}$-Hodge structure of weight k with unipotent monodromy on the poly-punctured disk Δ^{*r}. Write H for the limit mixed Hodge structure of $\mathcal{H}$. In the language of nearby cycles, $H = \psi_{s_1} \cdots \psi_{s_r} \mathcal{H}$. As a vector space, H is isomorphic to the fiber of $\mathcal{H}$ over any chosen point $s \in \Delta^{*r}$. For $i = 1, \dots, r$, we write T_i for the monodromy operator and $N_i = \log T_i$. The N_i then induce morphisms $N_i : H \to H(-1)$ of the mixed Hodge structure H. We write $\mathrm{H}^p(\mathcal{H}) = \mathrm{H}^p(\Delta^{*r}, \mathcal{H})$ and $\mathrm{IH}^p(\mathcal{H}) = \mathrm{IH}^p(\Delta^r, \mathcal{H})$ for the cohomologies of the local system underlying the variation $\mathcal{H}$. By [10, 28], these are computed in terms of complexes $K(\mathcal{H}) = H \otimes \wedge^* E$ and $B(\mathcal{H})$ as in §6.3. But here the vector space E is viewed as the pure, weight 2, Hodge structure $\mathbb{F}(-1)^r$. The map

$$d : K^p(\mathcal{H}) = H \otimes \wedge^p E \to K^{p+1}(\mathcal{H}) = H \otimes \wedge^{p+1} E$$

is then a morphism of mixed Hodge structures.

Write $0 \in \Delta^r$ for the origin. Then we have equalities of Hodge structures $\mathrm{H}^p(\mathcal{H}) = \mathrm{H}^p_0(\mathcal{H})$ and $\mathrm{IH}^p(\mathcal{H}) = \mathrm{IH}^p_0(\mathcal{H})$ where the groups $\mathrm{H}^p_0(\mathcal{H})$ and $\mathrm{IH}^p_0(\mathcal{H})$ groups defined in 5.2 and 5.3 respectively.

6.8. Write F for the Hodge filtration on H. Then the data $(N_1, \dots, N_r; F)$ defines a nilpotent orbit. This data in turn defines a variation of Hodge structure $\mathcal{H}_{\mathrm{nilp}}$ of weight k on Δ_b^{*r} for some $b > 0$ (see (49)). Explicitly, $\mathcal{H}_{\mathrm{nilp}}$ is the variation with local normal form $e^{\sum_{i=1}^r z_i N_i} F$ where the z_i are points in $\mathbb{C}$ such that $|e^{2\pi i z_i}| < b$. We then have equalities of complexes of mixed Hodge structure: $K(\mathcal{H}) = K(\mathcal{H}_{\mathrm{nilp}})$, $B(\mathcal{H}) = B(\mathcal{H}_{\mathrm{nilp}})$. In particular, $H^p(\mathcal{H})$ and $\mathrm{IH}^P(\mathcal{H}))$ depend only on the nilpotent orbit associated to $\mathcal{H}$. More precisely: $\mathcal{H}$ and $\mathcal{H}_{\mathrm{nilp}}$ restrict to the same local system over Δ_b^{*r}, and hence the complexes $B^\bullet(\mathcal{H})$ and $B^\bullet(\mathcal{H}_{\mathrm{nilp}})$ are equal. Likewise, all Hodge theoretic data attached to these complexes depends only the limit mixed Hodge structure, which agrees for $\mathcal{H}$ and $\mathcal{H}_{\mathrm{nilp}}$ by construction.

We now prove versions of Lemma 114 and Theorem 113 for the normal crossing case.

Lemma 119. *Suppose $r = 1$. Then $W_{k+1}\mathrm{H}^1(\mathcal{H}) = 0$.*

Proof. The proof is essentially the same as the proof of Lemma 114. Set $N = N_1$. As in the proof of Lemma 114, we get an exact sequence of mixed Hodge structures

$$0 \to \ker N \to H \xrightarrow{N} H(-1) \to (\operatorname{coker} N)(-1) \to 0.$$

Then we use the same reasoning as in Lemma 114 to show that $W_{k+1}\mathrm{H}^1(\mathcal{H}) = 0$. □

Proposition 120. *Let $\alpha = \sum_{i=1}^r \alpha_i \otimes e_i \in Z^1(B(\mathcal{H}))$ be a representative of an intersection cohomology class $\bar{\alpha} \in \mathrm{IH}^1\,\mathcal{H}$. Then, for every $t \in \mathbb{F}^r_{\geq 0}$, there is an element $l(t) \in H$ such that*

$$\alpha(t) = N(t)l(t). \tag{121}$$

Proof. Suppose $t \in \mathbb{F}^r_{\geq 0}$. Then, if F denotes the Hodge filtration of H, the triple $(H, F, N(t))$ defines a nilpotent orbit in one variable, and, thus, a $\mathbb{F}$-variation of Hodge structure on the disk Δ^*. Write $\mathcal{H}_t$ for this variation. Then, by 119, $W_{k+1}H^1(\mathcal{H}_t) = 0$.

As in 117, we get a morphism of complexes $\phi_t^\dagger : K(\mathcal{H}) \to K(\mathcal{H}_t)$. Moreover, it is easy to see that $\phi_t^\dagger$ is, in fact, a morphism of complexes of mixed Hodge structures. By §6.5, the composition $B^1(\mathcal{H}) \to K(\mathcal{H}) \to K(\mathcal{H}_t)$ takes a class $\alpha \in B^1(\mathcal{H})$ to $\alpha(t) \otimes e$.

By the purity theorem of [10], $W_{k+1}\,\mathrm{IH}^1(\mathcal{H}) = \mathrm{IH}^1(\mathcal{H})$. So, it follows from Lemma 119 that the composition $\mathrm{IH}^1(\mathcal{H}) \to \mathrm{H}^1(\mathcal{H}) \to \mathrm{H}^1(\mathcal{H}_t)$ is zero.

So suppose $\alpha \in B^1(\mathcal{H})$ represents a class in $\mathrm{IH}^1(\mathcal{H})$. Then $\phi_t^\dagger\alpha = \alpha(t) \otimes e = dl(t) = N(t)l(t)$ for some $l(t) \in H$. The result follows. □

6.9. Suppose $X \in E^*$. Then contraction (or interior product) with X gives a map $\iota_X : \wedge^p E \to \wedge^{p-1}E$. We then get a $(-1,-1)$ morphism of mixed Hodge structures $\delta_X : K^p(\mathcal{H}) \to K^{p-1}(\mathcal{H})$ by setting $\delta_X(h \otimes \omega) = h \otimes \iota_X(\omega)$. Note that δ_X preserves the restricted Koszul complex $B(\mathcal{H}) \subset K(\mathcal{H})$. It is also easy to see that $\delta_X^2 = 0$.

Suppose $t = (t_1, \dots, t_r) \in \mathbb{F}^r$. Then we write $X(t) = \sum t_i e_i^* \in E^*$ and $\delta_t = \delta_{X(t)}$. We write

$$\triangle_t := d\delta_t + \delta_t d. \tag{122}$$

Note that, for $\alpha = \sum \alpha_i \otimes e_i \in K^1(\mathcal{H})$, we have

$$\delta_t\alpha = \sum t_i\alpha_i = \alpha(t). \tag{123}$$

The following lemma is a variant of [10, Lemma 3.7]. (Indeed, it can easily be deduced from [10, Lemma 3.7], but it is also easy enough to give a direct proof.)

Lemma 124. *For $t \in \mathbb{F}^r$, write $N(t) : K^p(\mathcal{H}) \to K^p(\mathcal{H})$ for the operator given by $N(t)(h \otimes \omega) = (N(t)h) \otimes \omega$. Then $\triangle_t = N(t)$.*

Proof. Suppose $\alpha = h \otimes \omega \in K^p(\mathcal{H})$ with $h \in H$ and $\omega \in \wedge^p E$. Then

$$\begin{aligned}\triangle_t \alpha &= d\delta_t \alpha + \delta_t d\alpha \\ &= d(h \otimes \iota_{X(t)}\omega) + \delta_t(\sum_{i=1}^r N_i h \otimes e_i \wedge \omega) \\ &= \sum_{i=1}^r N_i h \otimes e_i \wedge \iota_{X(t)}\omega + \sum_{i=1}^r N_i h \otimes \iota_{X(t)}(e_i \wedge \omega) \\ &= \sum_{i=1}^r N_i h \otimes e_i \wedge \iota_{X(t)}\omega + \sum_{i=1}^r N_i h \otimes t_i \omega - \sum_{i=1}^r N_i h \otimes e_i \wedge \iota_{X(t)}\omega \\ &= \sum_{i=1}^r t_i N_i h \otimes \omega = N(t)\alpha.\end{aligned}$$

□

6.10. **A pairing on the B complex.** Suppose $I = (i_1, \ldots, i_k)$ is a multi-index in $\{1, \ldots, r\}$. For $t \in \mathbb{F}^r$, set $t_I = t_{i_1} t_{i_2} \cdots t_{i_k}$. Using §6.2, we get a pairing

$$q_t : B^p(\mathcal{H}) \otimes B^p(\mathcal{H}^*) \to \mathbb{F}(-p) \text{ given by} \tag{125}$$

$$q_t(N_I h \otimes e_I, N_J^* \lambda \otimes e_J) = \begin{cases} t_I(h, N_J^* \lambda), & I = J; \\ 0, & \text{else.} \end{cases}$$

Here $h \in H$ and $\lambda \in H^*$, and $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ are multi-indices with $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_l$.

Remark 126. Note that, for $h \in H$ and $\mu \in H^*$, $(N_I h, \mu) = (h, N_I^* \mu)$. It follows that

$$q_t(N_I h \otimes e_I, N_I^* \lambda \otimes e_I) = t_I(N_I h, \lambda).$$

As we mentioned in §6.1, the monodromy logarithms on $\mathcal{H}^*$ are give by $-N_i^*$. So let $\mathrm{sw} : B^p\mathcal{H} \otimes B^p\mathcal{H}^* \to B^p\mathcal{H}^* \otimes B^p\mathcal{H}$ be the map switching the factors. From this it follows that $q_t \circ \mathrm{sw} = (-1)^p q_t$.

Proposition 127. *For each $t \in \mathbb{F}^r$, the pairing q_t in* (125) *is a morphism of mixed Hodge structure. If $t \in (\mathbb{F}^\times)^r$, then q_t is a non-degenerate pairing.*

Proof. The fact that the pairing is a morphism of Hodge structures follows from the fact that each N_i induces a morphism $N_i : H \to H(-1)$ of mixed Hodge structures (plus a little bit of bookkeeping about the number of Tate twists). The fact that the pairing is non-degenerate for $t \in (\mathbb{F}^\times)^r$ follows from 6.2. □

Proposition 128. *For $\alpha \in B(\mathcal{H})$, $\beta \in B(\mathcal{H}^*)$ and $t \in \mathbb{F}^r$, we have*

$$q_t(d\alpha, \beta) = q_t(\alpha, \delta_t \beta); \tag{129}$$

$$q_t(\delta_t \alpha, \beta) = q_t(\alpha, d\beta). \tag{130}$$

Proof. By the symmetry in the definition (or Remark 126), it suffices to prove (129). For this, suppose $h \in H$ and $\lambda \in H^*$, and set $\alpha = N_J h \otimes e_J$, $\beta = N_K^* \lambda \otimes e_K$ for multi-indices $J, K \subset \{1, \ldots, r\}$. Then both sides of (129) are 0 unless there is an element $i \in K \setminus J$ and an $\epsilon \in \{\pm 1\}$ such that $e_i \wedge e_J = \epsilon e_K$.

Now, we compute

$$\begin{aligned} q_t(d\alpha, \beta) &= q_t(N_i N_J h \otimes e_i \wedge e_J, N_K^* \lambda \otimes e_K) \\ &= \epsilon q_t(N_K h \otimes e_K, N_K^* \lambda \otimes e_K) \\ &= \epsilon t_K(h, N_K^* \lambda). \end{aligned}$$

On the other hand,

$$\begin{aligned} q_t(\alpha, \delta_t \beta) &= q_t(N_J h \otimes e_J, \delta_t N_K^* \lambda \otimes e_K) \\ &= q_t(N_J h \otimes e_J, \delta_t N_K^* \lambda \otimes (\epsilon e_i \wedge e_J)) \\ &= q_t(N_J h \otimes e_J, t_i N_K^* \lambda \otimes (\epsilon e_J)) \\ &= \epsilon t_i q_t(N_J h \otimes e_J, N_K^* \lambda \otimes e_J) \\ &= \epsilon t_i t_J(h, N_K^* \lambda) \\ &= \epsilon t_K(h, N_K^* \lambda). \end{aligned}$$

□

Lemma 131. *Suppose $t \in \mathbb{F}^r_{\geq 0}$. Then, every class in $\mathrm{IH}^1 \mathcal{H}$ has a representative $\alpha \in Z^1(B\mathcal{H})$ with $\delta_t \alpha = 0$.*

Proof. Suppose $\alpha' \in Z^1(B\mathcal{H})$. Using Proposition 120, find $l = l(t) \in H$ such that $N(t)l = \alpha'(t)$. Then set $\alpha = \alpha' - dl$. We have $\delta_t(\alpha) = \delta_t \alpha' - \delta_t dl = \alpha'(t) - \triangle_t l = \alpha'(t) - N(t)l = 0$. □

Corollary 132. *Set $L^p_t \mathcal{H} = \ker \delta_t \cap \ker d \subset B^P(\mathcal{H})$. For each p, the canonical map $L^p_t \mathcal{H} \to \mathrm{IH}^p \mathcal{H}$ is a morphism of mixed Hodge structure. The map $L^1_t \mathcal{H} \to \mathrm{IH}^1 \mathcal{H}$ is surjective. In fact, the map $W_{k+1} L^1_t \mathcal{H} \to \mathrm{IH}^1 \mathcal{H}$ is surjective as well.*

Proof. Since $\mathrm{IH}^p \mathcal{H} = H^p(B\mathcal{H})$ as a mixed Hodge structure, the map $L^p_t \mathcal{H} \to \mathrm{IH}^p \mathcal{H}$ is a morphism of mixed Hodge structure. The second assertion follows from Lemma 131. The last assertion follows from the purity theorem [10, 28] and the strictness of morphisms of mixed Hodge structure with respect to the weight filtration. □

6.11. The restriction of q_t to L^1_t gives a pairing

$$q_t : L^1_t(\mathcal{H}) \otimes L^1_t(\mathcal{H}^*) \to \mathbb{F}(-1). \tag{133}$$

Since L^1_t is a sub-mixed Hodge structure of $B^1(\mathcal{H})$, this is a morphism of mixed Hodge structures. Using the notation that $\mathcal{H}^\vee = \mathcal{H}^*(1)$, we get a pairing $q_t : L^1_t \mathcal{H} \otimes L^1_t \mathcal{H}^\vee \to \mathbb{F}$.

Lemma 134. *Suppose $\alpha \in L^1_t(\mathcal{H}), \lambda \in B^0(\mathcal{H}^*) = \mathcal{H}^*$. Then $q_t(\alpha, d\lambda) = 0$. Similarly, if $h \in B^0\mathcal{H}$ and $\beta \in L^1_t(\mathcal{H}^*)$, then $q_t(dh, \beta) = 0$.*

Proof. For the first assertion, we have $q_t(\alpha, d\lambda) = q_t(\delta_t \alpha, \lambda) = 0$. The second assertion has the same proof (or follows by symmetry). □

Theorem 135. *Suppose $t \in \mathbb{F}^r_{\geq 0}$. The pairing $q_t : L^1_t \mathcal{H} \otimes L^1_t \mathcal{H}^\vee \to \mathbb{F}$ descends to a pairing*

$$h(t) : \mathrm{IH}^1 \mathcal{H} \otimes \mathrm{IH}^1 \mathcal{H}^\vee \to \mathbb{F}.$$

This pairing, which we call the asymptotic height pairing *is a morphism of mixed Hodge structures. Therefore if $\mathcal{H}$ has weight k, the pairing factors through a pairing*

$$\bar{h}(t) : \mathrm{Gr}^W_{k+1} \mathrm{IH}^1 \mathcal{H} \otimes \mathrm{Gr}^W_{-k-1} \mathrm{IH}^1 \mathcal{H}^\vee \to \mathbb{F}.$$

If $c \in \mathbb{F}_{\geq 0}$, then $h(ct) = ch(t)$.

Proof. The first assertion follows from Corollary 132 and Lemma 134. The fact that $h(t)$ is a morphism of Hodge structures follows from the fact that (133) is a morphism of mixed Hodge structures. If $\mathcal{H}$ is pure of weight k, then $\mathrm{IH}^1\mathcal{H}$ has weights in the interval $(-\infty, k+1]$. Similarly, $\mathrm{IH}^1\mathcal{H}^\vee$ has weighs in the interval $(-\infty, -k-1]$. So $h(t)$ factors through through a pairing $\bar{h}(t)$ by weight considerations.

To prove the last assertion, note that $q_{ct} = cq_t$ as a pairing on from $B^1(\mathcal{H}) \otimes B^1(\mathcal{H}^\vee)$ to $\mathbb{F}$. And $\delta_{ct} = c\delta_t$ on $B^1(\mathcal{H})$. It follows that $L^1_{ct}(\mathcal{H}) = L^1_t(\mathcal{H})$ and similarly $L^1_{ct}(\mathcal{H}^\vee) = L^1_c(\mathcal{H}^\vee)$. Therefore $h(ct) = c(t)$. □

Proposition 136. *The pairing*

$$h(t) : \mathrm{IH}^1\mathcal{H} \otimes \mathrm{IH}^1\mathcal{H}^\vee \to \mathbb{F} \tag{137}$$

can be computed as follows. Suppose $\alpha = \sum_{i=1}^r \alpha_i \otimes e_i \in B^1(\mathcal{H})$ and $\beta = \sum_{i=1}^r \beta_i \otimes e_i \in B^1(\mathcal{H}^\vee)$ represent intersection cohomology classes $\bar{\alpha}$ and $\bar{\beta}$ respectively. Fix t as above and $l(t)$ such that $N(t)l(t) = \alpha(t)$. Then

$$h(t)(\alpha, \beta) = \sum_{i=1}^r t_i(\alpha_i - N_i l(t), \beta_i)_{N_i}. \tag{138}$$

Proof. Given $l(t)$ as above, set $\alpha' = \alpha - dl(t)$. Then $\delta_t\alpha' = \alpha(t) - \triangle_t l(t) = 0$. If we have $\alpha_i = N_i h_i$ for $h_i \in H$, then

$$\begin{aligned} h(t)(\bar{\alpha}, \bar{\beta}) &= q_t(\alpha', \beta) = q_t(\alpha - dl(t), \beta) \\ &= \sum_{i=1}^r t_i(h_i - l(t), \beta) = \sum_{i=1}^r t_i(\alpha_i - N_i l(t), \beta_i)_{N_i} \end{aligned}$$

as desired. □

Example 139. Let $L = \mathbb{Q}^2$ with basis $u = (1,0)$ and $v = (0,1)$, and write u^*, v^* for the dual basis. Set

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Pick a positive integer r and let $N_i = N$ for $1 \leq i \leq r$. Let $\mathcal{L}$ denote the local system on Δ^{*r} with monodromy logarithms N_i. Then $\mathrm{IH}^1(\mathcal{L}) = \mathbb{Q}^r/\mathbb{Q}$ with the copy of $\mathbb{Q}$ embedded diagonally. An element of $\mathrm{IH}^1(\mathcal{L})$ is represented by a sum $\alpha = \sum_{i=1}^r a_i v \otimes e_i$ with $a_i \in \mathbb{Q}$. Similarly, and element of $\mathrm{IH}^1(\mathcal{L}^*)$ is given by a sum $\beta = \sum_{i=1}^r b_i u^* \otimes e_i$. Since $\mathcal{L}$ underlies a pure variation of Hodge structure of weight -1, α is admissible. We can take

$$l(t) = \frac{\sum_{i=1}^r a_i t_i}{\sum_{i=1}^r t_i} u.$$

Then $N(t)l(t) = \alpha(t)$. We claim that

$$h(t)(\alpha, \beta) = \frac{\sum_{i<j}(a_i - a_j)(b_i - b_j)t_i t_j}{\sum t_i}. \tag{140}$$

To see this, we compute

$$\begin{aligned}
h(t)(\alpha,\beta) &= \sum_i t_i(\alpha_i - N_i l(t), \beta_i)_{N_i} \\
&= \sum_i t_i(a_i v - \frac{\sum_j a_j t_j}{\sum_j t_j} N_i u, b_i u^*)_{N_i} \\
&= \sum_i t_i((a_i - \frac{\sum_j a_j t_j}{\sum_j t_j})v, b_i v^*) \\
&= \sum_i t_i(a_i - \frac{\sum_j a_j t_j}{\sum_j t_j})b_i \\
&= \frac{\sum_{i,j} a_i b_i t_i t_j - a_j b_i t_i t_j}{\sum_j t_j} \\
&= \frac{\sum_{i,j}(a_i - a_j) b_i t_i t_j}{\sum_j t_j}
\end{aligned}$$

This is easily manipulated into the form in (140).

7. Higher dimensional pairing and polarizations

In this section $\mathcal{H}$ is a pure variation of $\mathbb{F}$-Hodge structure of weight k with unipotent monodromy on Δ^{*r} as in §6.7. We fix a polarization

$$Q : \mathcal{H} \otimes \mathcal{H} \to \mathbb{F}(-k).$$

This induces a morphism $a_Q : \mathcal{H} \to \mathcal{H}^*(-k)$ by $a_Q(h_1)(h_2) = Q(h_1, h_2)$. By composing $h(t)$ with a_Q we get a pairing on $\mathrm{IH}^1\,\mathcal{H}$ with values in $\mathbb{F}(-k-1)$. In fact, we will show that for $t \in \mathbb{F}^r_+$ and any $p \in \mathbb{Z}$, we get a pairing on $\mathrm{IH}^p\,\mathcal{H}$. The goal of this section is to show that these pairings are, in fact, polarizations. We remark that, for $t = (1, \dots, 1)$, this is essentially contained in the paper by Cattani, Kaplan and Schmid on Intersection Cohomology [10].

Definition 141. For each $t \in \mathbb{F}^r$ and each $p \in \mathbb{Z}$, write

$$R^p_t(\mathcal{H}) = \ker d \cap \ker \triangle_t \cap B^p\mathcal{H}.$$

Note that, for each $t \in \mathbb{F}^r$, $R^p_t\mathcal{H}$ is a mixed Hodge substructure of $B^p\mathcal{H}$ containing the Hodge substructure $L^p_t\mathcal{H}$.

7.1. **Real splittings.** Let $\mathcal{D}$ be a classifying space of pure Hodge structures of k which are polarized by a morphism $Q : V \otimes V \to \mathbb{R}(-k)$. Let θ be a nilpotent orbit with values in $\mathcal{D}$ generated by $(N_1, \dots, N_r; F)$. Recall by Theorem (54), the monodromy weight filtration $W(N)$ is constant on the monodromy cone $\mathcal{C}$ of positive $\mathbb{R}$-linear combinations of $N_1, \dots, N_r$. Moreover, if $W = W(N)[-k]$ then (F, W) is a mixed Hodge structure for which each element of $\mathcal{C}$ is a $(-1,-1)$-morphism. The associated δ-splitting of θ is the nilpotent orbit $\tilde{\theta}$ generated by $(N_1, \dots, N_r; \tilde{F})$ where $(\tilde{F}, W) = (e^{-i\delta}F, W)$ is the Deligne δ-splitting of (F, W).

In particular, since θ and $\tilde{\theta}$ have the same monodromy logarithms $N_1, \dots, N_r$ they determine the same underlying complex $B = B(N_1, \dots, N_r)$. Likewise, since δ acts trivially on Gr^W it follows that (F, W) and $(\tilde{F}, W)$ induce the same Hodge structure on $Gr^W(B)$. In [10], Cattani, Kaplan and Schmid use this method to reduce questions about the intersection cohomology groups of a general nilpotent orbit to the case where the nilpotent orbit has limit split over $\mathbb{R}$.

Theorem 142. *For each $p \in \mathbb{Z}$ and each $t \in \mathbb{F}^r_+$, the map $R^p_t\mathcal{H} \to \mathrm{IH}^p\,\mathcal{H}$ is onto. Moreover, $R^p_t\mathcal{H}$ has weights in the interval $(-\infty, p+k]$.*

Proof. It suffices to prove this for real nilpotent orbits. When $t = (1, 1, \dots, 1)$ and the limit is split over $\mathbb{R}$, the first assertion is [10, (3.8)]. We can reduce to the case where the limit is split over $\mathbb{R}$ as in §7.1. Then we can reduce to the case to $t = (1, 1, \dots, 1)$ by scaling the N_i using Lemma (53). The second assertion follows from the definition of the weight filtration W in terms of $N(t)$. □

Definition 143. For each $t \in \mathbb{F}^r$, define a pairing Q_t on $B^p\mathcal{H}$ by setting

$$Q_t(N_I u \otimes e_I, N_J v \otimes e_J) = \begin{cases} t_I Q(u, N_J v), & I = J; \\ 0, & \text{else.} \end{cases}$$

When $t = (1, \dots, 1)$, this pairing is used in [10, (3.4)]. (They write S instead of Q for the polarization.)

Proposition 144. *For each $t \in (\mathbb{F}^\times)^r$, Q_t defines a pairing*

$$Q_t : B^p\mathcal{H} \otimes B^p\mathcal{H} \to \mathbb{F}(-p-k)$$

which is a non-degenerate morphism of mixed Hodge structure. For $t \in \mathbb{F}^r_+$, Q_t restricts to a polarization on $\mathrm{Gr}^W_{p+k} R^p_t \mathcal{H}$.

Proof. Again it suffices to check this for real nilpotent orbits. The non-degeneracy follows from §6.2. The restriction of the pairing to $R^p_t\mathcal{H}$ then gives a pairing

$$R^p_t\mathcal{H} \otimes R^p_t\mathcal{H} \to \mathbb{F}(-p-k). \tag{145}$$

Since $R^p_t\mathcal{H} \subset W_{p+k}B^p\mathcal{H}$, the pairing (145) vanishes on $W_{p+k-1}R^p_t\mathcal{H}$ for weight reasons. In the case that $t = (1, \dots, 1)$ and the limit is split over $\mathbb{R}$ this pairing is a polarization by [10]. We reduce to that case by applying the δ-splitting of §7.1 and rescaling. □

Note that a polarization on a pure Hodge structure induces a canonical polarization on any sub Hodge structure (by restriction), and it also induces a canonical polarization on any quotient Hodge structure (by orthogonal projection). Proposition 144 therefore gives a canonical polarization

$$\overline{Q}_t : \mathrm{Gr}^W_{p+k} \mathrm{IH}^p\,\mathcal{H} \otimes \mathrm{Gr}^W_{p+k} \mathrm{IH}^p\,\mathcal{H} \to \mathbb{F}(-p-k) \tag{146}$$

for $t \in \mathbb{F}^r_+$. Since $\mathrm{IH}^p\,\mathcal{H}$ is concentrated in weights $(-\infty, p+k]$, the pairing (146) gives a pairing

$$\mathrm{IH}^p\,\mathcal{H} \otimes \mathrm{IH}^p\,\mathcal{H} \to \mathbb{F}(-p-k) \tag{147}$$

also denoted by $\overline{Q}_t$ (with kernel $W_{p+k-1}\,\mathrm{IH}^p\,\mathcal{H}$).

Lemma 148. *For $\alpha, \beta \in B(\mathcal{H})$ we have $Q_t(d\alpha, \beta) = Q_t(\alpha, \delta_t\beta)$.*

Proof. The proof is essentially the same as the proof of Proposition 128. □

Corollary 149. *Suppose $\alpha \in L^p_t\mathcal{H}$ and $\beta \in \ker(R^p_t\mathcal{H} \to \mathrm{Gr}^W_{p+k} \mathrm{IH}^p\,\mathcal{H})$. Then $Q_t(\alpha, \beta) = 0$.*

Proof. Since $\beta \in R^p_t\mathcal{H}$, we have $\beta \in W_{p+k}B^p\mathcal{H}$. Then, the hypothesis implies that $\beta = d\gamma + \eta$ where $\gamma \in W_{p+k}B^{p-1}\mathcal{H}$ and $\eta \in W_{p+k-1}B^p\mathcal{H}$. So $Q_t(\alpha, \beta) = Q_t(\alpha, d\gamma + \eta) = Q_t(\delta_t\alpha, \gamma) + Q_t(\alpha, \eta) = Q_t(\alpha, \eta) = 0$ for weight reasons. □

7.2. Suppose $\mathcal{H}$ is pure of weight -1. Then the map $a_Q : \mathcal{H} \to \mathcal{H}^*(1) = \mathcal{H}^\vee$ induces maps $B^p\mathcal{H} \to B^p(\mathcal{H}^\vee)$ and $\mathrm{IH}^p\,\mathcal{H} \to \mathrm{IH}^p(\mathcal{H}^\vee)$, which we also denote by a_Q. Write $h_Q(t) : \mathrm{IH}^1\,\mathcal{H} \otimes \mathrm{IH}^1\,\mathcal{H} \to \mathbb{Q}$ for the pairing

$$([\alpha], [\beta]) \mapsto h(t)(a_Q[\alpha], [\beta]). \tag{150}$$

Theorem 151. *Suppose* $[\alpha], [\beta] \in \mathrm{IH}^1\,\mathcal{H}$ *with* $\mathcal{H}$ *pure of weight* -1.

$$h_Q(t)([\alpha], [\beta]) = \overline{Q}_t([\alpha], [\beta]).$$

Proof. We can find $\alpha, \beta \in L^1_t\mathcal{H}$ representing $[\alpha]$ and $[\beta]$ respectively. Then, by Corollary 149 and the definition of $\overline{Q}_t$, we have $h(t)(a_Q[\alpha], [\beta]) = q_t(a_Q\alpha, \beta) = \overline{Q}_t([\alpha], [\beta])$. □

The remainder of this section concerns the pairing $h(t)$ in a special case which will be useful for studying the Ceresa cycle.

Lemma 152. *Suppose L is a local system with unipotent monodromy on Δ^{*2}. Write γ_i, $i = 1, 2$ for the counterclockwise loops around $s_i = 0$. (So that $\pi_1(\Delta^{*2})$ is the free abelian group generated by γ_1 and γ_2.) Suppose that the monodromy logarithms N_1 and N_2 are equal. Write $N = N_1$ and suppose further that $N^2 = 0$. Let $i : \mathbb{Z} \to \pi_1(\Delta^{*2})$ be given by $1 \mapsto \gamma_2 - \gamma_1$. Then*

(i) *The map $NL \to B^1(L)$ given by $Nv \mapsto (0, Nv)$ induces an isomorphism $NL \to \mathrm{IH}^1(L)$.*
(ii) *The composition*

$$NL \to \mathrm{IH}^1(L) \to H^1(\Delta^{*2}, L) \xrightarrow{i^*} H^1(\mathbb{Z}, L) = L$$

is the inclusion.

Proof. Let $C(L)$ denote the complex $\ker N \to NL$ where the differential is 0. Then under the hypotheses, there is a morphism $C(L) \to B(L)$ which is the obvious inclusion on C^0 and where $C^1(L) \to B^1(L)$ is given by $Nv \mapsto (0, Nv)$. It is easy to see that this morphism is a quasi isomorphism of complexes. This proves (i).

To prove (ii), we note that $H^1(\mathbb{Z}, L)$ is computed by the Koszul complex $K(L)$ given by $L \to L$ with the 0 differential. Then the composition in (ii) is the map induced on the first cohomology by the following composition of complexes:

$$\begin{array}{ccc}
\ker N & \xrightarrow{0} & 0 \oplus NL \\
\downarrow & & \downarrow \\
L & \xrightarrow{d} & L \oplus L \\
\downarrow & & \downarrow {\scriptstyle(-\mathrm{id},\mathrm{id})} \\
L & \xrightarrow{0} & L.
\end{array}$$

(Here d denotes the morphism in the Koszul complex.) □

Proposition 153. *Suppose $\mathcal{H}$ is a variation of Hodge structure of weight -1 on Δ^{*2} with unipotent monodromy logarithms. Suppose $N_1 = N_2 = N$ and $N^2 = 0$. Let Q be a polarization of $\mathcal{H}$. Identify NH with $\mathrm{IH}^1(H)$ using Lemma 152. Then, for each $t \in \mathbb{Q}^2_{\geq 0}$, the pairing*

$$h_Q(t) : \mathrm{IH}^1\,\mathcal{H} \otimes \mathrm{IH}^1\,\mathcal{H} \to \mathbb{Q}$$

amounts to a paring

$$h_Q(t) : NH \otimes NH \to \mathbb{Q}.$$

This pairing is given, for $(t_1, t_2) \neq (0, 0)$ *by*

$$h_{\mathbb{Q}}(t)(Nh, Nk) = \frac{t_1 t_2}{t_1 + t_2} Q(h, Nk),$$

and vanishes for $(t_1, t_2) = (0, 0)$.

Proof. Under the isomorphism $NH \to \mathrm{IH}^1 \mathcal{H}$ of Lemma 152, the element Nh (resp. Nk) corresponds to the class $\alpha = Nh \otimes e_2$ (resp. $\beta = Nk \otimes e_2$) in $B^1 H$. For, $t \in \mathbb{Q}^2_{\geq 0} \setminus \{(0, 0)\}$, set

$$\begin{aligned} \beta(t) &= Nk \otimes e_2 - d\left(\frac{t_2}{t_1 + t_2} k\right) \\ &= -\frac{t_2}{t_1 + t_2} Nk \otimes e_1 + \frac{t_1}{t_1 + t_2} Nk \otimes e_2. \end{aligned}$$

And set $\beta(t) = Nk \otimes e_2$ for $t = (0, 0)$. Then $\delta_t \beta = 0$. So $\beta(t) \in L^1_t \mathcal{H}$. Therefore, the pairing vanishes when $t = (0, 0)$, and, when t is non-zero, $h_{\mathbb{Q}}(t)(\alpha, \beta) = h(t)(a_{\mathbb{Q}} \alpha, \beta) = q_t(a_{\mathbb{Q}} \alpha, \beta(t)) = t_2 Q(h, \frac{t_1}{t_1 + t_2} Nk) = \frac{t_1 t_2}{t_1 + t_2} Q(h, Nk)$. □

8. Comparison Theorem

In this section we relate the asymptotic height pairing (135) and the asymptotics of the height of the biextension line bundle. We begin with a minor modification of Remark 20.

Lemma 154. *Let* $V_{\mathbb{Q}} = \mathbb{Q} v_0 \oplus U_{\mathbb{Q}} \oplus \mathbb{Q} v_{-2}$ *be a finite dimensional vector space. Suppose that* N *is a nilpotent endomorphism of* $V_{\mathbb{Q}}$ *such that*

$$N(v_0) = \mu v_{-2}, \qquad N(U_{\mathbb{Q}}) \subseteq U_{\mathbb{Q}}, \qquad N(v_{-2}) = 0.$$

Let $e_0 = v_0 + u + c v_{-2}$ *be another element of* $V_{\mathbb{Q}}$ *such that* $N(e_0) = \mu' v_{-2}$ *where* $u \in U_{\mathbb{Q}}$. *Then,* $\mu = \mu'$.

Proof. By definition,

$$(\mu' - \mu) v_{-2} = N(e_0 - v_0) = N(u + c v_{-2}) = N(u).$$

Since $N(u) \in U_{\mathbb{Q}}$ and $U_{\mathbb{Q}} \cap \mathbb{Q} v_{-2} = 0$ it follows that both the right and left hand sides of the previous equation vanish. □

Theorem 155. *Let* $\mathcal{V}$ *be an admissible biextension variation with unipotent monodromy over* Δ^{*r} *with underlying normal functions* $\nu \in \mathrm{ANF}(\Delta^{*r}, \mathcal{H})$ *and* $\omega \in \mathrm{ANF}(\Delta^{*r}, \mathcal{H}^\vee)$. *Let* $\mathrm{sing}(\nu) = \alpha$ *and* $\mathrm{sing}(\omega) = \beta$. *Define* $\mu(\mathcal{V})$ *as in equation* (19). *Define* $\mu_1, \dots, \mu_r$ *as in Remark* (20). *Then, for any* $t \in \mathbb{Z}^r_{\geq 0}$,

$$h(t)(\alpha, \beta) = -\mu(\mathcal{V})(t) + \sum_j t_j \mu_j. \tag{156}$$

In particular, $h(t)(\alpha, \beta) \equiv -\mu(\mathcal{V})(t)$ *modulo a linear function of* t.

Proof. Let V be the $\mathbb{Q}$-vector space corresponding to a reference fiber of $\mathcal{V}$. Pick a splitting of the weight filtration W of V so that $V = \mathbb{Q} e_0 \oplus H \oplus \mathbb{Q} e_{-2}$ where:

(i) the image of e_0 is the canonical generator of $\mathrm{Gr}^W_0 V = \mathbb{Q}$;
(ii) H maps isomorphically to Gr^W_{-1};
(iii) e_{-2} is the canonical generator of $W_{-2} V = \mathbb{Q}(1)$.

Write $\bar{N}_i$ for the i-th monodromy operator on H and write N_i for the i-th monodromy operator on V. Then, in terms of the splitting of W chosen above, we have

$$N_i = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_i & \bar{N}_i & 0 \\ \gamma_i & -\beta_i & 0 \end{pmatrix}$$

where $\gamma_i \in \mathbb{Q}(1)$ and $(\alpha_1, \dots, \alpha_r)$ (resp. $(\beta_1, \dots, \beta_r)$) is a representative of the class α (resp. β) in $Z^1(B(H))$ (resp. $Z^1(B(H^\vee))$. (§6.1 explains why β_i appears with a minus sign in the matrix of N_i.)

Set $\bar{N}(t) = \sum t_i \bar{N}_i, \gamma(t) = \sum t_i \gamma_i$, etc. Let $l(t) \in H$ be an element such that $\bar{N}(t)l(t) = \alpha(t)$. Set $e_0(t) = e_0 - l(t)$. Then,

$$\begin{aligned} N(t)e_0(t) &= \alpha(t) + \gamma(t)e_{-2} - \bar{N}(t)l(t) + (l(t), \beta(t))e_{-2} \\ &= (\gamma(t) + (l(t), \beta(t)))e_{-2}. \end{aligned}$$

Therefore, by Remark (20) and Lemma (154), $\mu(\mathcal{V})(t) = \gamma(t) + (l(t), \beta(t))$. Likewise, we have $\mu_i = \gamma_i + (l_i, \beta_i)$ where $\bar{N}_i(l_i) = \alpha_i$.

On the other hand, by (138)

$$h(t)(\alpha, \beta) = \sum_{i=1}^{r} t_i(\alpha_i - N_i l(t), \beta_i)_{N_i}.$$

Consequently,

$$\begin{aligned} \mu(\mathcal{V})(t) + h(t)(\alpha, \beta) &= \gamma(t) + (l(t), \beta(t)) + \sum_{i=1}^{r} t_i(\alpha_i - N_i l(t), \beta_i)_{N_i} \\ &= \gamma(t) + (l(t), \beta(t)) + \sum_{i=1}^{r} t_i(\alpha_i, \beta_i)_{N_i} - \sum_{i=1}^{r} t_i(N_i l(t), \beta_i)_{N_i} \\ &= \gamma(t) + (l(t), \beta(t)) + \sum_{i=1}^{r} t_i(\alpha_i, \beta_i)_{N_i} - \sum_{i=1}^{r} t_i(l(t), \beta_i) \\ &= \gamma(t) + (l(t), \beta(t)) + \sum_{i=1}^{r} t_i(\alpha_i, \beta_i)_{N_i} - (l(t), \beta(t)) \\ &= \gamma(t) + \sum_{i=1}^{r} t_i(\alpha_i, \beta_i)_{N_i} \\ &= \gamma(t) + \sum_{i=1}^{r} t_i(l_i, \beta_i) \\ &= \sum_{i=1}^{r} t_i \mu_i \end{aligned}$$

□

9. Boundary Values

Let $\mathcal{V} \to \Delta^{*r}$ be an admissible biextension variation with unipotent monodromy, and $C = \mathbb{R}^r_{>0}$. Then, as noted in Remark (20), the function $\mu(\mathcal{V})(m)$ is defined on the closure $\bar{C}$ of C.

Lemma 157. *Let $J \subset \{1, \dots, r\}$. Then $\mu(\mathcal{V})(m_1, \dots, m_r)$ is given by a rational function on the subset $\bar{C}_J \subset \bar{C}$ on which $m_j > 0$ if $j \in J$ and $m_j = 0$ if $j \notin J$.*

Proof. The proof Theorem (155) shows that $\mu(\mathcal{V})(m) = \gamma(m) + (\ell(m), \beta(m))$. Accordingly, the content of the lemma boils down to showing we can find a solution to the equation

$$N(m)\ell(m) = \alpha(m)$$

using rational functions of m. Without loss of generality, we assume that $J = \{1, \dots, a\}$. Let $\mathcal{H} = Gr^{W}_{-1}\mathcal{V}$ with reference fiber H. Then, the nilpotent orbit of $\mathcal{H}$ determines an associated nilpotent orbit

$$\tilde{\theta}(z) = e^{\sum_k z_k N_k}\tilde{F}$$

with limit mixed Hodge structure split over $\mathbb{R}$ as in §2. Moreover, $\tilde{\theta}(z)$ takes values in the appropriate classifying space of pure Hodge structure as soon as $\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_r) > 0$. Therefore,

$$\tilde{\theta}_a(z_1, \dots, z_a) = \tilde{\theta}(z_1, \dots, z_a, i, \dots, i)$$

is a nilpotent orbit. By [9], we therefore obtain an sl_2-triple $(N(m), H, N^+(m))$ for each $m \in \bar{C}_J$ where H is a fixed semisimple element.

To continue, we note that the fact that H is constant reflects the fact that the monodromy weight filtration $W_J := W(N(m))$ is constant on $\bar{C}_J$. Moreover, $\alpha(m)$ and $\beta(m) \in W_{-1}(J)$. Let $\alpha_{-1}(m)$ denote the projection of $\alpha(m)$ to the -1-eigenspace of H. Then, since $N(m) : Gr^{W_J}_1 \to Gr^{W_J}_{-1}$ is an isomorphism, there exists a unique element $\ell_1(m)$ in the $+1$ eigenspace of H such that $N(m)\ell_1(m) = \alpha_{-1}(m)$. For weight reasons,

$$(\ell(m), \beta(m)) = (\ell_1(m), \beta(m))$$

Accordingly, the rationality of $\mu(m)$ boils down to the rationality of the inverse map $N(m)^{-1} : Gr^{W_J}_{-1} \to Gr^{W_J}_1$. □

Given this lemma, it is natural to ask of if $\mu(\mathcal{V})$ has a continuous extension from C to $\bar{C}$. To establish this, let $\mathcal{G}$ be a class of sequences of points in C with the property that any sequence of points $\{m(p)\}$ in C which converges to $m_o \in \bar{C} - C$ contains a subsequence which belongs to $\mathcal{G}$.

Lemma 158. *Let $f : C \to \mathbb{R}$ be a function which has limit L along every $\mathcal{G}$-sequence which converges to $m_o \in \bar{C} - C$. Then, f has limit L along every sequence in C converging to m_o.*

Proof. Let $m(p)$ be a sequence in C which converges to m_o. Then, there exists a real number B and an index p' such that $|f(m(p))| < B$ for all $p > p'$. Indeed, if this not true we can find a subsequence $m(p_i)$ such that $|f(m(p_i)| > i$ for all i sufficiently large. By the defining property of $\mathcal{G}$, we can find a $\mathcal{G}$-subsequence of $m(p_i)$ which converge to m_o. By hypothesis, f has limit L along $\mathcal{G}$-sequences through m_o.

Suppose now that $\lim_{p\to\infty} f(m(p)) \neq L$. Then, there exists an $\epsilon > 0$ and a subsequence $m(p_i)$ such that $|f(m(p_i)) - L| \geq \epsilon$ for all i. Passage to a $\mathcal{G}$-subsequence of $m(p_i)$ again produces a contradiction. □

To apply this lemma, we recall that in [4] the authors defined a notion of sl_2-sequences on

$$I = \{(z_1, \dots, z_r) \mid \mathrm{Re}(z) \in [0,1], \quad \mathrm{Im}(z) \in [1, \infty)\}$$

which has the property that every sequence in I contains an sl_2-sequence. Given a point $m_o \in \bar{C} - C$ and a sequence of points $m(p)$ in C which converges to m_0, define

$$\tilde{m}(p) = m(p) / \min(m_1(p), \dots, m_r(p)) \tag{159}$$

keeping in mind that each $m_j(p) > 0$. Then, $\sqrt{-1}\tilde{m}(p)$ is a sequence of points in I. We say that $m(p)$ is a $\mathcal{G}$-sequence if $i\tilde{m}(p)$ is a sl_2-sequence.

As our next preliminary, given a mixed Hodge structure (F, W) let $\hat{Y}_{(F,W)}$ denote the Deligne grading of the sl_2-splitting of (F, W). Next we note (cf. [4]) that if $(N_1, \ldots, N_r; F, W)$ generate an admissible nilpotent orbit θ with limit mixed Hodge structure split over $\mathbb{R}$ then

$$\hat{Y}_{(e^{icN}F,W)} = \hat{Y}_{(e^{iN}F,W)} \tag{160}$$

for any positive real number c and any $N \in \mathcal{C} = \{\sum_j y_j N_j \mid y_1, \ldots, y_r > 0\}$.

Finally, suppose that θ is a nilpotent orbit of biextension type with reference fiber V and let $e_0 \in V$ project to $1 \in Gr^W_0$ and e_{-2} be the generator of Gr^W_{-2}. Then, for any $m \in C$

$$\mu(\theta)(m)e_{-2} = \frac{1}{2}[N(m), \hat{Y}_{(e^{iN(m)}F,W)}]e_0. \tag{161}$$

Indeed, by the properties of Deligne systems and the short length of W, the grading $\hat{Y}_{(e^{iN}F,W)}$ produces a lift v_0 of $1 \in Gr^W_0$ such that $N(v_0)$ belongs to W_{-2}, so we can apply Lemma (154) to conclude the stated formula.

Lemma 162. *Suppose $f : \bar{C} \to \mathbb{R}$ is a function, and suppose that, for every sequence $m(p)$ in C converging to $m_0 \in \bar{C}$, $\lim f(m(p)) = f(m_0)$. Then f is continuous on $\bar{C}$.*

Proof. By the definition of continuity, f is continuous on C. Suppose that $m(p)$ is a sequence in $\bar{C}$ converging to m_0. For each p, we can find a sequence $\{n(p,q)\}_{q=1}^{\infty}$ of points in C converging to $m(p)$. By our hypothesis, for each p, we have $\lim_{q\to\infty} f(n(p,q)) = f(m(p))$. So we can find a $q \in \mathbb{Z}_{>0}$ such that

$$\max(|n(p,q) - m(p)|, |f(n(p,q)) - f(m(p))|) < 2^{-p}$$

where we write $|n(p,q) - m(p)|$ for the usual Euclidean norm. Set $n(p) := n(p,q)$. Then $n(p) \to m_0$, so, by our hypotheses, $f(n(p)) \to f(m_0)$. Moreover, $\lim_{p\to\infty} f(n(p)) = \lim_{p\to\infty} f(m(p))$. So, since $m(p)$ was an arbitrary sequence in $\bar{C}$ converging to m_0, this proves that f is continuous. □

Theorem 163. *$\mu(\mathcal{V})$ has a continuous extension from C to $\bar{C}$.*

Proof. The first step is to observe that $\mu(\mathcal{V}) = \mu(\theta)$ where θ is the nilpotent orbit generated by $(N_1, \ldots, N_r; \hat{F}, M)$ where (F, M) is sl_2-splitting of the limit mixed Hodge structure of $\mathcal{V}$. Indeed, by Remark 20, the value of μ depends only on structure of the local monodromy logarithm.

To continue, let $m(p)$ be a sequence in C which converges to $m_o \in \bar{C} - C$ and $\tilde{m}(p)$ be the corresponding sequence (159) in I. Then, by property (160) we can rewrite (161) as

$$\mu(\theta)(m(p))e_{-2} = \frac{1}{2}[N(m(p)), \hat{Y}_{(e^{iN(\tilde{m}(p))}F,W)}]e_0$$

By Theorem (2.30) of [4], the grading $\hat{Y}_{(e^{iN(\tilde{m}(p))}F,W)}$ has limits of the form

$$Y(N(\theta^1), Y(N(\theta^2), \ldots, Y_{(F,M)}))$$

along sl_2-sequences (see [4] for notation). The first key thing to know about this limit is that $N(\theta^1) = N(m_o)$ since $N(m_0)/\min(m_j(p))$ is the dominant term of $N(\tilde{m}(p))$. The second key thing to know is that

$$[N(m_o), Y(N(m_o), Y(N(\theta^2), \ldots, Y_{(F,M)}))]$$

belongs to $W_{-2}\mathrm{End}(V)$. Therefore,

$$\mu(\theta)(m_o)e_{-2} = \frac{1}{2}[N(m_o), Y(N(m_o), Y(N(\theta^2), \ldots, Y_{(F,M)}))]e_0$$

which proves that $\lim_{m\to m_o} \mu(\theta)(m) = \mu(\theta)(m_o)$ for sequences $m(p) \in C$ tending to $m_o \in \bar{C} - C$. Now apply Lemma 162. □

10. Mixed Extensions and Biextensions

In [41, IX.9.3, pp. 421–426], Grothendieck develops the theory of mixed extensions (extensions panachées) in an abelian category. This turns out to be a convenient way to think about mixed extensions of normal functions and the variations of Hodge structure which we have been calling biextension variations. In this section, we briefly review this notion and its relation to the Mumford-Grothendieck's concept of a biextension.

10.1. Suppose G is a group acting on a set X. If X is isomorphic to G regarded as a G-set with the action of left multiplication, then X is called a torsor for G. If X is either empty or isomorphic to G, then X is called an *pseudo-torsor* for G.

Similarly a sheaf $\mathcal{F}$ is a called a *pseudo-torsor* for a sheaf of groups $\mathcal{G}$ if there is an action of $\mathcal{G}$ on $\mathcal{F}$ and, for each open U, either $\mathcal{F}(U)$ is empty or $\mathcal{G}(U)$ acts simply transitively on $\mathcal{F}(U)$. (See the Stacks Project [42, Tag 03AH].)

10.2. Suppose Q_0, Q_1 and Q_2 are three objects in an abelian category $\mathbf{C}$. A *mixed extension* of Q_0 by Q_1 by Q_2 is an object X of $\mathbf{C}$ with an increasing filtration W_iX such that $W_{-3}X = 0$ and $W_0X = X$ together with isomorphisms $p_i : \mathrm{Gr}^W_{-i} X \xrightarrow{\sim} Q_i$.

Write $\mathrm{EXTPAN}(Q_0, Q_1, Q_2)$ for the category of mixed extensions of Q_0 by Q_1 by Q_2. Here a morphism in $\mathrm{EXTPAN}(Q_0, Q_1, Q_2)$ from X to X' is a morphism of objects in $\mathbf{C}$ commuting with the isomorphisms from p_i. Let $\mathrm{Extpan}(Q_0, Q_1, Q_2)$ denote the set of isomorphism classes in $\mathrm{EXTPAN}(Q_0, Q_1, Q_2)$.

From a mixed extension X of Q_0 by Q_1 by Q_2, we get an extension $W_0X/W_{-2}X$ of Q_0 by Q_1 and an extension $W_{-1}X$ of Q_1 by Q_2. In fact, we get a functor

$$\mathrm{EXTPAN}(Q_0, Q_1, Q_2) \to \mathrm{EXT}(Q_0, Q_1) \times \mathrm{EXT}(Q_1, Q_2). \tag{164}$$

Here $\mathrm{EXT}(Q_0, Q_1)$ denotes the usual category of extensions of Q_0 by Q_1. Given extensions E_0 of Q_0 by Q_1 and E_1 of Q_1 by Q_2, Grothendieck lets $\mathrm{EXTPAN}(E_0, E_1)$ denote the category of all mixed extensions of Q_0 by Q_1 by Q_2 together with isomorphisms $\pi_i : W_{-i}X/W_{-i-2}X \xrightarrow{\sim} E_i$ for $i = 0, 1$. The morphisms in $\mathrm{EXTPAN}(E_0, E_1)$ are morphisms $X \to X'$ in $\mathbf{C}$ commuting with the isomorphism π_i for $i = 0, 1$.

10.3. **Action of** $\mathrm{Ext}(Q_0, Q_2)$**.** Given an extension G of Q_0 by Q_2 and an object X of $\mathrm{EXTPAN}(E_0, E_1)$, Grothendieck defines a mixed extension $G \wedge X$ as follows. First, regard X as an object of the category $\mathrm{EXT}(Q_0, E_1)$ of extensions of Q_0 by E_1. The sequence

$$0 \to Q_2 \xrightarrow{i} E_1 \to Q_1 \to 0$$

gives a functor i_* from the category $\mathrm{EXT}(Q_0, Q_2)$ of extensions of Q_0 by Q_2 to $\mathrm{EXT}(Q_0, E_1)$. Then using Baer sum in $\mathrm{EXT}(Q_0, E_1)$ we can define $G \wedge X$ as $i_*G + E_1$. This gives an action of $\mathrm{EXT}(Q_0, Q_2)$ on $\mathrm{EXTPAN}(E_0, E_1)$.

Alternatively we can define $G \wedge X$ as follows. Suppose G is given by

$$0 \to Q_2 \xrightarrow{\iota} G \xrightarrow{\pi} Q_0 \to 0.$$

Then $G \wedge X$ is the cohomology of the complex

$$Q_2 \to G \oplus X \to Q_0 \tag{165}$$

where the first map is the one sending q to $(\iota q, -p_2^{-1}(q))$ and the second map sends (g, x) to $\pi(g) - p_0(x)$.

If we give G the filtration W_iG where $W_{-3}G = 0, W_{-2}G = W_{-1}G = Q_2$ and $W_0G = G$, then this induces a filtration on $G \oplus X$ making the morphisms in the complex (165) strict (for the obvious filtrations on Q_0 and Q_2). We then get a filtration on $G \wedge X$ making $G \wedge X$ into an object in $\mathrm{EXTPAN}(E_0, E_1)$. The isomorphism $\pi_0 : G \wedge X/W_{-2}(G \wedge X) \xrightarrow{\sim} E_0$ is induced from $\pi_0 : X/W_{-2}X \xrightarrow{\sim} E_0$. It sends a pair (g, x) to $\pi_0 x$.

A class in $W_{-1}(G \wedge X)$ can be represented by an element $(g, x) \in W_{-1}G \oplus W_{-1}X$. Since $g \in W_{-1}G$, we can consider g as an element in $Q_2 \subset E_1$. Then $\pi_1 : W_{-1}(G \wedge X) \xrightarrow{\sim} E_1$ sends (g, x) to $g + \pi_1(x)$.

It is easy to check that the action $(G, X) \mapsto G \wedge X$ described above gives a functorial action; that is, a functor $\mathrm{EXT}(Q_0, Q_2) \times \mathrm{EXTPAN}(E_0, E_1) \to \mathrm{EXTPAN}(E_0, E_1)$. The same prescription gives a functor $\mathrm{EXT}(Q_0, Q_2) \times \mathrm{EXTPAN}(Q_0, Q_1, Q_2) \to \mathrm{EXTPAN}(Q_0, Q_1, Q_2)$.

Proposition 166 (Grothendieck)**.** *Suppose Q_i are as above for $i = 0, 1, 2$ and E_i are as above for $i = 0, 1$.*

(i) $\mathrm{EXTPAN}(E_0, E_1)$ *is a groupoid; that is, every morphism in* $\mathrm{EXTPAN}(E_0, E_1)$ *is an isomorphism.*

(ii) *The set* $\mathrm{Extpan}(E_0, E_1)$ *of isomorphism classes of objects in* $\mathrm{EXTPAN}(E_0, E_1)$ *is a pseudo-torsor for* $\mathrm{Ext}(Q_0, Q_2)$. *In other words,* $\mathrm{Extpan}(E_0, E_1)$ *is either empty or a torsor for* $\mathrm{Ext}(Q_0, Q_2)$ *under the action $(G, X) \mapsto G \wedge X$.*

(iii) *For $E \in \mathrm{EXT}(Q_0, Q_1)$ and $F \in \mathrm{EXT}(Q_1, Q_2)$ as above. Consider the long exact sequence*

$$\mathrm{Ext}^1(Q_0, Q_2) \to \mathrm{Ext}^1(E, Q_2) \to \mathrm{Ext}^1(Q_1, Q_2) \xrightarrow{\partial} \mathrm{Ext}^2(Q_0, Q_2)$$

arising from the short exact sequence

$$0 \to Q_1 \to E \to Q_0 \to 0.$$

Write $\xi \in \mathrm{Ext}^1(Q_1, Q_2)$ for the class of F and define $c(E, F) = \partial(\xi)$. Then $c(E, F) = 0$ if and only if $\mathrm{Extpan}(E, F)$ *is non-empty.*

10.4. There is an obvious map $\mathrm{Extpan}(Q_0, Q_1, Q_2) \to \mathrm{Ext}^1(Q_0, Q_1) \times \mathrm{Ext}^1(Q_1, Q_2)$ and if $(E_0, E_1) \in \mathrm{Ext}^1(Q_0, Q_1) \times \mathrm{Ext}^1(Q_1, Q_2)$ we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Extpan}(E_0, E_1) & \longrightarrow & \{(E_0, E_1)\} \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ \mathrm{Extpan}(Q_0, Q_1, Q_2) & \xrightarrow{\pi} & \mathrm{Ext}^1(Q_0, Q_1) \times \mathrm{Ext}^1(Q_1, Q_2). \end{array}$$

We also get an action of $\mathrm{Hom}(Q_0, Q_1) \times \mathrm{Hom}(Q_1, Q_2)$ on the $\mathrm{Extpan}(E_0, E_1)$ as follows. Take $X \in \mathrm{Extpan}(E_0, E_1)$ and $(f_0, f_1) \in \mathrm{Hom}(Q_0, Q_1) \times \mathrm{Hom}(Q_1, Q_2)$. Then

$$G(f_0, f_1) := f_1 \cup [E_0] + [E_1] \cup f_0 \in \mathrm{Ext}^1(Q_0, Q_2). \tag{167}$$

Since $\mathrm{Extpan}(E_0, E_1)$ is a $\mathrm{Ext}^1(Q_0, Q_2)$-torsor, $G(f_0, f_1) \wedge X$ is and element of $\mathrm{Extpan}(E_0, E_1)$.

Proposition 168. *The map ϕ surjects onto $\pi^{-1}(E_0, E_1)$. Moreover, if $X, Y \in$* $\mathrm{Extpan}(E_0, E_1)$, *then $\phi(X) = \phi(Y)$ if and only if $Y = G(f_0, f_1) \wedge X$ for some $(f_0, f_1) \in$* $\mathrm{Hom}(Q_0, Q_1) \times \mathrm{Hom}(Q_1, Q_2)$.

Proof. The first statement is obvious. For the second, suppose $X \in \mathrm{EXTPAN}(E_0, E_1)$ and $Y = G(f_0, 0) \wedge X$. The extension $G := G(f_0, 0)$ is given by the following pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_2 & \longrightarrow & E_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle f_0} & & \\ 0 & \longrightarrow & Q_2 & \longrightarrow & G & \longrightarrow & Q_0 & \longrightarrow & 0 \end{array}$$

of the extension E_1 by the map f_0. We get a map $\psi : G \wedge X \to X$ sending a class represented by (g, x) to $x - h(g)$. This map is well-defined on $G \wedge X$ because, for $q \in Q_2$, $\psi(q, -q) = q - h(q) = 0$. The map ψ does not, in general commute with the maps π_0 to E_0: we have $\pi_0(\psi(g, x)) = \pi_0(x - h(g))$ while $\pi_0(g, x) = \pi_0(x)$. However, using the fact that $h(g) \in W_{-1}X$, we see that ψ

does commute with p_0. Then it is easy to see that ψ commutes with p_1 and p_2. Thus ψ induces a morphism in $\mathrm{EXTPAN}(Q_0, Q_1, Q_2)$ which is easily seen to be an isomorphism. So $G \wedge X$ is isomorphic to X in $\mathrm{EXTPAN}(Q_0, Q_1, Q_2)$.

The proof that $G(0, f_1) \wedge X$ is isomorphic to X in $\mathrm{EXTPAN}(Q_0, Q_1, Q_2)$ is similar. This proves the "if" part of the proposition.

For the "only if" part, suppose that X and Y are objects in $\mathrm{EXTPAN}(E_0, E_1)$ and $\psi : Y \to X$ is an isomorphism between them as objects in $\mathrm{EXTPAN}(Q_0, Q_1, Q_2)$. Using π_i, identify $W_{-i}X/W_{-i-2}X$ and $W_{-i}Y/W_{-i-2}Y$ with E_i for $i = 0, 1$. Then there exist morphisms $f_0 : Q_0 \to Q_1$ and $f_1 : Q_1 \to Q_2$ such that $(W_{-i}/W_{-i-2})(f) = \mathrm{id}_{E_i} + f_i$. From this it is not hard to see that $Y \cong G(f_0, f_1) \wedge X$. □

Corollary 169. *Suppose that* $\mathrm{Hom}(Q_0, Q_1) = \mathrm{Hom}(Q_1, Q_2) = 0$. *Then the fiber of the map*

$$\mathrm{Extpan}(Q_0, Q_1, Q_2) \to \mathrm{Ext}^1(Q_0, Q_1) \times \mathrm{Ext}^1(Q_1, Q_2)$$

over (E_0, E_1) *is* $\mathrm{Extpan}(E_0, E_1)$.

Proof. Obvious. □

10.5. **Pseudo-biextensions.** Grothendieck's concept of mixed extension interacts in a rather interesting way with the Mumford-Grothendieck concept of a biextension [41, pp. 156–159]. To explain this fix objects Q_0, Q_1 and Q_2 as above.

Suppose $E_1, E_2 \in \mathrm{Ext}^1(Q_0, Q_1)$ and $F_1, F_2 \in \mathrm{Ext}^1(Q_1, Q_2)$. There are natural composition laws

$$+_1 : \mathrm{Extpan}(E_1, F_1) \times \mathrm{Extpan}(E_2, F_1) \to \mathrm{Extpan}(E_1 + E_2, F_1)$$
$$+_2 : \mathrm{Extpan}(E_1, F_1) \times \mathrm{Extpan}(E_1, F_2) \to \mathrm{Extpan}(E_1, F_1 + F_2)$$

satisfying various compatibilities. This is explained in C. Hardouin's 2005 thesis [23, Theorem 4.4.1]. For the convenience of the reader we explain the operations $+_1$ and $+_2$.

10.6. Suppose $X_i \in \mathrm{Extpan}(E_i, F)$ for $i = 1, 2$. Write $\rho_i : X_i \to Q_0$ for the composition

$$X_i \to \mathrm{Gr}_0^W X_i \stackrel{p_0}{\to} Q_0$$

and write $j_i : F \to X_i$ for the canonical injection induced by π_1^{-1}. Then Hardouin defines $X_1 +_1 X_2$ to be the middle homology group of the complex

$$F \xrightarrow{\begin{pmatrix} j_1 \\ -j_2 \end{pmatrix}} X_1 \oplus X_2 \xrightarrow{\begin{pmatrix} \rho_1 & -\rho_2 \end{pmatrix}} Q_0$$

where we write the arrows in matrix notation. We then get maps

$$\begin{pmatrix} j_1 & 0 \end{pmatrix} : F \to X_1 +_1 X_2,$$
$$\begin{pmatrix} \rho_1 \\ 0 \end{pmatrix} : X_1 +_1 X_2 \to Q_0.$$

Combining these maps, we obtain an exact sequence

$$0 \to F \to X_1 +_1 X_2 \to Q_0 \to 0.$$

And it is not hard to see that this sequence makes $X_1 +_1 X_2$ into an element of $\mathrm{Extpan}(E_1 + E_2, F)$ in a canonical way.

10.7. Here is another way to think about the composition $+_1$. An object $X_i \in \mathrm{EXTPAN}(E_i, F)$ can be thought of as an object of $\mathrm{EXT}^1(Q_0, F)$ whose image in $\mathrm{EXT}^1(Q_0, Q_1)$ is E_i. In other words, if we let $\pi : \mathrm{EXT}^1(Q_0, F) \to \mathrm{EXT}^1(Q_0, Q_1)$ denote the functor induced by pushforward along the canonical map $F \to Q_1$, then $\mathrm{EXTPAN}(E_i, F)$ is the fiber category $\pi^{-1}(E_i)$. If we take the Baer sum of X_1 and X_2 regarded as objects in $\mathrm{Ext}^1(Q_0, F)$, then we clearly get an element of $\pi^{-1}(E_1 + E_2)$. It is easy to check that this element is, in fact, $X_1 +_1 X_2$ as defined above.

10.8. The construction of $+_2$ is similar, and we explain it in the language of §10.7. Suppose X_i are objects in $\mathrm{EXTPAN}(E, F_i)$ for $i = 1, 2$. Write $\iota : \mathrm{EXT}^1(E, Q_2) \to \mathrm{EXT}^1(Q_1, Q_2)$ for the canonical functor (induced by the inclusion $Q_1 \to E$). Then we can regard X_i as object in the fiber category $\iota^{-1}(F_i)$. We then define the sum $X_1 +_2 X_2$ to be the Baer sum of the two X_i in $\mathrm{EXT}^1(E, Q_2)$. It is easy to see that $X_1 +_2 X_2$ lies in the fiber category $\iota^{-1}(F_1 + F_2) = \mathrm{EXTPAN}(E, F_1 + F_2)$.

Definition 170. Suppose B, C and A are three abelian groups. A *pseudo-biextension* of $B \times C$ by A is a set E equipped with

(i) an A-action and a map $\pi : E \to B \times C$ giving E the structure of an A pseudo-torsor over $B \times C$;
(ii) commutative operations $+_1 : E \times_B E \to E$ and $+_2 : E \times_C E \to E$.

Write $p_B = \mathrm{pr}_1 \circ \pi$ and $p_C = \mathrm{pr}_2 \circ \pi$. For each $b \in B$, write $E_b = p_B^{-1}(b)$ and, for each C, write $E_c = p_C^{-1}(c)$. Then the above data is assumed to satisfy the following properties

(iii) For each $b \in B$, the operation $+_1$ makes E_b into an abelian group and the canonical map $E_b \to C$ into a group homomorphism. Moreover, the action of A on $\ker(E_b \to C)$ coming from the structure of E as an A pseudo-torsor induces a group isomorphism between A and $\ker(E_b \to C)$.
(iv) For each $c \in C$, the operation $+_2$ makes E_c into an abelian group and the canonical map $E_c \to B$ into a group homomorphism with kernel equal to A as in (iii).
(v) Suppose $\pi(X_{ij}) = (E_i, F_j)$ for $i = 1, 2$. Then

$$(X_{11} +_1 X_{12}) +_2 (X_{21} +_1 X_{22}) = (X_{11} +_2 X_{21}) +_1 (X_{12} +_1 X_{22}).$$

10.9. A surjective pseudo-biextension $\pi : E \to B \times C$ is called a *biextension*. Grothendieck studied the notion of biextensions in the category of sheaves of abelian groups in a topos T in [41]. Suppose B, C and A are three sheaves of abelian groups in T. We define a *pseudo-biextension* of $B \times C$ by A to be a sheaf E in T equipped with a morphism $\pi : E \to B \times C$ along with an A-action making E into an A-torsor over $B \times C$ and operations $+_1, +_2$ satisfying the same axioms as in Definition 170 above. A pseudo-biextension $\pi : E \to B \times C$ is a *biextension* if π is surjective as a morphism of sheaves.

Remark 171. Suppose A, B and C are abelian groups, and U is an extension of $B \otimes C$ by A. Write E for the pull-back of U to $B \times C$ via the natural map $B \times C \to B \otimes C$ given by $(b, c) \mapsto b \otimes c$. The group A acts on U via right translation in such a way that U becomes an A-torsor over $B \otimes C$. Therefore, the pull-back E of U becomes an A-torsor over $B \times C$. Write p for the map $U \to B \otimes C$ and π for the map $E \to B \times C$. Suppose $b_i \in B$, $c_i \in C$ for $i = 1, 2$ and $u_{ij} \in p^{-1}(b_i \otimes c_j)$. Then $u_{11} + u_{12} \in p^{-1}(b_1 \times C)$. So we can use the addition in U to define a map

$$+_1 : E \times_B E \to E.$$

We can similarly define $+_2 : E \times_C E \to E$. And it is slightly tedious but not hard to show that E becomes a biextension of $B \times C$ by A.

In fact, Grothendieck showed that all biextensions arise in this way provided that $\mathrm{Tor}_1(B, C) = 0$. More generally we have an exact sequence

$$0 \to \mathrm{Ext}^1(B \otimes C, A) \to \mathrm{Biext}(B \times C, A) \to \mathrm{Hom}(\mathrm{Tor}_1(B \otimes C), A)$$

where $\mathrm{Biext}(B \times C, A)$ denotes the set of isomorphisms of biextensions of $B \times C$ by A. [41, p.220].

Theorem 172 (Hardouin). *Suppose* $\mathbf{C}$ *is an abelian category and* Q, R, P *are objects satisfying* $\mathrm{Hom}(Q, R) = \mathrm{Hom}(R, P) = 0$. *Then the map*

$$\mathrm{Extpan}(Q, R, P) \to \mathrm{Ext}^1(Q, R) \times \mathrm{Ext}^1(R, P)$$

together with the action of $\mathrm{Ext}^1(Q, P)$ *and the operations* $+_1$ *and* $+_2$ *described above makes* $\mathrm{Extpan}(Q, R, P)$ *into a pseudo-biextension of* $\mathrm{Ext}^1(Q, R) \times \mathrm{Ext}^1(R, P)$ *by* $\mathrm{Ext}^1(Q, P)$.

Explanation. The main point is to check that the $\mathrm{Ext}^1(Q, P)$ action gives $\mathrm{Extpan}(Q, R, P)$ the structure of a pseudo-torsor. This follows from the condition that $\mathrm{Hom}(Q, R) = \mathrm{Hom}(R, P) = 0$ and Corollary 169. The rest of the verification is straightforward. □

10.10. **Sheaf theoretic version.** Suppose now that T is a topological space and $\mathcal{C}$ is a stack of abelian categories over T. So for each open subset U of T we have an abelian category $\mathcal{C}(U)$ and for each inclusion $i : V \to U$ we have restriction functor $i^* : \mathcal{C}(U) \to \mathcal{C}(V)$, which we also write as $X \rightsquigarrow X_{|V}$. As in [45, p. 5396], we assume that i^* is exact. The main example we have in mind is when T is a complex manifold and $\mathcal{C}(U)$ is the category of mixed Hodge modules on T.

For any objects $A, B \in \mathcal{C}(T)$, we get a sheaf $\mathbf{Hom}(A, B)$ sending on open U to $\mathrm{Hom}(A_{|U}, B_{|U})$.

Lemma 173. *Suppose that* A, B *are objects in* $\mathcal{C}(T)$ *satisfying* $\mathbf{Hom}(A, B) = 0$.

(i) *If* E *is an extension of* A *by* B *in* $\mathcal{C}(T)$, *then the identity is the only automorphism of* E *as an extension.*
(ii) *The assignment* $U \rightsquigarrow \mathrm{Ext}^1(A_{|U}, B_{|U})$, *which is naturally a presheaf by the exactness of the restriction functors, is a sheaf.*

Proof. (i) If ϕ is an automorphism then $\phi - \mathrm{id}$ induces an element of $\mathrm{Hom}(A, B)$ which must be 0.

(ii) Suppose $\{V_i\}$ is an open cover of U and, for each i, $E_i \in \mathrm{Ext}^1(A_{|V_i}, B_{|V_i})$ such that, for all i, j, $(E_i)_{|V_i \cap V_j} \cong (E_j)_{|V_i \cap V_j}$. It follows from (i) that there is a *unique* isomorphism $\varphi_{ij} : (E_i)_{|V_i \cap V_j} \cong (E_j)_{|V_i \cap V_j}$. In particular, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for all i, j, k. Therefore the E_i glue together to form an object $E \in \mathrm{Ext}^1(A_{|U}, B_{|U})$. □

We write $\mathbf{Ext}^1(A, B)$ for the resulting sheaf when $\mathbf{Hom}(A, B) = 0$.

Definition 174. We call a triple $Q_\bullet = (Q_0, Q_1, Q_2)$ *disjoint* if $\mathbf{Hom}(Q_0, Q_1) = \mathbf{Hom}(Q_0, Q_2) = \mathbf{Hom}(Q_1, Q_2) = 0$.

The main example we have in mind here is a sequence Q_i of torsion-free variations of pure Hodge structure of weight $-i$.

If $Q_\bullet$ is disjoint then the same argument as above shows that the presheaf

$$U \rightsquigarrow \mathrm{Extpan}(Q_0|U, Q_1|U, Q_2|U)$$

is a sheaf. Similarly, for $E \in \mathrm{Ext}^1(Q_0, Q_1)$ and $F \in \mathrm{Ext}^1(Q_1, Q_2)$, the presheaf $U \rightsquigarrow \mathrm{Extpan}(E_{|U}, F_{|U})$ is a sheaf. We write $\mathbf{Extpan}(Q_0, Q_1, Q_2)$ for the first sheaf and $\mathbf{Extpan}(E, F)$ for the second. By Grothendieck's result, $\mathbf{Extpan}(E, F)$ is a $\mathbf{Ext}^1(Q_0, Q_2)$ pseudo-torsor. Moreover, $\mathbf{Extpan}(E, F)$ is a $\mathbf{Ext}^1(Q_0, Q_2)$ torsor if the stalks of $\mathbf{Extpan}(E, F)$ are non-empty.

Using Theorem 172 we see that

$$\mathbf{Extpan}(Q_0, Q_1, Q_2) \to \mathbf{Ext}^1(Q_0, Q_1) \times \mathbf{Ext}^1(Q_1, Q_2)$$

together with the action of $\mathbf{Ext}^1(Q_0, Q_2)$ and the operations $+_1$ and $+_2$ (which become sheaf morphisms) is a pseudo-biextension of the sheaf $\mathbf{Ext}^1(Q_0, Q_1) \times \mathbf{Ext}^1(Q_1, Q_2)$ by $\mathbf{Ext}^1(Q_0, Q_2)$. It is a biextension if the stalks of $\mathbf{Extpan}(Q_0, Q_1, Q_2)$ are non-empty.

11. Mixed Extensions of normal functions

11.1. **The sheaf of mixed extensions.** Suppose $j : S \to \bar{S}$ is the inclusion of one complex manifold in another as a Zariski open set. By analogy with §5.1, for an A-variation of mixed Hodge structure $\mathcal{H}$ on S with $W_{-1}\mathcal{H} = \mathcal{H}$, write $\mathrm{NF}(S, \mathcal{H})$ for $\mathrm{Ext}^1_{\mathrm{VMHS}(S)}(A, \mathcal{H})$ and $\mathrm{ANF}(S, \mathcal{H})_{\bar{S}}$ for $\mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(A, \mathcal{H})$. If $\mathcal{H}$ is torsion free and pure of weight -1, duality gives an isomorphism $\mathrm{NF}(S, \mathcal{H}^*(1)) = \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathcal{H}, A)$ and sending $\mathrm{ANF}(S, \mathcal{H}^*(1))$ to $\mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathcal{H}, A)$.

Suppose now that $\mathcal{H}$ is pure of weight -1 and torsion-free. Since the triple $(A, \mathcal{H}, A(1))$ is disjoint, we get sheaves on S: $U \rightsquigarrow \mathrm{NF}(U, \mathcal{H}_{|U}), U \rightsquigarrow \mathrm{NF}(U, \mathcal{H}^\vee_{|U})$ and $U \rightsquigarrow \mathrm{NF}(U, A(1))$. We also get a sheaf $U \rightsquigarrow \mathrm{Extpan}(A, \mathcal{H}_{|U}, A(1))$. Write $\mathbf{NF}(\mathcal{H})$ and $\mathbf{NF}(A(1))$ for these sheaves on S. We also get sheaves $\mathbf{ANF}(\mathcal{H})$ and $\mathbf{ANF}(A(1))$ on $\bar{S}$ given by admissible normal functions.

Fix $\nu \in \mathrm{ANF}(S, \mathcal{H})_{\bar{S}}$ and $\omega \in \mathrm{ANF}(S, \mathcal{H}^\vee)_{\bar{S}}$. Then, for any open set $U \subset S$, we write $\mathcal{B}_A(U)$ for the set of isomorphism classes of mixed extensions of A by $\mathcal{H}_A$ by $A(1)$ in $A - \mathrm{VMHS}(U)$. Similarly, if $U \subset \bar{S}$, we write $\mathcal{B}^{\mathrm{ad}}_A(U)$ for the set of isomorphism classes of mixed extensions of A by H_A by $A(1)$ in $A - \mathrm{VMHS}(U \cap S)^{\mathrm{ad}}_U$. The normal functions ν and ω give rise to extensions $\nu_A \in \mathrm{Ext}^1_{A-\mathrm{VMHS}(U\cap S)^{\mathrm{ad}}_U}(A, \mathcal{H}_A)$ and $\omega_A \in \mathrm{Ext}^1_{A-\mathrm{VMHS}(U\cap S)^{\mathrm{ad}}_U}(\mathcal{H}^\vee_A, A(1))$. We write $\mathcal{B}_A(\nu, \omega)(U)$ (resp. $\mathcal{B}^{\mathrm{ad}}_A(\nu, \omega)(U)$) for set of isomorphism classes of mixed extensions of ν_A by ω_A in $A - \mathrm{VMHS}(U)$ (resp. $A - \mathrm{VMHS}(U \cap S)^{\mathrm{ad}}_U$). When $A = \mathbb{Z}$ we drop the subscript A from the notation and simply write $\mathcal{B}$ (resp. $\mathcal{B}^{\mathrm{ad}}$).

By pullback of variations, $\mathcal{B}_A$ (resp. $\mathcal{B}^{\mathrm{ad}}_A$) is presheaf on S (resp. $\bar{S}$).

Lemma 175. *The presheaf $\mathcal{B}_A$ (resp. $\mathcal{B}^{\mathrm{ad}}_A$) is a sheaf on S (resp. $\bar{S}$).*

Proof. This follows from the fact that $A, \mathcal{H}$ and $A(1)$ are disjoint in the category of A variations. □

Lemma 176. *Write $j^{\mathrm{mer}}_* \mathcal{O}^\times_S$ for the sheaf of meromorphic functions on $\bar{S}$ which are regular and non-vanishing on S. Then*

(i) *The functor*

$$U \rightsquigarrow \mathrm{Ext}^1_{\mathrm{VMHS}(U)}(\mathbb{Z}, \mathbb{Z}(1)) = \mathrm{NF}(U, \mathbb{Z}(1))$$

defines a sheaf on S which is canonically isomorphic to $\mathcal{O}^\times_S$.

(ii) *The functor*

$$U \rightsquigarrow \mathrm{Ext}^1_{\mathrm{VMHS}(U\cap S)^{\mathrm{ad}}_U}(\mathbb{Z}, \mathbb{Z}(1)) = \mathrm{ANF}(U \cap S, \mathbb{Z}(1))_U$$

defines a sheaf on $\bar{S}$ which is canonically isomorphic to $j^{\mathrm{mer}}_* \mathcal{O}^\times_S$.

Proof. Both of these statements are local. The first is well known and follows essentially from the canonical isomorphism $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, \mathbb{Z}(1)) = \mathbb{C}^\times$.

In the case that $\bar{S} = \Delta^{a+b}$ and $S = \Delta^{*a} \times \Delta^b$ for non-negative integers a and b, the second statement follows from the local normal form of an admissible normal function. Since the statement is local on $\bar{S}$, this proves that (ii) holds when $Y := \bar{S} \setminus S$ is a normal crossing divisor.

For the general case, by using Hironaka, we can find a proper morphism $p : \bar{S}' \to \bar{S}$ such that p is an isomorphism over S and $D = p^{-1}Y$ is a normal crossing divisor (with $Y = \bar{S} \setminus S$ as above). Write $S' = p^{-1}S$, and $j' : S' \to \bar{S}'$ for the inclusion. Then, if U is open in $\bar{S}$, by M. Saito we have $\mathrm{ANF}(U \cap S, \mathbb{Z}(1))_U = \mathrm{ANF}(p^{-1}U \cap S', \mathbb{Z}(1))_{p^{-1}U}$. Since $j_*^{\mathrm{mer}}\mathcal{O}_S^\times = p_* j_*'^{\mathrm{mer}}\mathcal{O}_{S'}^\times$, this proves that (ii) holds in general. □

Corollary 177. *The sheaf $\mathcal{B}$ (resp. $\mathcal{B}^{\mathrm{ad}}$) is a biextension (resp. pseudo-biextension) of $\mathbf{NF}(S, \mathcal{H}) \times \mathbf{NF}(S, \mathcal{H}^\vee)$ by $\mathcal{O}_S^\times$ (resp. of $\mathbf{ANF}(S, \mathcal{H})_{\bar{S}} \times \mathbf{ANF}(S, \mathcal{H}^\vee)_{\bar{S}}$ by $j_*^{\mathrm{mer}}\mathcal{O}_S^\times$).*

Proof. The fact that $\mathcal{B}$ and $\mathcal{B}^{\mathrm{ad}}$ are pseudo-biextensions follows directly from Lemma 176 and Theorem 172. The fact that $\mathcal{B}$ is a biextension follows from Theorem 81 (applied to the case where $D = \emptyset$). □

Remark 178. If $\bar{S} \setminus S$ is a normal crossing divisor, then Theorem 81 implies that $\mathcal{B}^{\mathrm{ad}}$ in Corollary 177 is a biextension. We will prove a stronger result in Theorem 233.

Corollary 179. *The sheaf $\mathcal{B}(\nu, \omega)$ (resp. $\mathcal{B}^{\mathrm{ad}}(\nu, \omega)$) is a torsor (resp. pseudo-torsor) for $\mathcal{O}_S^\times$ (resp. $j_*^{\mathrm{mer}}\mathcal{O}_S^\times$).*

12. The torsion pairing

Suppose $\mathcal{L}$ is a torsion-free local system of $\mathbb{Z}$ modules on Δ^*. Write $\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}}$ for the torsion elements of $\mathrm{H}^1(\mathcal{L}) := \mathrm{H}^1(\Delta^*, \mathcal{L})$. Note that $\mathrm{H}^1(\mathcal{L}^*)$ is canonically isomorphic to $\mathrm{Ext}^1_{\Delta^*}(\mathcal{L}, \mathbb{Z})$ via the map taking an extension to its dual. Let M denote the category of mixed extensions of $\mathbb{Z}$ by $\mathcal{L}$ by $\mathbb{Z}$ (in the category of sheaves of abelian groups over Δ^*). We call a mixed extension X in M *restricted* if $X/W_{-2}X \in \mathrm{H}^1(\mathcal{L})_{\mathrm{tors}}$ and $W_{-1}X \in \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}$. Write $\mathcal{R}$ for the set of isomorphism classes of such mixed extensions. It comes equipped with an obvious map $\mathcal{R} \to \mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \times \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}$.

Proposition 180. *We have an action of $\mathbb{Z} = \mathrm{Ext}^1_{\Delta^*}(\mathbb{Z}, \mathbb{Z})$ on $\mathcal{R}$ along with operations $+_1$ and $+_2$ on R. These make $\mathcal{R}$ into a biextension of $\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \times \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}$ by $\mathbb{Z}$.*

Proof. Suppose E_0 is an extension of $\mathbb{Z}$ by $\mathcal{L}$ and E_1 is an extension of $\mathcal{L}$ by $\mathbb{Z}$. If E_0 and E_1 have torsion cohomology classes, then, for $(f_0, f_1) \in \mathrm{Hom}(\mathbb{Z}, \mathcal{L}) \times \mathrm{Hom}(\mathcal{L}, \mathbb{Z})$, the class $G(f_0, f_1) \in \mathrm{Ext}^1_{\Delta^*}(\mathbb{Z}, \mathbb{Z}) = \mathrm{H}^1(\Delta^*, \mathbb{Z})$ (given by (167)) is trivial. It follows by Proposition 168 that $\mathrm{Extpan}(E_0, E_1)$ injects into $\mathrm{Extpan}(\mathbb{Z}, \mathcal{L}, \mathbb{Z})$ (where the Extpan sets are taken with respect to the category of sheaves of abelian groups on Δ^*). The set $\mathrm{Extpan}(E_0, E_1)$ is an $\mathrm{Ext}^1_{\Delta^*}(\mathbb{Z}, \mathbb{Z})$-torsor. In other words, it is a $\mathbb{Z}$-torsor. Consequently, $\mathcal{R}$ has an action of $\mathbb{Z}$ making it into a $\mathbb{Z}$-torsor over $\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \times \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}$.

The operations $+_1$ and $+_2$ are defined as in §10.5. The rest of the verification is left to the reader. □

By the results in [41] summarized in Remark 171, biextensions of $\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}$ by $\mathbb{Z}$ are classified by the group $\mathrm{Ext}^1(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Z})$. This group sits in an exact sequence

$$\begin{aligned} &\mathrm{Hom}(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Q}) \to \mathrm{Hom}(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z}) \\ &\to \mathrm{Ext}^1(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Z}) \to \mathrm{Ext}(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Q}). \end{aligned}$$

Since the first and last groups are 0, we have an isomorphism between the second and third groups. Thus, the biextension gives rise to a bilinear pairing

$$\tau : \mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}} \to \mathbb{Q}/\mathbb{Z}$$

which we call the *Grothendieck torsion pairing* or just the *torsion pairing*.

We want to compute the torsion pairing in an explicit way. To do this, write $\mathcal{R}_{\mathbb{Q}}$ for the set of isomorphism classes of mixed extensions X of $\mathbb{Q}$ by $\mathcal{L}_{\mathbb{Q}}$ by $\mathbb{Q}$ which are restricted in the sense that X/W_{-2} and $W_{-1}X$ are both 0 (in $\mathrm{H}^1(\mathcal{L}_{\mathbb{Q}})$ and $\mathrm{H}^1(\mathcal{L}^*_{\mathbb{Q}})$ respectively). The same argument as above shows that $\mathcal{R}_{\mathbb{Q}}$ has the structure of a biextension, but this time it is a biextension over the trivial group 0. Therefore, there is a canonical isomorphism

$$\mathcal{R}_{\mathbb{Q}} = \mathrm{Ext}^1_{\Delta^*}(\mathbb{Q},\mathbb{Q}) = \mathbb{Q}. \tag{181}$$

Moreover, tensoring with $\mathbb{Q}$ gives a morphism of biextensions $X \rightsquigarrow X_{\mathbb{Q}}$ from $\mathcal{R}$ to $\mathcal{R}_{\mathbb{Q}}$. (See [41, p. 162] for the notion of a morphism of biextensions.) So, for $X \in \mathcal{R}$, we get a rational number $\tilde{\tau}X$ given by the image of $X_{\mathbb{Q}}$ under (181).

Definition 182. Write $\mathcal{R}'$ for the set of triples $(\alpha,\beta,\gamma) \in \mathrm{H}^1(\mathcal{L}_{\mathrm{tors}}) \times \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}} \times \mathbb{Q}$ such that there exists $X \in R$ with $\pi X = (\alpha,\beta)$, $\tilde{\tau}X = \gamma$.

Since $X \to X_{\mathbb{Q}}$ is a morphism of biextensions from $\mathcal{R}$ to $\mathcal{R}_{\mathbb{Q}}$, we have, for $i = 1,2$,

$$\tilde{\tau}(X +_i X') = \tilde{\tau}(X) + \tilde{\tau}(X')$$

whenever $+_i$ is defined. Moreover, for $n \in \mathbb{Z}$, $\tilde{\tau}(n + X) = n + \tilde{\tau}X$.

From this, it is not hard to see that the group $\mathbb{Z}$ acts on $\mathcal{R}'$ by the rule $n + (\alpha,\beta,\gamma) = (\alpha,\beta,n+\gamma)$. Moreover, by setting

$$(\alpha,\beta,\gamma) +_1 (\alpha',\beta,\gamma') = (\alpha+\alpha',\beta,\gamma+\gamma')$$
$$(\alpha,\beta,\gamma) +_2 (\alpha,\beta',\gamma') = (\alpha,\beta+\beta',\gamma+\gamma')$$

we get the structure of a biextension on $\mathcal{R}'$. In fact, we get an isomorphism of biextensions

$$\phi : \mathcal{R} \to \mathcal{R}' \tag{183}$$

given by $X \mapsto (X/W_{-2}X, W_{-1}X, \tilde{\tau}X)$. All this leads to the following proposition.

Proposition 184. *Suppose* $(\alpha,\beta) = \pi X$ *for* $X \in \mathcal{R}$. *Then* $\tau(\alpha \otimes \beta) = \tilde{\tau}X \pmod{\mathbb{Z}}$.

Proof. Write t for the reduction of $\tilde{\tau}$ modulo $\mathbb{Z}$. Then t gives a map

$$t : \mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}} \to \mathbb{Q}/\mathbb{Z}, \tag{185}$$

and what we have to prove is that $\tau = t$. Using t and the isomorphism

$$\mathrm{Hom}(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^1(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Z}), \tag{186}$$

we see that t corresponds to the extension of $\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}$ by $\mathbb{Z}$ whose fiber over $\alpha \otimes \beta$ is the set of triples (α,β,γ) with $\gamma \in t(\alpha\otimes\beta)$ (where we view $t(\alpha\otimes\beta)$ as a coset of $\mathbb{Z}$ in $\mathbb{Q}$). But this is exactly the fiber of $\mathcal{R}'$ over $\alpha \times \beta$. And this shows that the $\mathcal{R}'$ is the image of t under the map

$$\mathrm{Ext}^1(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \otimes \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Z}) \to \mathrm{Biext}(\mathrm{H}^1(\mathcal{L})_{\mathrm{tors}} \times \mathrm{H}^1(\mathcal{L}^*)_{\mathrm{tors}}, \mathbb{Z}). \tag{187}$$

Since $\mathcal{R}$ is isomorphic to $\mathcal{R}'$, this shows that $\tau = t$. ☐

To write the torsion pairing explicitly, let L denote the fiber of $\mathcal{L}$ at a chosen point $s_0 \in \Delta^*$ and write $T \in \mathrm{Aut}\, L$ for the monodromy operator. Then $\mathrm{H}^1(\mathcal{L})$ is computed by the Koszul complex $K_{\mathbb{Z}}(L)$ given by

$$L \stackrel{T-1}{\to} L$$

in degrees 0 and 1. So that $H^0\mathcal{L} = L^T$ and $H^1\mathcal{L} = L/(T-1)L$. We can make the computation of $H^1\mathcal{L}$ explicit if we view and element of $H^1\mathcal{L}$ as an extension

$$0 \to L \to E \stackrel{p}{\to} \mathbb{Z} \to 0 \tag{188}$$

in the category of abelian groups equipped with a $\mathbb{Z}$-action (where the action on $\mathbb{Z}$ on itself is taken to be trivial).

If we pick an element $e \in E$ such that $p(e) = 1$, then $(T-1)e \in L$. We can change e to $e' := e + l$ for some $l \in L$. Then $(T-1)e' = (T-1)e + (T-1)l$. Therefore the class $[(T-1)e] \in L/(T-1)L$ is well defined. And it is easy to see that this identifies $H^1(\Delta^*, \mathcal{L})$ with $L/(T-1)L$.

Corollary 189. *Under the above identification, we have*

$$H^1(\mathcal{L})_{\text{tors}} = \frac{L \cap (T-1)L_{\mathbb{Q}}}{(T-1)L}.$$

Proof. Obvious. □

Suppose $X \in \mathcal{R}$ with $\pi(X) = (\alpha, \beta) \in \mathrm{H}^1(\mathcal{L})_{\text{tors}} \times \mathrm{H}^1(\mathcal{L}^*)_{\text{tors}}$. Write V for the fiber of X at s_0, and write $\tilde{T} \in \operatorname{Aut} V$ for the monodromy action. This action preserves the filtration W on V.

Proposition 190. *Suppose $X \in \mathcal{R}$ has monodromy matrix $\tilde{T}$. There exists an $e_0 \in V_{\mathbb{Q}}$ with the following properties*

(i) *the projection of e_0 to $\mathrm{Gr}^W V_{\mathbb{Q}}$ is equal to 1 under the identification $p_0 : \mathrm{Gr}^W_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}$;*
(ii) $(\tilde{T}-1)e_0 \in W_{-2}V_{\mathbb{Q}}$.

For any such element e_0, we have $\tilde{\tau}(X) = p_{-2}((\tilde{T}-1)(e_0))$.

Proof. When tensored with $\mathbb{Q}$, the extensions $X/W_{-2}X$ and $W_{-1}X$ in $\mathrm{H}^1(\mathcal{L})$ respectively $\mathrm{H}^1(\mathcal{L}^*)$ become trivial. The existence of $e_0 \in V_{\mathbb{Q}}$ follows from the triviality of $X_{\mathbb{Q}}/W_{-2}X_{\mathbb{Q}}$. Then the equality $\tilde{\tau}X = p_{-2}((\tilde{T}-1)(e_0))$ follows from the definition of $\tilde{\tau}$ as the image of $X_{\mathbb{Q}}$ in $\operatorname{Ext}^1_{\Delta^*}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$. □

Proposition 191. *Suppose $\alpha \in L \cap (T-1)L_{\mathbb{Q}}$ represents a class in $H^1(\mathcal{L})_{\text{tors}}$ and $\beta \in L^*$ represents a class in $H^1(\mathcal{L}^*)_{\text{tors}}$. Pick $\ell \in L_{\mathbb{Q}}$ such that $\alpha = (T-1)l$. Then*

$$\tau([\alpha], [\beta]) = -(\ell, \beta) \mod \mathbb{Z}.$$

Proof. We can represent a mixed extension $X \in p^{-1}(\alpha, \beta)$ by giving the monodromy with respect to a basis consisting of

(i) an element $e_0 \in X$ projecting to 1 under the isomorphism $\mathrm{Gr}^W_0 X = \mathbb{Z}$,
(ii) elements in $W_{-1}X$ lifting a basis of L under the map $\mathrm{Gr}^W_{-1} X = L$,
(iii) the generator e_{-2} of $W_{-2}X = \mathbb{Z}$.

In matrix form, we then have

$$\tilde{T} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & T & 0 \\ m & \beta & 1 \end{pmatrix}$$

for some $m \in \mathbb{Z}$. Over $\mathbb{Q}$ we can change e_0 to $e_0' = e_0 - \ell$. Then $\tilde{T}e_0' = e_0 + \alpha + me_{-2} - T\ell - (\ell, \beta)e_{-2} = e_0' + \alpha + me_{-2} - (T-1)\ell - (\ell, \beta)e_{-2} = e_0' + (m - (\ell, \beta))e_{-2}$. So if we change the basis by changing e_0 to e_0' the matrix for $\tilde{T}$ becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & T & 0 \\ m - (\ell, \beta) & \beta & 1 \end{pmatrix}.$$

So $\tilde{\tau}(X) = m - (\ell, \beta)$, and $\tau(X) = -(\ell, \beta)$ as desired. □

Suppose now that $\mathcal{H}$ is a torsion free weight -1 variation of Hodge structure on Δ^*. Fix admissible normal functions $\nu \in \mathrm{ANF}(\Delta^*, \mathcal{H})_\Delta$ and $\omega \in \mathrm{ANF}(\Delta^*, \mathcal{H}^\vee)_\Delta$. A biextension variation of Hodge structure $\mathcal{V} \in \mathcal{B}^{\mathrm{ad}}(\nu, \omega)$ then gives a mixed extension in the category of admissible variations. The underlying local system $\mathcal{V}_\mathbb{Z}$ is then a restricted mixed extension in $\mathcal{R}$.

Corollary 192. *Suppose $\mathcal{H}$ has unipotent monodromy. Then $\mu(\mathcal{V}) = \tilde{\tau}\mathcal{V}_\mathbb{Z}$.*

Proof. This follows from Proposition 190. □

13. Meromorphic Extensions

The goal of this section is to show that the biextension line bundle $\mathcal{L}(\nu, \mu)$ on S has a canonical extension to a meromorphic line bundle on $\bar{S}$. Essentially this is a consequence of Corollary 179 above, which shows that the sheaf of biextensions is a $j_*^{\mathrm{mer}}\mathcal{O}_S^\times$-torsor. (We remind the reader that the main work going into that Corollary was done in §4.) What remains to do is to recall the definition of a meromorphic extension, which we take from Deligne's book on differential equations [13, p. 65], and to show that $j_*^{\mathrm{mer}}\mathcal{O}_S^\times$-torsors are in one-one correspondence with meromorphic line bundles. Deligne considers meromorphic extensions of coherent sheaves, and we have followed this but, in order to understand the category of meromorphic sheaves better, we have rephrased his definition in the language of stacks. (This gives us a category of meromorphic sheaves, and we use the category in the case of meromorphic line bundles to recover the $j_*^{\mathrm{mer}}\mathcal{O}_S^\times$-torsor.) We also give a bit of background on the notion of meromorphic extensions and the (very significant) differences between the analytic and algebraic cases.

In the next section, §14, we prove that $\mathcal{L}(\nu, \mu)$ extends as a holomorphic line bundle. (In Deligne's language, we prove that the meromorphic extension is effective.) We felt that this section should go before §14 because the meromorphic extension is unique, while the holomorphic extension depends on some choices. However, §14 does not logically depend on this section. So, the reader may want to skip this section starting from subsection 13.3 (where we begin the study of meromorphic sheaves) at first reading.

13.1. **Notation.** Following [13, p. 61], we take X to be an analytic space with Y a closed analytic subset and $X^* = X \setminus Y$. We write $j : X^* \to X$ and $i : Y \to X$ for the inclusions.

13.2. **Extensions of line bundles.** Before bringing up the subject of extensions of coherent sheaves, we want to consider extensions of holomorphic line bundles from X^* to X to illustrate some of the differences between the analytic and algebraic settings.

Definition 193. Suppose $\mathcal{L}^*$ is a holomorphic line bundle on X^*, an *extension* of $\mathcal{L}^*$ to a line bundle on X is a pair $(\mathcal{L}, r)$ where $\mathcal{L}$ is a line bundle on X and $r : \mathcal{L}_{|X^*} \to \mathcal{L}^*$ is an isomorphism of line bundles on X^*. If $(\mathcal{L}_i, r_i)$ $(i = 1, 2)$ are two extensions of $\mathcal{L}^*$, then a *morphism* from $(\mathcal{L}_1, r_1)$ to $(\mathcal{L}_2, r_2)$ is a morphism of line bundles $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ commuting with the isomorphisms r_i. In this way, we get a category $\mathcal{P}(\mathcal{L}^*)$ of extensions of $\mathcal{L}^*$ to X.

Lemma 194. *Suppose X is complex manifold and Y has codimension at least 2 in X. Let $\mathcal{L}^*$ be a line bundle on X^* which extends to a line bundle X. Then this extension is unique up to isomorphism.*

Proof. Suppose $(\mathcal{L}_i, r_i)$ are two extensions. Then $r_2^{-1} \circ r_1 : j^*\mathcal{L}_1 \to j^*\mathcal{L}_2$ is an isomorphism of line bundles. By Hartog's theorem, it extends to an isomorphism $R : \mathcal{L}_1 \to \mathcal{L}_2$ commuting with the restrictions r_i. □

There are two main problematic differences between extensions of line bundles in the holomorphic and the algebraic case. The first is that not all holomorphic line bundles on X^* extend to X even when X is a manifold. For example, as we pointed out in Remark 8, there are infinitely many analytic line bundles on $\mathbb{C}^2 \setminus \{0\}$ which do not extend to $\mathbb{C}^2$. In fact, by the exponential exact sequence, it is easy to see that every line bundle on $\mathbb{C}^2$ is trivial. So the trivial line bundle is the only line bundle on $\mathbb{C}^2 \setminus \{0\}$ which extends to $\mathbb{C}^2$. By Lemma 194, in this case, the extension is unique.

The second difference is that analytic line bundles can have too many extensions. To be precise, they can have uncountably many extensions which are not even meromorphically equivalent. (We show in Proposition 205 that, when X is smooth, meromorphic extensions of line bundles in the algebraic setting are unique.) Before giving Deligne's definition of meromorphic equivalence, we want to illustrate this problem with an example.

Example 195. Let E be an (algebraic) elliptic curve over $\mathbb{C}$ and let p be a point in E. Set $U = E \setminus \{p\}$ considered as an algebraic curve. Then write X, X^* and Y for E, U and $\{p\}$ respectively regarded as analytic varieties.

We have an exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Pic} E \to \operatorname{Pic} U \to 0.$$

where the first non-trivial map sends 1 to the line bundle $\mathcal{O}_E([p])$ and the second sends a line bundle to its restriction to U. From this, it is not hard to see that any algebraic line bundle on U extends to E, and, while the extension is not unique, it is unique modulo tensoring with $\mathcal{O}_E([p])$. (Here we use the obvious algebraic analogue of Definition 193.)

The analytic case is very different. Here, by GAGA [39], we have $\operatorname{Pic} X = \operatorname{Pic} E$. However, from the exponential exact sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_{X^*} \to \mathcal{O}_{X^*}^{\times} \to 1$$

and the fact that X^* is Stein, it follows easily that $\operatorname{Pic} X^* = 0$. So the restriction of any line bundle on X to X^* is trivial. The trivial line bundle on X^* has uncountably many non-isomorphic extensions to X as every line bundle on X gives rise to an extension of $\mathcal{O}_{X^*}$. In fact, the situation is even worse than it seems: every line bundle $\mathcal{L}$ on X gives rise to uncountably many non-isomorphic extensions of $\mathcal{O}_{X^*}$. To understand this phenomenon we make the following definition.

Definition 196. With X, Y and X^* as in (13.1), write P for the set of isomorphism classes of extensions $(\mathcal{L}, r)$ of the trivial line bundle $\mathcal{O}_{X^*}$ to a line bundle on X. We give P an (abelian) group structure by setting $(\mathcal{L}_1, r_1)(\mathcal{L}_2, r_2) = (\mathcal{L}_1 \otimes \mathcal{L}_2, r_1 \otimes r_2)$.

Proposition 197. *Suppose that X is a complex manifold and X^* is a Zariski open subset. Then we have an exact sequence*

$$1 \to \frac{H^0(X^*, \mathcal{O}_{X^*}^{\times})}{H^0(X, \mathcal{O}_X^{\times})} \to P \to \operatorname{Pic} X \to \operatorname{Pic} X^*.$$

Proof. First we describe the maps in the sequence. The last one is restriction. The second-to-last sends an extension $(\mathcal{L}, r)$ to $\mathcal{L}$. The sequence is exact at $\operatorname{Pic} X$ by the definition of P (as the set of isomorphism classes of extensions of the trivial line bundle on X^*). We have a map $\phi : H^0(X, \mathcal{O}_{X^*}^{\times}) \to P$ given by $\phi(f) = (\mathcal{O}_X, f)$ where we think of f as an isomorphism from $\mathcal{O}_{X^*}$ to itself. The kernel of ϕ consists of function f with $(\mathcal{O}_X, f)$ isomorphic to $(\mathcal{O}_X, 1)$. This is exactly $H^0(X, \mathcal{O}_X^{\times})$. Since any element in the kernel of $P \to \operatorname{Pic} X$ can be written as $(\mathcal{O}_X, f)$ for some f as above, this finishes the proof of the proposition. □

Remark 198. In the case of the elliptic curve from Example 195 above, both $\operatorname{Pic} X$ and the group $H^0(X^*, \mathcal{O}_{X^*}^{\times})/H^0(X, \mathcal{O}_X^{\times}) = H^0(X^*, \mathcal{O}_{X^*}^{\times})/\mathbb{C}^{\times}$ are uncountable.

13.3. **Meromorphic Extensions of Coherent Analytic Sheaves.** For any analytic space T we write $\mathrm{Coh}\, T$ for the category of coherent sheaves on T.

Suppose $\mathcal{F}^*$ is a coherent analytic sheaf on X^*. An *extension of* $\mathcal{F}^*$ *to* X is a coherent analytic sheaf $\mathcal{F}$ on X together with an isomorphism $r_{\mathcal{F}} : \mathcal{F}_{|X^*} \to \mathcal{F}^*$. A morphism $\mathcal{F} \to \mathcal{F}'$ of extensions is then a morphism of coherent sheaves on X respecting the isomorphisms to $\mathcal{F}$ over X^*.

We now give a theorem from [13, p. 65]

Theorem 199 (Deligne)**.** *Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are extensions of $\mathcal{F}$. Then the following conditions are equivalent.*

(200) *there is an extension $\mathcal{F}_3$ of $\mathcal{F}$ along with morphisms from $\mathcal{F}_3$ to $\mathcal{F}_1$ and $\mathcal{F}_2$;*

(201) *there is an extension $\mathcal{F}_4$ of $\mathcal{F}$ along with a morphisms from $\mathcal{F}_1$ and $\mathcal{F}_2$ to $\mathcal{F}_4$;*

(202) *either* (200) *or* (201) *hold locally on X.*

Deligne says that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *meromorphically equivalent* if the conditions in Theorem 199 above hold.

Proof. For the convenience of the reader, we give a slightly expanded version of Deligne's proof of the equivalence of (200—202) above. First note that, if $\mathcal{G}$ is any coherent analytic sheaf on X, then the sheaf $\Gamma_Y\mathcal{G}$ of sections with support in Y is coherent [40, Proposition 3, p. 366]. Moreover, we have a short exact sequence

$$0 \to \Gamma_Y\mathcal{G} \to \mathcal{G} \to j_*j^{-1}\mathcal{G}. \tag{203}$$

So, the coherence of $\Gamma_Y\mathcal{G}$ implies the coherence of the image of the map $\mathcal{G} \to j_*j^{-1}\mathcal{G}$.

Now, suppose we have $\mathcal{F}_3$ as in (200). Set $\mathcal{F}_4 = (\mathcal{F}_1 \oplus \mathcal{F}_2)/\mathcal{F}_3$ where the embedding of $\mathcal{F}_3$ is the diagonal embedding. Then $\mathcal{F}_4$ with the obvious morphisms satisfies (201). On the other hand, if we have $\mathcal{F}_4$ as in (201), setting $\mathcal{F}_3 = \mathcal{F}_1 \cap \mathcal{F}_2$ gives an extension satisfying (200).

Finally, suppose (200) holds locally. Set $\mathcal{F}_4$ equal to the sum of the images of $\mathcal{F}_1$ and $\mathcal{F}_2$ in $j_*\mathcal{F}$. The morphisms $r_i^{-1} : \mathcal{F} \to j^*\mathcal{F}_i$ for $i = 1, 2$ induce morphisms $a_i : \mathcal{F} \to j^*\mathcal{F}_4$. We need to show that $\mathcal{F}_4$ is coherent and that the a_i ($i = 1, 2$) are two idenitical isomorphisms.

Fortunately, both statements above are local on X. So we can assume that (200) holds globally. Then set $\mathcal{F}_3'$ equal to the image of $\mathcal{F}_3$ in $\mathcal{F}_1 \oplus \mathcal{F}_2$. We get an exact sequence

$$0 \to \Gamma_Y(\frac{\mathcal{F}_1 \oplus \mathcal{F}_2}{\mathcal{F}_3'}) \to \frac{\mathcal{F}_1 \oplus \mathcal{F}_2}{\mathcal{F}_3'} \to \mathcal{F}_4 \to 0. \tag{204}$$

This shows that $\mathcal{F}_4$ is coherent. And applying j^* proves the rest. □

We say that two holomorphic line bundles $(\mathcal{L}_i, r_i)$ are meromorphically equivalent if they are meromorphically equivalent as coherent analytic sheaves.

We can also make the same definition of meromorphic equivalence in the algebraic setting replacing $X^*, X, \mathcal{F}^*$ and $\mathcal{F}$ with algebraic spaces and coherent algebraic sheaves. Then we have the following proposition.

Proposition 205. *Suppose X is a smooth complex algebraic variety and X^* is a Zariski open subset. Any two algebraic extensions $(\mathcal{L}_1, r_1)$ and $(\mathcal{L}_2, r_2)$ of an algebraic line bundle $\mathcal{L}^*$ on X^* are meromorphically equivalent.*

Proof. As in the analytic case, by (202), the question is local on X. So pick a point $x \in X$. We can find an affine open neighborhood of x where $\mathcal{L}_1$ and $\mathcal{L}_2$ are trivial, i.e., isomorphic to $\mathcal{O}_X$. Replacing X with this affine open neighborhood, we can regard $f := r_2^{-1} \circ r_1 : \mathcal{O}_{X^*} \to \mathcal{O}_{X^*}$ as a meromorphic function on X with poles and zeros only on $Y = X \setminus X^*$. Since X is smooth, $\mathcal{O}_{X,x}$ is a UFD. So, after possibly shrinking X further about x, we can assume that $f = g/h$ with g and

h non-zero regular functions which are non-vanishing off of Y. Now, set $(\mathcal{L}_3, r_3) := (\mathcal{O}_X, r_1/h)$. Multiplication by h (resp. g) gives an inclusion of $\mathcal{L}_3$ in $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) compatible with the isomorphisms r_i. So, by (200), $(\mathcal{L}_1, r_1)$ and $(\mathcal{L}_2, r_2)$ are meromorphically equivalent. □

Remark 206. In the analytic case, there are, in general, uncountably many non-meromophically equivalent extensions of a given holomorphic line bundle. This is not hard to see directly from Example 195.

Lemma 207. *Suppose $\mathcal{F}$ is an extension to X of a coherent analytic sheaf $\mathcal{F}^*$ on X^*. Set $\mathcal{G} := \mathcal{F}/\Gamma_Y\mathcal{F}$. Then the composition*

$$\mathcal{F}^* \stackrel{r_{\mathcal{F}}}{\to} j^*\mathcal{F} \to j^*\mathcal{G} \tag{208}$$

gives $\mathcal{G}$ the structure of an extension of $\mathcal{F}^*$. Moreover, $\Gamma_Y\mathcal{G} = 0$, and $\mathcal{G}$ is meromorphically equivalent to $\mathcal{F}$.

Proof. Obvious. □

In [13, p. 65], Deligne gives the following definition.

Definition 209. A coherent analytic sheaf on X^* meromorphic along Y is a coherent analytic sheaf on X^* together with a locally defined system of equivalence classes of extensions to X.

We want to reformulate this definition in a way that lends itself to defining a category of meromorphic extensions. For this, suppose U is an open subset of X. Set $U^* = U \cap X^*$, $Y_U = U \cap Y$ and write $j_U : U^* \to U$ for the inclusion. Write $\operatorname{Coh}_{(Y)} U$ for the full subcategory of $\operatorname{Coh} U$ consisting of sheaves supported on Y. This is a thick subcategory. Set $\operatorname{Coh}^{\mathit{effm}} U := \operatorname{Coh} U / \operatorname{Coh}_{(Y)} U$, the quotient category. We call this the category of *effective meromorphic extensions on U*. Since $j^* \operatorname{Coh}_{(Y)} U = 0$, the restriction functor $\operatorname{Coh} U \to \operatorname{Coh} U^*$ factors as $\operatorname{Coh} U \stackrel{q}{\to} \operatorname{Coh}^{\mathit{effm}} U \to \operatorname{Coh} U^*$ naturally (where q is the quotient functor).

Proposition 210. *Suppose $V \subset U$ is the inclusion of an open set. Then the restriction functor* $\operatorname{Coh} U \to \operatorname{Coh} V$ *induces an exact functor* $\operatorname{Coh}^{\mathit{effm}} U \to \operatorname{Coh}^{\mathit{effm}} V$.

Proof. Easy (so we leave it to the reader). □

It follows that we get a category $\operatorname{Coh}^{\mathit{effm}}_X$, or simply $\operatorname{Coh}^{\mathit{effm}}$, fibered over the category X^{top} of open subsets of X (whose fiber over U is $\operatorname{Coh}^{\mathit{effm}} U$). (See [16] for the notion of fibered categories.)

Lemma 211. *Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are two extensions to X of a coherent analytic sheaf $\mathcal{F}^*$ on X^*. Then the following are equivalent*

(a) *$\mathcal{F}_1$ and $\mathcal{F}_2$ are meromorphically equivalent.*
(b) *There is an isomorphism $q(\mathcal{F}_1) \to q(\mathcal{F}_2)$ commuting with the isomorphisms $j^*\mathcal{F}_i \to \mathcal{F}^*$.*

Proof. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are meromorphically equivalent. Let $\mathcal{F}_3$ be as in (200). Then $\mathcal{F}_i/\mathcal{F}_3$ is supported on Y for $i = 1, 2$. So the induced map $q(\mathcal{F}_3) \to q(\mathcal{F}_i)$ is an isomorphism for $i = 1, 2$. Then (b) follows.

Now assume that (b) holds. If we set $\mathcal{G}_i := \mathcal{F}_i/\Gamma_Y\mathcal{F}_i$, for $i = 1, 2$, then the quotient map $\mathcal{F}_i \to \mathcal{G}_i$ induces both an equivalence of meromorphic extensions and an isomorphism in $\operatorname{Coh}^{\mathit{effm}} X$. So, by replacing $\mathcal{F}_i$ with $\mathcal{G}_i$, we see that we can assume that neither $\mathcal{F}_1$ nor$\mathcal{F}_2$ has a non-trivial subsheaf supported on Y.

Assume then that $f : q\mathcal{F}_1 \to q\mathcal{F}_2$ is an isomorphism commuting as in (b). By the definition of a quotient category, we have

$$\operatorname{Hom}_{\operatorname{Coh}^{\mathit{effm}} X}(q\mathcal{F}_1, q\mathcal{F}_2) = \varinjlim_{\mathcal{F}_1', \mathcal{F}_3'} \operatorname{Hom}(\mathcal{F}_3, \mathcal{F}_2/\mathcal{F}_4)$$

where the limit runs over all pairs $\mathcal{F}_3, \mathcal{F}_4$ of coherent subsheaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively such that $\mathcal{F}_1/\mathcal{F}_3$ and $\mathcal{F}_4$ are supported on Y. Since $\mathcal{F}_2$ has no non-trivial subsheaf supported on Y, f is represented by a morphism $\tilde{f} : \mathcal{F}_3 \to \mathcal{F}_2$. Since $f = q(\tilde{f})$ is an isomorphism, $\ker \tilde{f}$ and coker $\tilde{f}$ are both supported on Y. Since $\mathcal{F}_1$ has no non-trivial subsheaf supported on Y, this implies that $\tilde{f} : \mathcal{F}_3 \to \mathcal{F}_2$ is a mono-morphism with cokernel supported on Y. From this (a) follows directly. □

Proposition 212. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are coherent analytic sheaves on X. Then the natural morphism*

$$\rho : \operatorname{Hom}_{\operatorname{Coh}^{effm} X}(q\mathcal{F}, q\mathcal{G}) \to \operatorname{Hom}_{\operatorname{Coh} X^*}(j^*\mathcal{F}, j^*\mathcal{G})$$

induced by restriction is a monomorphism.

Proof. We can assume that $\Gamma_Y\mathcal{F} = \Gamma_Y\mathcal{G} = 0$. Then $\operatorname{Hom}_{\operatorname{Coh}^{effm} X}(q\mathcal{F}, q\mathcal{G}) = \varinjlim_{\mathcal{F}'} \operatorname{Hom}_{\operatorname{Coh} X}(\mathcal{F}', \mathcal{G})$ where the limit is taken over all subsheaves $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}/\mathcal{F}'$ is supported on Y. Suppose f is a homomorphism from $q\mathcal{F}$ to $q\mathcal{G}$ represented by a morphism $f' : \mathcal{F}' \to \mathcal{G}$. If $j^*(f') = 0$, then $f'(\mathcal{F}')$ is supported on Y. But, since $\Gamma_Y\mathcal{G} = 0$, this implies that $f' = 0$. □

Definition 213. We write Coh_X^m for the stackification of the fibered category $\operatorname{Coh}_X^{effm}$ [16, p. 76]. This is the category of *coherent analytic sheaves on X meromorphic along Y.*

If Coh_X denotes the stack of coherent analytic sheaves on X^{top}, then we get a sequence of morphisms of fibered categories

$$\operatorname{Coh}_X \to \operatorname{Coh}_X^{effm} \to \operatorname{Coh}_X^m . \tag{214}$$

We call the composition Q.

For any open subset $U \subset X$, we get a restriction functor $\operatorname{Coh}_X^{effm}(U) \to \operatorname{Coh}_{X^*}(U^*)$ (with $U^* = U \cap X$). It is easy to see that this functor factors naturally through $\operatorname{Coh}_X^m(U^*)$. By abuse of notation, we write

$$j^* : \operatorname{Coh}_X^m(U) \to \operatorname{Coh}_X(U^*) \tag{215}$$

for the the induced restriction functor.

Proposition 216. *A coherent analytic sheaf on X^* meromorphic along Y is the same thing as an object of $\operatorname{Coh}_X^m(X)$.*

Proof. By the definition of stackification, an object $\mathcal{F}$ of $\operatorname{Coh}_X^m(X)$ consists of a family of objects $\mathcal{F}_\alpha$ in $\operatorname{Coh}_X^{effm}(U_\alpha)$ for an open cover $\{U_\alpha\}$ of X together with descent data for gluing the data together. Explicitly, this descent data consists of isomorphisms

$$\phi_{\alpha,\beta} : \mathcal{F}_{\alpha|U_\alpha \cap U_\beta} \to \mathcal{F}_{\beta|U_\alpha \cap U_\beta} \tag{217}$$

in the category $\operatorname{Coh}_X^{effm}(U_\alpha \cap U_\beta)$ which are compatible in the sense that $\phi_{\gamma,\beta}\phi_{\alpha,\beta} = \phi_{\gamma,\alpha}$.

Since the restriction of the descent data to X^* gives descent data for a coherent sheaf on X^*, the object $\mathcal{F}$ gives rise to a sheaf $\mathcal{F}^*$ on X^*. Moreover, it is not difficult to see that this sheaf $\mathcal{F}^*$ is independent of the choice of presentation of $\mathcal{F}$ in terms of descent data. This gives the functor $\operatorname{Coh}_X^m(X) \to \operatorname{Coh} X^*$ explicitly. The sheaves $\mathcal{F}_\alpha$ are then by definition meromorphic extensions of $\mathcal{F}_{U_\alpha^*}$. So, from this, we see that an object $\mathcal{F}$ in Coh_X^m gives rise to a coherent sheaf $\mathcal{F}^*$ on X^* along with a meromorphic extension of $\mathcal{F}^*$ to X in Deligne's sense.

On the other hand, suppose we start with a coherent sheaf $\mathcal{F}^*$ on X^*, an open covering $\{U_\alpha\}$ of X and a family $(\mathcal{F}_\alpha, r_\alpha)$ of extensions of $\mathcal{F}^*$ from U_α^* to U_α. Assume that $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ have meromorphically equivalent restrictions to $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Then there is a morphism $\phi_{\alpha,\beta}$ in $\operatorname{Coh}^{effm}(U_{\alpha\beta})$ as in (217) commuting with the restrictions r_α and r_β. The fundamental point to make now is that, in fact, by Proposition 212, there is a *unique* such morphism $\phi_{\alpha,\beta}$. This implies that the resulting family $\{\phi_{\alpha,\beta}\}$ automatically satisfies the descent condition required to give an object $\mathcal{F}$ in $\mathrm{Coh}^m_X(X)$.

We leave the rest of the verification (e.g., the fact that $\mathcal{F}$ is independent of the presentation $(\mathcal{F}_\alpha, r_\alpha)$) to the reader. □

Lemma 218. *Suppose X is a complex manifold, and $\mathcal{F} \in \mathrm{Coh}^{\mathit{effm}}_X(X)$ is is a coherent analytic sheaf on X such that $j^*\mathcal{F}$ is an invertible sheaf. Then the double dual $\mathcal{F}^{\vee\vee}$ is an invertible sheaf on X and the canonical morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism in $\mathrm{Coh}^{\mathit{effm}}_X(X)$.*

Proof. The double dual $\mathcal{F}^{\vee\vee}$ is reflexive. (This is proved in the algebraic setting in [24], and the same proof works in the analytic setting.) Since $\mathcal{F}^{\vee\vee}$ is rank 1 and reflexive, it is a line bundle [32, Lemma 1.1.15, p. 154]. The kernel and cokernel of the canonical morphism $\phi : \mathcal{F} \to \mathcal{F}^{\vee\vee}$ is clearly supported on Y. So ϕ is an isomorphism in $\mathrm{Coh}^{\mathit{effm}}_X(X)$. □

Corollary 219. *Suppose X is a complex manifold, and $\hat{\mathcal{F}}$ is an object in Coh^m_X such that $j^*(\hat{\mathcal{F}})$ is a line bundle. Then $\hat{\mathcal{F}}$ is locally isomorphic to a line bundle.*

Proof. Since this is (by definition) a local question, we can assume that the object $\hat{\mathcal{F}}$ is represented by a coherent analytic sheaf $\mathcal{F}$ on X. Then the result follows from Lemma 218. □

Definition 220. We call an object $\hat{\mathcal{F}} \in \mathrm{Coh}^m_X$ which is locally isomorphic to a line bundle a *meromorphic line bundle (along Y)*.

Write $\widehat{\mathrm{Pic}}_Y(X)$ for the set of isomorphism classes of meromorphic line bundles along Y. Obviously restriction gives a map $\widehat{\mathrm{Pic}}_Y(X) \to \mathrm{Pic}(X^*)$, and we get a map $\mathrm{Pic}\, X \to \widehat{\mathrm{Pic}}_Y(X)$ by associating to each line bundle on X its associated meromorphic line bundle. Moreover, the composition $\mathrm{Pic}\, X \to \widehat{\mathrm{Pic}}_Y\, X \to \mathrm{Pic}\, X^*$ is just the usual restriction.

Suppose $\hat{\mathcal{L}}$ and $\hat{\mathcal{M}}$ are meromorphic line bundles. Then, in the language of Deligne, $\hat{\mathcal{L}}$ and $\hat{\mathcal{M}}$ are meromorphic extensions of $\mathcal{L} := j^*\hat{\mathcal{L}}$ and $\mathcal{M} := j^*\hat{\mathcal{M}}$ respectively. We can find a covering $\{U_\alpha\}_{\alpha\in I}$ of X and extensions $\mathcal{L}_\alpha$, $\mathcal{M}_\alpha$ of $\mathcal{L}_{|U^*_\alpha}$ and $\mathcal{M}_{U^*_\alpha}$ respectively representing the meromorphic equivalence classes $\hat{\mathcal{L}}$ and $\hat{\mathcal{M}}$. By Corollary 219, we can, moreover, assume that $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$ are line bundles. It is not hard to check that, for $\alpha, \beta \in I$, $\mathcal{L}_\alpha \otimes \mathcal{M}_\alpha$ is meromorphically equivalent to $\mathcal{L}_\beta \otimes \mathcal{M}_\beta$ as an extensions of $\mathcal{L} \otimes \mathcal{M}$ from $U_\alpha \cap U_\beta \cap X^*$ to $U_\alpha \cap U_\beta$. Thus, $\{\mathcal{L}_\alpha \otimes \mathcal{M}_\alpha\}_{\alpha\in I}$ represents a meromorphic extension of $\mathcal{L} \otimes \mathcal{M}$. It is then not hard to check that the meromorphic equivalence class of this extension is independent of the choice of the covering $\{U_\alpha\}$ and the choices of the $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$ representing $\hat{\mathcal{L}}$ and $\hat{\mathcal{M}}$ respectively. So it makes sense to define the tensor product $\hat{\mathcal{L}} \otimes \hat{\mathcal{M}}$ to be the meromorphic extension of $\mathcal{L} \otimes \mathcal{M}$ represented by the data $\{\mathcal{L}_\alpha \otimes \mathcal{M}_\alpha\}_{\alpha\in I}$. This gives rise to an abelian group structure on $\widehat{\mathrm{Pic}}_Y\, X$ (defined by taking isomorphism classes of tensor product). Moreover, it is easy to see that the maps $\mathrm{Pic}\, X \to \widehat{\mathrm{Pic}}_Y\, X \to \mathrm{Pic}\, X^*$ are group homomorphisms under this tensor product.

Adjunction gives a morphism of sheaves $\mathcal{O}_X \to j_*j^{-1}\mathcal{O}_X$ (which is a monomorphism if X is a complex manifold and Y is nowhere dense). Then $j^{\mathrm{mer}}_*\mathcal{O}_{X^*}$ is the subsheaf of $j_*j^{-1}\mathcal{O}_X$ consisting of sections which can be locally written in the form g/h with $g, h \in \mathcal{O}_X$ and h invertible outside of Y. We then get a morphism of sheaves $\mathcal{O}_X \to j^{\mathrm{mer}}_*\mathcal{O}_{X*}$. For a coherent sheaf $\mathcal{F}$ on X, restriction gives a morphism $R : \mathcal{F} \otimes_{\mathcal{O}_X} j^{\mathrm{mer}}_*\mathcal{O}_{X^*} \to j_*j^{-1}\mathcal{F}$ (which is a monomorphism for $\mathcal{F}$ locally free). Write $\mathcal{F}^{\mathrm{mer}}$ for the image of R.

Note that $\mathcal{O}^{\mathrm{mer}}_X = j^{\mathrm{mer}}_*\mathcal{O}_{X^*}$.

Proposition 221. *Suppose $\mathcal{F}$ is a reflexive sheaf on a smooth complex analytic space X. Then*

$$\mathrm{Hom}_{\mathrm{Coh}^m_X}(Q\mathcal{O}_{\mathcal{X}}, Q\mathcal{F}) = \mathcal{F}^{\mathrm{mer}}.$$

Proof. Restriction gives a morphism of sheaves

$$\rho : \mathrm{Hom}_{\mathrm{Coh}^m_X}(Q\mathcal{O}_X, Q\mathcal{F}) \to j_* j^{-1}\mathcal{F}. \tag{222}$$

Using Proposition 212, we see that this is a monomorphism. To see that it factors through $\mathcal{F}^{\mathrm{mer}}$ is a local question. So suppose $\varphi : Q\mathcal{O}_X \to Q\mathcal{F}$ is a morphism in Coh^m_X defined near a point $x \in X$. We then have a coherent analytic subsheaf $\iota : \mathcal{I} \to \mathcal{O}_X$ and a morphism $\psi : \mathcal{I} \to \mathcal{F}$ such that $\mathcal{O}_X/\mathcal{I}$ is supported on Y and $\varphi = Q(\psi) \circ Q(\iota)^{-1}$. Taking double duals and using the assumption that $\mathcal{F}$ is reflexive, we get a map $\sigma : \mathcal{I}^{\vee\vee} \to \mathcal{F}$. Then $J := \mathcal{I}^{\vee\vee}$ is a rank 1 reflexive sheaf, and, therefore, invertible. Moreover, we have $\mathcal{I} \subset \mathcal{J} \subset \mathcal{O}_X$. So $\mathcal{O}_X/\mathcal{J}$ is also supported on Y. By replacing X with a suitably small open neighborhood of x, we can assume that $\mathcal{J} = h\mathcal{O}_X$ is a principal ideal. Writing $\iota_{\mathcal{J}} : \mathcal{J} \to \mathcal{O}_X$ for the inclusion, it follows that $\varphi = Q(\sigma) \circ Q(\iota_J)^{-1}$. We then get that $\rho(\varphi) = \sigma(h)/h \in \mathcal{F}^{\mathrm{mer}}$.

To show that ρ induces an isomorphism with $\mathcal{F}^{\mathrm{mer}}$, take $s \in \mathcal{F}^{\mathrm{mer}}(U)$ for some open set U. We can work locally near a point $x \in X$, so we can assume that $s = a/h$ where a and h are holomorphic near x and h is a unit off of Y. Set $\mathcal{I} = h\mathcal{O}_X$, and let $\iota : \mathcal{I} \to \mathcal{O}_X$ denote the inclusion. Write $\alpha : \mathcal{I} \to \mathcal{F}$ for the $\mathcal{O}_X$ linear morphism sending h to a. Then we see that $s = \rho(Q(\alpha)) \circ Q(\iota)^{-1})$. □

Definition 223. Suppose $\hat{\mathcal{L}}$ is a meromorphic line bundle on X along Y. We write

$$\hat{\mathcal{L}}^{\mathrm{mer}} := \mathrm{Hom}_{\mathrm{Coh}^m_X}(Q\mathcal{O}_X, \hat{\mathcal{L}}^{\mathrm{mer}}).$$

By Proposition 221, the sheaf $\hat{\mathcal{L}}^{\mathrm{mer}}$, which we call the *sheaf of meromorphic sections* of $\hat{\mathcal{L}}$, is a locally rank one $j^{\mathrm{mer}}_*\mathcal{O}_X$-module.

Suppose X is smooth and $\hat{\mathcal{L}}$ is a meromorphic line bundle along Y. Write $\hat{\mathcal{O}}_X$ for the object $Q(\mathcal{O}_X)$ in Coh^m_X, and write $\mathrm{Isom}_{\mathrm{Coh}^m_X}(\hat{\mathcal{O}}_X, \hat{\mathcal{L}})$ for the subsheaf of $\mathrm{Hom}_{\mathrm{Coh}^m_X}(\hat{\mathcal{O}}_X, \hat{\mathcal{L}})$ consisting of isomorphisms. Then $\mathrm{Isom}_{\mathrm{Coh}^m_X}(\hat{\mathcal{O}}_X, \hat{\mathcal{L}})$ is a torsor for the sheaf $\mathrm{Aut}_{\mathrm{Coh}^m_X}(\hat{\mathcal{O}}_X)$. By Proposition 221, $\mathrm{Aut}_{\mathrm{Coh}^m_X}(\hat{\mathcal{O}}_X)$ is identified with the subsheaf $j^{\mathrm{mer}}_*\mathcal{O}^\times_{X^*}$ consisting of invertible sections of $j^{\mathrm{mer}}_*\mathcal{O}_{X^*}$. So, this gives a map

$$c : \widehat{\mathrm{Pic}}_Y X \to H^1(X, j^{\mathrm{mer}}_*\mathcal{O}^\times_{X^*}). \tag{224}$$

On the other hand, suppose E is a $j^{\mathrm{mer}}_*\mathcal{O}^\times_{X^*}$-torsor. The restriction, E^*, of E to X^* gives a line bundle $\mathcal{L}^*$ on X^* via the Borel construction: Explicitly

$$\mathcal{L}^* := E^* \times^{\mathcal{O}^\times_{X^*}} \mathcal{O}_{X^*}. \tag{225}$$

In other words, for U open in X^*, $\mathcal{L}^*(U)$ is the quotient of $E^* \times \mathcal{O}_X$ by the diagonal action of $\mathcal{O}^\times_X$ (acting by $f(e,g) = (f^{-1}e, fg)$). We get a map of sheaves $E^* \to \mathcal{L}^*$ (given explicitly by $e \mapsto (e,1)$) sending the sections of E isomorphically onto the non-vanishing sections of $\mathcal{L}^*$.

Now, if U is an open set in X and $s \in E(U)$, we get an extension $\mathcal{L}_s$ of $\mathcal{L}^*$ from $U^* = U \cap X^*$ to U essentially by declaring the image of s in $\mathcal{L}^*$ to be a generator of $\mathcal{L}_s$. To be more precise, we set $\mathcal{L}_s = \mathcal{O}_U$ and choose an isomorphism $j^*\mathcal{L}_s \to \mathcal{L}$ by sending 1 to s. If we pick a different $t \in E(U)$, then $\mathcal{L}_t$ and $\mathcal{L}_s$ are meromorphically equivalent because s/t is meromorphic and non-vanishing off of Y. So we can glue together the different choices of $\mathcal{L}_s$ for $s \in E(U)$ (varying s and U), to get a meromorphic line bundle $b(E)$. This gives us a map

$$b : H^1(X, j^{\mathrm{mer}}_*\mathcal{O}^\times_{X^*}) \to \widehat{\mathrm{Pic}}_Y X. \tag{226}$$

Proposition 227. *The maps b and c above are inverse to each other. So, when X is smooth, the sets $\widehat{\mathrm{Pic}}_Y X$ and $H^1(X, j^{\mathrm{mer}}_*\mathcal{O}^\times_X)$ are isomorphic.*

Proof. This is just a matter of wading through the definitions, and we feel it is best to leave it to the reader. □

Suppose now that X is smooth and, for simplicity, assume that Y is nowhere dense in X. A *meromorphic extension* of a line bundle $\mathcal{L}^*$ on X^* is then the same thing as a pair $(\hat{\mathcal{L}}, r)$ with $\hat{\mathcal{L}}$ a meromorphic line bundle and $r : j^*\hat{\mathcal{L}} \to \mathcal{L}^*$ is an isomorphism.

Lemma 228. *Suppose X is smooth and Y is nowhere dense. Let $\mathcal{L}^*$ be a line bundle on X^*. Then there is a one-one correspondence between meromorphic extensions of $\mathcal{L}^*$ and pairs (E, ρ) where E is a $j_*^{\mathrm{mer}}\mathcal{O}_{X^*}^\times$ torsor and $\rho : E^* \to \mathcal{L}^{*\times}$ is an $\mathcal{O}_{X^*}^\times$-equivariant isomorphism from the restriction of E to X^* to the sheaf of non-vanishing sections of $\mathcal{L}^*$.*

Proof. If $(\hat{\mathcal{L}}, r)$ is a meromorphic extension of $\mathcal{L}^*$, we get a torsor $E := \mathrm{Isom}_{\mathrm{Coh}_X^m}(\hat{\mathcal{O}}_X, \hat{\mathcal{L}})$ and the map $r : j^*\hat{\mathcal{L}} \to \mathcal{L}^*$ provides us with the isomorphism from E^* to the non-vanishing sections of $\mathcal{L}^*$. On the other hand, suppose we are given a pair (E, ρ) as above. Then E gives rise to a meromorphic line bundle $\hat{\mathcal{L}} = b(E)$ as in Proposition 226 above. And, the map ρ provides an isomorphism of the holomorphic bundle $E^* \times^{\mathcal{O}_{X^*}^\times} \mathcal{O}_{X^*}$ with $\mathcal{L}^*$. It is easy to check that these two correspondences are mutually inverse. So the lemma follows. □

Theorem 229. *Suppose $\nu, \omega, S, \bar{S}$ and $\mathcal{L} = \mathcal{L}(\nu, \omega)$ are as in Q6. Set $Y = \bar{S} \setminus S$. Then there is a unique (up to isomorphism) meromorphic extension $\hat{\mathcal{L}}$ of $\mathcal{L}$ to $\bar{S}$ whose non-vanishing meromorphic sections are the admissible biextensions.*

Proof. The $j_*^{\mathrm{mer}}\mathcal{O}_S^\times$-torsor $\mathcal{B}^{\mathrm{ad}}(\nu, \omega)$ of Corollary 179 gives us a torsor E and the restriction isomorphism from $\mathcal{B}^{\mathrm{ad}}(\nu, \omega)$ (which is a torsor on $\bar{S}$) to $\mathcal{B}(\nu, \omega)$ (which is the $\mathcal{O}_S^\times$ torsor of invertible section of $\mathcal{L}$) gives us an isomorphism ρ as in Lemma 228. So we see that (E, ρ) gives us a meromorphic extension $\hat{\mathcal{L}}$ of $\mathcal{L}$ to $\bar{S}$.

The extension is clearly unique up to isomorphism because $\mathcal{B}^{\mathrm{ad}}(\nu, \omega)$ determines both the torsor E and the map ρ. □

14. Extension Theorem

In this section, the goal is to show that the meromorphic extension of $\mathcal{L}(\nu, \omega)$ constructed in Theorem 229 actually gives rise to a line bundle $\overline{\mathcal{L}}(\nu, \omega)$ extending $\mathcal{L}(\nu, \omega)$. The extensions $\overline{\mathcal{L}}(\nu, \omega)$ is not unique by the choices involved are easily understood.

14.1. **Setup and torsion pairings of smooth divisors.** We take $\bar{S}$ to be a complex manifold, $j : S \to \bar{S}$ the inclusion of a Zariski open and $\mathcal{H}$ to be a weight -1 variation of pure Hodge structure without torsion on S. We write $Y := \bar{S} \setminus S$, $Y^{(1)} = \{Y_1, \ldots, Y_k\}$ for the set of components of Y which are codimension 1 in $\bar{S}$. We write $Y^{(2)}$ for the union of the codimension 2 components of Y and the singular locus of Y. We write $S' := \bar{S} \setminus Y^{(2)}$, and $Y' = Y \cap S'$. Then Y' is a smooth divisor in S' with components $Y_i' := Y_i \cap S'$.

Definition 230. Suppose $\mathcal{L}$ is a $\mathbb{Z}$ local system on S. Set

$$\mathrm{IH}^1_{\mathbb{Z}}\,\mathcal{L} := \{\alpha \in \mathrm{H}^1(S, \mathcal{L}) : \alpha_{\mathbb{Q}} \in \mathrm{IH}^1(S, \mathcal{L}_{\mathbb{Q}})\}.$$

By Theorem 110, the map $\mathrm{cl} : \mathrm{ANF}(S, \mathcal{H})_{\bar{S}} \to \mathrm{H}^1(S, \mathcal{H})$ factors through $\mathrm{IH}^1_{\mathbb{Z}}\,\mathcal{H}$.

For each $Y_i \in Y^{(1)}$ we are going to define a pairing

$$\tau_i : \mathrm{IH}^1_{\mathbb{Z}}\,\mathcal{H} \otimes \mathrm{IH}^1_{\mathbb{Z}}\,\mathcal{H}^\vee \to \mathbb{Q}/\mathbb{Z} \tag{231}$$

which we will call the i-th torsion pairing.

We define (231) first in the case that $\bar{S} = \Delta$, $S = \Delta^*$ and $Y = \{0\}$. Note that, in this case, $\mathrm{IH}^1_{\mathbb{Z}}(\Delta, \mathcal{H}) = \{\alpha \in \mathrm{H}^1(\Delta^*, \mathcal{H}) : \alpha_{\mathbb{Q}} = 0\}$. In other words, it is just the set of torsion elements of $\mathrm{H}^1(\Delta^*, \mathcal{H})$. So, in this case the pairing in (231) is just given by the torsion pairing τ.

In general we take a test curve $\bar{\varphi} : \Delta \to S'$ intersecting $Y_i \in Y^{(1)}$ transversally at a point $\bar{\varphi}(0)$. By Theorem 113, the pull back of $\mathrm{IH}^1_{\mathbb{Z}}\,\mathcal{H}$ to Δ^* consists of torsion classes. Then we can define τ_i to be the torsion pairing restricted to the test curve. It is easy to see that this definition does not depend on the test curve.

14.2. Extension of the line bundle $\mathcal{L}(\nu, \omega)$.

Lemma 232. *Suppose $j : S \to \bar{S}$ is an in §14.1. Suppose that* $\operatorname{codim}_{\bar{S}} Y \geq 2$. *Let $\pi : \bar{T} \to \bar{S}$ be a proper morphism from a complex manifold, $\bar{T}$, which is an isomorphism over S, and suppose $\mathcal{M}$ is a line bundle on $\bar{T}$. Set $\mathcal{L} := (\pi_*\mathcal{M})^{\vee\vee}$, the reflexive hull of $\pi_*\mathcal{M}$. Then $\mathcal{L}$ is a line bundle on $\bar{S}$ and there is a canonical isomorphism $\mathcal{L}_{|S} \cong \mathcal{M}_{|S}$. (See Remark 234.)*

Proof. Write $i : S \to \bar{T}$ for the open immersion lifting j. Since π is proper, $\pi_*\mathcal{M}$ is coherent. We have $j^*\pi_*\mathcal{M} = i^*\mathcal{M}$ by smooth base change. So $j^*\mathcal{L} = j^*(\pi_*(\mathcal{M})^{\vee\vee}) = (i^*\mathcal{M})^{\vee\vee} = i^*\mathcal{M}$.

Now, $\mathcal{L}$ is reflexive and agrees with a line bundle outside of codimension 2. It is therefore a rank 1 reflexive sheaf. Therefore, since $\bar{S}$ is a complex manifold, $\mathcal{L}$ is a line bundle. (This is in [32].) ☐

Theorem 233. *Suppose S, $\bar{S}$, Y and $\mathcal{H}$ are as in the beginning of §14.1, so that $\mathcal{H}$ is a torsion free variation of Hodge structure on S of weight -1 and $\bar{S}$ is a smooth partial compactification of S. Suppose*

$$(\nu, \omega) \in \mathrm{ANF}(S, \mathcal{H})_{\bar{S}} \times \mathrm{ANF}(S, \mathcal{H}^\vee)_{\bar{S}}.$$

Then each choice of coset representative $\tilde{\tau}_i$ of $\tau_i := \tau_i(\mathrm{cl}\,\nu, \mathrm{cl}\,\omega) \in \mathbb{Q}/\mathbb{Z}$ determines a unique extension $\overline{\mathcal{L}}$ of $\mathcal{L} := \mathcal{L}(\nu, \omega)$ to $\bar{S}$ whose non-vanishing sections are admissible biextension variations $\mathcal{V}$ with the following property: for every test curve $\bar{\varphi} : \Delta \to S'$ interesting Y_i transversally at $\bar{\varphi}(0)$, $\mu(\varphi^*\mathcal{V}) = \tilde{\tau}_i$.

Proof. First assume that Y is a normal crossing divisor. Then, by Corollary 177, the sheaf $\mathcal{B}^{\mathrm{ad}}(\nu, \omega)$ of admissible biextension variations is a pseudo-torsor for the group of non-vanishing meromorphic functions with poles along Y. By Theorem 81, this pseudo-torsor is actually a torsor for the sheaf of non-vanishing meromorphic functions. For every section $\mathcal{V}$ of $\mathcal{B}^{\mathrm{ad}}(\nu, \omega)$ and every test curve $\bar{\varphi} : \Delta \to S'$ intersecting the divisor Y transversally at $y = \bar{\varphi}(0)$ with y a point in Y_i, $\mu(\varphi^*\mathcal{V}) = \tilde{\tau}\varphi^*\mathcal{V}$ is a coset representative of τ_i. Fixing the $\tilde{\tau}_i$ then reduces the torsor to a $\mathcal{O}^\times_{\bar{S}}$ torsor $\overline{\mathcal{B}}^{\mathrm{ad}}(\nu, \omega)$. Equivalently, it gives a line bundle $\overline{\mathcal{L}}$ as desired.

Now, in the general case, set $Y' = Y \cap S'$. Then, as Y' is a normal crossing divisor in S', there is a extension $\mathcal{L}'$ of $\mathcal{L}$ to S'. Now, use Hironaka to find a proper morphism $\pi : \bar{T} \to \bar{S}$ which is an isomorphism over S' such that the inverse image of Y under π is a normal crossing divisor D. Write $D = \cup_{i=1}^m D_i$ in such a way that the D_i ($i = 1, \ldots, k$) are the strict transforms of the Y_i. Pick rational numbers $\tilde{\tau}_i$ lifting the τ_i keeping them the same as for the Y_i for $i = 1, \ldots k$ (and making arbitrary choices for $i = k+1, \ldots, m$). Then we get a unique extension $\overline{\mathcal{M}}$ of $\mathcal{L}'$ to $\bar{T}$. Finally Lemma 232 produces the desired extension on $\bar{S}$.

The uniqueness follows from Hartog's theorem and the property in the statement of Theorem 233, which defines $\overline{\mathcal{L}}$ on S'. The point is that two extensions of $\mathcal{L}$ which agree outside of a codimension 2 set are equal by Hartog's theorem. ☐

Remark 234. We get a canonical choice of extension $\overline{\mathcal{L}}_{\mathrm{can}} \in \operatorname{Pic}\bar{S} \otimes \mathbb{Q}$ defined by taking all the $\tilde{\tau}_i = 0$.

15. The Ceresa cycle and the Hain-Reed bundle

15.1. Fix an integer $g > 1$ and let $\mathcal{T}_g$ denote the Teichmüller space of a smooth, projective genus g Riemann surface X. Write Γ_g for the mapping class group, the space of orientation preserving diffeomorphisms of X taken modulo isotopy. Then, $\mathcal{T}_g$ is a complex manifold, which is isomorphic as a real manifold to $\mathbb{R}^{6g-6}$. Moreover, Γ_g acts on $\mathcal{T}_g$ with orbifold quotient $\mathcal{M}_g$, the moduli stack of smooth, projective genus g curves. Write $H = \mathrm{H}_1(X,\mathbb{Z})$ and write $Q : \wedge^2 H \to \mathbb{Z}$ for the intersection pairing. Fix a basis $e_1, \dots, e_g, f_1, \dots, f_g$ for H with the property that

$$Q(e_i, f_j) = \delta_{ij}, Q(e_i, e_j) = Q(f_i, f_j) = 0$$

for $1 \le i, j \le g$. Write $\mathbf{Sp}_{2g}(\mathbb{Z}) = \mathbf{Sp}(H)$ for the group of automorphisms of H with determinant one preserving Q. The action of an element of Γ_g on H determines a surjection $\Gamma_g \to \mathbf{Sp}_{2g}(\mathbb{Z})$. The kernel T_g of this surjection is called the Torelli group.

The pairing Q induces pairings $Q_k : \wedge^k H \otimes \wedge^k H \to \mathbb{Z}$ for all non-negative integers k. These are $\mathbf{Sp}_{2g}(\mathbb{Z})$-equivariant and $(-1)^k$-symmetric. If we set $\theta := \sum_{i=1}^{g} e_i \wedge f_i \in \wedge^2 H$, then θ is $\mathbf{Sp}_{2g}(\mathbb{Z})$-invariant. It follows that the maps $u : \wedge^k H \to \wedge^{k+2} H$ induced by $v \mapsto v \wedge \theta$ are $\mathbf{Sp}_{2g}(\mathbb{Z})$-equivariant as well.

15.2. **Boundary components of $\overline{\mathcal{M}}_g$.** Suppose $g > 2$ and h is an integer such that $1 \le h \le \lfloor g/2 \rfloor$. Then D_h denotes the Zariski closure of the locus of stable curves consisting of a smooth curve of genus h and another smooth curve of genus $g - h$ meeting at one point. D_0 denotes the Zariski closure of the locus of stable curves consisting of a curve of geometric genus $g - 1$ with one node. Then $D = D_0 \cup \cdots \cup D_{\lfloor g/2 \rfloor}$ is a normal crossing divisor whose support is the complement of $\mathcal{M}_g$ in $\overline{\mathcal{M}}_g$ (the Deligne-Mumford compactification of $\mathcal{M}_g$).

The divisor D_0 intersects itself in components which we will label as $D_{0,h}$ for $0 \le h \le \lfloor g/2 \rfloor$. ($D$ is not a strict normal crossing divisor.) For $h > 0$, the component $D_{0,h}$ is the Zariski closure of the locus of stable curves consisting of two smooth curves of genus h and $g - h - 1$ respectively meeting at two points. The generic point of the component $D_{0,0}$ is a curve of geometric genus $g - 2$ with 2 nodes.

15.3. **Dehn twists and bounding pairs.** If γ is any simple closed curve, we let T_γ denote the Dehn twist of X determined by γ. This is an element of the mapping class group.

A simple closed curve γ in X is said to be *bounding* if $X \setminus \gamma$ is a union of two open Riemann surfaces. A pair (γ, δ) of homologous simple closed curves, which are not homologously trivial, is said to be a *bounding pair* if γ and δ are disjoint, homologous and not homologically trivial. (See Johnson [26].) Contracting the simple closed curves γ and δ in a bounding pair to two distinct points produces a curve C which is a union of two smooth curves of genus $h =: h(\gamma, \delta)$ and $g - h - 1$ respectively for $1 \le h \le g - 1$ meeting at two points. Thus contracting the simple closed curves produces a curve in the interior of $D_{0,h}$. It is possible to pick the symplectic basis from §15.1 in such a way that $\{e_1, \dots, e_h, f_1, \dots, f_h\}$ and $\{e_{h+2}, \dots, e_g, f_{h+2}, \dots, f_g\}$ are symplectic bases for the cohomology of the two components. We say that such a symplectic basis is *adapted to the bounding pair*.

15.4. **Johnson Homomorphism.** The lattice $\wedge^3 H$ decomposes as a sum of two sublattices as follows [See [21] or [26]]: Let $u : H \to \wedge^3 H$ and $c : \wedge^3 H \to H$ be the $\mathbf{Sp}(H)$-equivariant maps given by

$$u(x) = \theta \wedge x, \qquad c(x \wedge y \wedge z) = Q(x,y)z + Q(y,z)x + Q(z,x)y$$

Direct computation shows $c \circ u(x) = (g-1)x$. Define $I : \wedge^3 H \to \wedge^3 H$ by the rule

$$I(\omega) = (g-1)\omega - u \circ c(\omega)$$

Lemma 235. $\ker(I) = \mathrm{im}(u)$. *Moreover, there exists a subgroup L of $\wedge^3 H$ such that $\wedge^3 H = u(H) \oplus L$.*

Proof. The assertion that $\operatorname{im} u \subset \ker I$ follows from the fact that $c \circ u(x) = (g-1)x$. To see that $\ker I \subset \operatorname{im} u$, suppose $\omega \in \ker I$. Then $(g-1)\omega = u \circ c(\omega)$. So $(g-1)\omega \in \operatorname{im} u$. So it suffices to show that $\operatorname{im} u$ has a complement, L, in $\wedge^3 H$.

The required subgroup L is generated by the following elements:

(i) all products of the form $v_i \wedge v_j \wedge v_k$ where each v_l is either e_l of f_l and $i < j < k$,
(ii) all products of the form $v_i \wedge e_j \wedge f_j$ where v_i is either e_i or f_i and $i - j$ is not congruent to 0 or 1 modulo g.

□

Define $V = \wedge^3 H / \operatorname{im}(u)$. By the previous lemma, the quotient map $\wedge^3 H \to V$ restricts to an isomorphism $I(\wedge^3 H) \to V$. Let $j : V \to I(\wedge^3 H)$ denote the inverse isomorphism. We note that, since $V \cong L$, V is torsion free.

Theorem 236 (Johnson)**.** *Suppose $g > 1$. There is a surjective group homomorphism $\tau : T_g \to V$. If (γ, δ) is a bounding pair then there is a symplectic basis adapted to (γ, δ) such that*

$$\tau(T_\gamma T_\delta^{-1}) = [(\sum_{i=1}^{h} e_i \wedge f_i) \wedge f_{h+1}] \in V$$

where $h = h(\gamma, \delta)$.

15.5. **The Variation V.** By abuse of notation, we can view H as a variation of Hodge structure of weight -1 on $\mathcal{M}_g$. We get an exact sequence of variations Hodge structure of weight -1

$$0 \to H \xrightarrow{u} (\wedge^3 H)(-1) \to V \to 0 \tag{237}$$

where $V \cong \operatorname{im}(I)(-1)$ as in the previous section.

Suppose now that $C \in D_{0,h}$ is a curve obtained by contracting a bounding pair (γ, δ) as in §15.3. Since γ and δ are homotopic, T_γ and T_δ act identically on H and, thus, on V. The action of $T = T_\gamma$ on H is given by $h \mapsto h + Q(h, \gamma)\gamma$. So $T = \operatorname{id} + N$ is unipotent with monodromy logarithm N given by $h \mapsto Q(h, \gamma)\gamma$. We have $N^2 = 0$.

We can find a polydisk $P = \Delta^{3g-3}$ and an étale map $j : P \to \overline{\mathcal{M}}_g$ such that $j(0) = C$ and $P' := j^{-1}\mathcal{M}_g \cong \Delta^{*2} \times \Delta^{3g-5}$. If C has no automorphisms, then (by shrinking P if necessary) we can arrange it so that j is an isomorphism onto its image. Then the monodromy action of $\mathbb{Z}^2 = \pi_1(P')$ on the pullback of the universal curve to P' is given $(a, b) \mapsto T_\gamma^a T_\delta^b$. In particular, if we write N_1 and N_2 for the logarithms of the monodromy on H, or, rather, its pullback to P', we see that $N_1 = N_2$. Moreover, if $N = N_1$, then $N^2 = 0$.

Corollary 238. *We have* $\mathrm{IH}^1(P, V) = NV$.

15.6. **Normal functions on $\mathcal{M}_g$.**

Theorem 239. [22] *There is an element $\xi \in H^1(\Gamma_g, V)$, which is the class of a normal function $\nu \in \mathrm{ANF}(\mathcal{M}_g, V)_{\bar{\mathcal{M}}_g}$. The restriction of ξ to T_g under the map*

$$H^1(\Gamma_g, V) \to H^1(T_g, V) = \operatorname{Hom}(T_g, V)$$

is twice the Johnson homomorphism.

Theorem 240. *Suppose $C \in \overline{\mathcal{M}}_g$ is a curve without automorphism obtained by contracting a bounding pair (γ, δ) as in §15.3. Then, in terms of a symplectic basis adapted to the bounding pair, we have*

$$\operatorname{sing}_C \xi = [2(\sum_{i=1}^{h} e_i \wedge f_i) \wedge f_{h+1}] \in NV.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{IH}^1(\mathcal{M}_g, V) & \searrow & \\ \downarrow & & \\ \mathrm{IH}^1(P, V) & & H^1(T_g, V) = \mathrm{Hom}(T_g, V) \\ \downarrow & & \downarrow \\ H^1(\mathbb{Z}^2, V) = H^1(P', V) & \longrightarrow & H^1(\mathbb{Z}, V) = V. \end{array}$$

The map along the bottom row is induced by the map $\mathbb{Z} \to \mathbb{Z}^2 = \pi_1(P')$ sending the generator to $(1, -1)$. The map on the right is induced by the homomorphism $\mathbb{Z} \to T_g$ which sends the generator to $T_\gamma T_\delta^{-1}$.

By Theorem (239), the image of ξ in $\mathrm{Hom}(T_g, V)$ is the homomorphism which takes $T_\gamma T_\delta^{-1}$ to $[2(\sum_{i=1}^h e_i \wedge f_i) \wedge f_{h+1}]$. By restriction, the image of the Johnson homomorphism in $H^1(\mathbb{Z}, V) = V$ is precisely $[2(\sum_{i=1}^h e_i \wedge f_i) \wedge f_{h+1}]$. The result then follows from Lemma 152. □

The variation V has a polarization $q : V \otimes V \to \mathbb{Z}(1)$ defined by

$$q(u, v) := \frac{1}{g-1} Q(j(u), j(v)).$$

(See [21, p. 203].) This polarization gives an isomorphism $a_q : V \to V^\vee$ and thus a normal function $\nu^\vee = a_q(\nu) \in \mathrm{ANF}(\mathcal{M}_g, V^\vee)_{\overline{\mathcal{M}_g}}$. This in turn gives a metrized line bundle $\mathcal{L} := \mathcal{L}(\nu, \nu^\vee)$ on $\mathcal{M}_g$.

Theorem 241. *Suppose C is a generic curve in $D_{0,h}$. Then*

$$h_q(\mathrm{sing}_C \xi, \mathrm{sing}_C \xi) = \frac{4t_1 t_2}{t_1 + t_2}(g - h - 1)h$$

Proof. To simplify the notation below, we let $h' = g - h - 1$, and recall that

$$\mathrm{sing}_C \xi = 2[\sum_{i=1}^h e_i \wedge f_i \wedge f_{h+1}] = 2N[\sum_{i=1}^h e_i \wedge f_i \wedge e_{h+1}]$$

where $N = N_1 = N_2$ is the monodromy logarithm around the two branches of D_0 intersecting at C.

Invoking Proposition (153), it follows that:

$$\begin{aligned} h_q(\mathrm{sing}_C \xi, \mathrm{sing}_C \xi) &= \frac{t_1 t_2}{t_1 + t_2} q(2[\sum_{i=1}^h e_i \wedge f_i \wedge e_{h+1}], 2[\sum_{i=1}^h e_i \wedge f_i \wedge f_{h+1}]) \\ &= \frac{4t_1 t_2}{(g-1)(t_1 + t_2)} Q(j[(\sum_{i=1}^h e_i \wedge f_i) \wedge e_{h+1}], j[(\sum_{i=1}^h e_i \wedge f_i) \wedge f_{h+1}]). \end{aligned}$$

Now, for $v = e_{h+1}$ or f_{h+1}, $c((\sum_{i=1}^{h} e_i \wedge f_i) \wedge v) = h\theta \wedge v$. So using this to compute j, we see that

$$\begin{aligned} h_q(\operatorname{sing}_C \xi, \operatorname{sing}_C \xi) &= \frac{4t_1t_2}{(g-1)(t_1+t_2)} Q\left(h' \sum_{i=1}^{h} e_i \wedge f_i \wedge e_{h+1} - h \sum_{i=h+2}^{g} e_i \wedge f_i \wedge e_{h+1},\right. \\ &\qquad \left. h' \sum_{i=1}^{h} e_i \wedge f_i \wedge f_{h+1} - h \sum_{i=h+2}^{g} e_i \wedge f_i \wedge f_{h+1}\right) \\ &= \frac{4t_1t_2}{(g-1)(t_1+t_2)} [(h')^2 h + h^2(h')] \\ &= \frac{4t_1t_2}{(g-1)t_1+t_2} (h')h(h+h') \\ &= \frac{4t_1t_2}{t_1+t_2} (g-h-1)h \end{aligned}$$

since $h + h' = g - 1$. □

16. General jump pairing

16.1. **General Pairing.** The goal of this section is to give a definition of the jump pairing on local intersection cohomology without the assumption that the boundary divisor is normal crossing.

To this end, we fix our usual notation that $\mathcal{H}$ is a variation of pure Hodge structure with $\mathbb{Q}$ coefficients on a complex manifold S which is a Zariski open subset of another manifold $\bar{S}$. Write $j : S \to \bar{S}$ for the embedding. Since we will only be concerned with the local situation, we will assume that $\bar{S} = \Delta^r$ is a polydisk.

The intermediate extension $j_{!*}\mathcal{H}_{\mathbb{Q}}[r]$ gives rise to a perverse sheaf $\mathrm{IC}(\mathcal{H})$ on $\bar{S}$. Write $\mathcal{P}_{\bar{S}}$ $\mathcal{P}$ for the set of all isomorphism classes mixed extensions of $\mathrm{IC}(\mathbb{Q}_{\bar{S}})$ by $\mathrm{IC}(\mathcal{H})$ by $\mathrm{IC}(\mathbb{Q}_{\bar{S}})$ in the category of perverse sheaves on $\bar{S}$. For a local system of $\mathbb{Q}$-vector spaces $\mathcal{L}$ on S, we can write $\mathrm{IH}^k(\mathcal{L})$ for $\mathrm{IH}^k_0(\mathcal{L})$. By shrinking $\bar{S}$ if necessary, we can assume that $\mathrm{IH}^k_0(\mathcal{L}) = \mathrm{IH}^k(\bar{S}, \mathcal{L})$.

We get a map

$$\pi : \mathcal{P} \to \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}}) \times \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}}^{\vee})$$

by identifying the intersection cohomology groups with the appropriate extension groups. Moreover, since $\mathrm{IH}^1(\bar{S}, \mathbb{Q}) = \mathrm{IH}^2(\bar{S}, \mathbb{Q}) = 0$, Proposition 168 shows that π is surjective and, for any pair

$$(\alpha, \beta) \in \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}}) \times \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}}^{\vee}), \tag{242}$$

the set $\mathrm{Extpan}(\alpha, \beta)$ injects into $\mathcal{P}$. It follows that $\mathcal{P}$ has the structure of a biextension of $\mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}}) \times \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}}^{\vee})$ by $\mathrm{IH}^1(\bar{S}, \mathbb{Q}) = 0$.

Corollary 243. *For each pair (α, β) as in* (242)*, there exists a unique element $X = X(\alpha, \beta)$ of $\mathcal{P}$ with $\pi(X) = (\alpha, \beta)$.*

Proposition 244. *Suppose $(\bar{\varphi}, \varphi) : (\Delta, \Delta^*) \to (\bar{S}, S)$ is a test curve. Then $\varphi^* X(\alpha, \beta)$ is in the biextension $\mathcal{R}_{\mathbb{Q}}$ of* (181)*. In other words, both $\varphi^*\alpha$ and $\varphi^*\beta$ vanish.*

Proof. This follows directly from Theorem 113. □

Definition 245. The *jump $j(\alpha, \beta, \bar{\varphi})$* of α and β along a test curve $\bar{\varphi}$ is the number $\tilde{\tau}\varphi^* X(\alpha, \beta)$.

Let $\mathcal{Q}_S$ (or simply $\mathcal{Q}$ if S is clear) denote the set of isomorphisms of mixed extensions of $\mathbb{Q}$ by $\mathcal{H}$ by $\mathbb{Q}$ on S such that the associated classes $\alpha \in \mathrm{H}^1(S, \mathcal{H})$ and $\beta \in \mathrm{H}^1(S, \mathcal{H}^{\vee})$ lie in $\mathrm{IH}^1(\mathcal{H})$ and $\mathrm{IH}^1(\mathcal{H}^{\vee})$ respectively. Here we view α and β as extension classes (of $\mathbb{Q}$ by $\mathcal{H}$ and of $\mathcal{H}$ by $\mathbb{Q}$ respectively).

Lemma 246. *For $(\alpha,\beta) \in \mathrm{IH}^1(\mathcal{H}) \times \mathrm{IH}^1(\mathcal{H}^\vee)$, $\mathrm{Extpan}(\alpha,\beta)$ is the fiber of $\mathcal{Q}$ over (α,β).*

Proof. We need to show that $G(f_0, f_1) = 0$ for every pair $(f_0, f_1) \in \mathrm{Hom}(\mathbb{Q}, \mathcal{H}) \times \mathrm{Hom}(\mathcal{H}, \mathbb{Q})$. By symmetry and additivity of $G(f_0, f_1)$, it suffices to show that the composition

$$\mathrm{Hom}(\mathcal{H}, \mathbb{Q}) \times \mathrm{IH}^1(\mathcal{H}) \to \mathrm{Ext}^1(\mathbb{Q}, \mathbb{Q}) = \mathrm{H}^1(S, \mathbb{Q})$$

induced by the cup product vanishes. (Here the Hom and Ext groups are taken in the category of local systems on S.)

To see this, note that $\mathrm{Hom}(\mathcal{H}, \mathbb{Q}) = \mathrm{Hom}(\mathrm{IC}(\mathcal{H}), \mathrm{IC}(\mathbb{Q}))$ (by restriction). So we get a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\bar{S}}(\mathrm{IC}(\mathcal{H}), \mathrm{IC}(\mathbb{Q})) \times \mathrm{Ext}^1_{\bar{S}}(\mathrm{IC}(\mathcal{H}), \mathrm{IC}(\mathbb{Q}))) & \longrightarrow & \mathrm{Ext}^1_{\bar{S}}(\mathrm{IC}(\mathbb{Q}), \mathrm{IC}(\mathbb{Q})) & = & \mathrm{H}^1(\bar{S}, \mathbb{Q}) = 0 \\
\| & & \downarrow & & \\
\mathrm{Hom}_S(\mathcal{H}, \mathbb{Q}) \times \mathrm{IH}^1(\mathcal{H}) & \longrightarrow & \mathrm{Ext}^1_S(\mathbb{Q}, \mathbb{Q}) & = & \mathrm{H}^1(S, \mathbb{Q})
\end{array}$$

where the downward arrows are restriction. The result follows immediately. □

Corollary 247. *The map $\mathcal{Q} \to \mathrm{IH}^1(\mathcal{H}) \times \mathrm{IH}^1(\mathcal{H}^\vee)$ makes $\mathcal{Q}$ into a biextension of $\mathrm{IH}^1(\mathcal{H}) \times \mathrm{IH}^1(\mathcal{H}^\vee)$ by $\mathrm{H}^1(S, \mathbb{Q})$. Moreover, the map $\mathcal{P} \to \mathcal{Q}$ induced by restriction is a morphism of biextensions.*

Explanation. Here the operations $+_1$ and $+_2$ are the obvious ones coming from §10.5 as is the action of $\mathrm{H}^1(S, \mathbb{Q}) = \mathrm{Ext}^1_S(\mathbb{Q}, \mathbb{Q})$. □

Now suppose $\bar{\varphi} : \Delta \to \bar{S}$ is a test curve. Since the classes α and β vanish on restriction to Δ^* via φ, we get a homomorphism of biextensions $\varphi^* : \mathcal{Q}_S \to \mathcal{Q}_{\Delta^*}$. Via the isomorphism $\tilde{\tau} : \mathcal{Q}_{\Delta^*} \to \mathbb{Q}$, we then get a number $\tilde{\tau}\varphi^* X$ for any isomorphism class $X \in \mathcal{Q}$.

16.2. **Comparison with the asymptotic height pairing.** Now, we want to compare the general jump pairing from Definition 245 with the asymptotic height pairing defined earlier. To do this, we first want to generalize Theorem 155 to the local systems $X \in \mathcal{Q}_S$ where $S = \Delta^{*r}$. Pick $a \in \Delta^*$. Then, for each r-tuple, $t = (t_1, \dots, t_r) \in \mathbb{Z}_{\geq 0}$, we can write $\bar{\varphi}_t : \Delta \to \Delta^r$ for the test curve $s \mapsto a(s^{t_1}, \dots, s^{t_r})$. Write

$$\tilde{\tau}_t X := \tilde{\tau}\varphi^* X.$$

It is easy to see that this rational number does not depend on the choice of a. Recall that $\epsilon_i = (0, \dots, 0, 1, \dots, 0)$ with the 1 in the i-th place.

For any non-negative integer r, write $\mathrm{Perv}(\Delta^r)$ for the category of perverse sheaves on Δ^r and, following Saito's notation from [36, §3.1], write $\mathrm{Perv}(\mathbb{Q}_X)_{nc}$ for the full subcategory consisting of perverse sheaves which are constructible with respect to the stratification induced by the coordinate hyperplanes.

Lemma 248. *Suppose a and b are non-negative integers with $a + b = r$, $y = (y_1, \dots, y_a) \in (\Delta^*)^a$ and write $\bar{\varphi} : \Delta^b \to \Delta^r$ for the map sending $z = (z_1, \dots, z_b)$ to $(y_1, \dots, y_a, z_1, \dots, z_b)$. If $\mathcal{F}$ is a perverse sheaf in $\mathrm{Perv}(\Delta^r)_{nc}$ then $\bar{\varphi}^*\mathcal{F}$ is an object in $\mathrm{Perv}(\Delta^b)_{nc}$. Consequently the functor $\bar{\varphi}^* : \mathrm{Perv}(\Delta^r)_{nc} \to \mathrm{Perv}(\Delta^b)_{nc}$ is exact.*

Proof. As in [36, §3.1], the perverse sheaves in $\mathrm{Perv}(\Delta^r)_{nc}$ consist of perverse sheaves on Δ^r with characteristic variety contained in the conormal bundles of the intersections of the coordinate hyperplanes. Consequently, the map $\bar{\varphi} : \Delta^b \to \Delta^r$ is non-characteristic. The result then follows from Kashiwara's theorem on non-characteristic restriction. □

Corollary 249. *We have $\tilde{\tau}_{\epsilon_i} X(\alpha, \beta) = 0$ for all integers i with $1 \leq i \leq n$.*

Proof. Pick $a \in \Delta^*$ to define $\bar{\varphi}_t : \Delta \to \Delta^r$ as above for any $t \in \mathbb{Z}_{\geq 0}$. Since $\bar{\varphi}_{\epsilon_i}^* : \mathrm{Perv}(\Delta^r)_{nc} \to \mathrm{Perv}(\Delta)_{nc}$ is exact, $\varphi_{\epsilon_i}^* X(\alpha, \beta)$ is in the trivial biextension $\mathcal{P}_\Delta$. So $\tilde{\tau}\varphi_{\epsilon_i}^* X(\alpha, \beta) = 0$. ☐

Corollary 250. *Suppose $X \in \mathcal{Q}_{\Delta^{*r}}$, and $t \in \mathbb{Z}_{\geq 0}^r$. Then*

$$\tilde{\tau}_t X = \tilde{\tau}_t X(\alpha, \beta) + \sum_{i=1}^{r} \tilde{\tau}_{\epsilon_i} X.$$

Proof. We have $X = X(\alpha, \beta) + E$ for some element $E \in \mathrm{Ext}^1_{\Delta^{*r}}(\mathbb{Q}, \mathbb{Q}) = \mathrm{H}^1(\Delta^{*r}, \mathbb{Q}) = \mathbb{Q}^r$. Consequently, $\tilde{\tau}_t X = \tilde{\tau}_t X(\alpha, \beta) + \tilde{\tau}_t E$. The result follows from the (easy) fact that $\tilde{\tau}_t E = \sum \tilde{\tau}_{\epsilon_i}(E) t_i$. ☐

We now state an analogue of Theorem 155.

Proposition 251. *Suppose $X \in \mathcal{Q}_{\Delta^{*r}}$ with $\pi(X) = (\alpha, \beta)$. Then*

$$h(t)(\alpha, \beta) = -\tilde{\tau}_t(X) + \sum_{i=1}^{r} \tilde{\tau}_{\epsilon_i} t_i.$$

Proof. The proof is essentially the same as the proof of Theorem 155. ☐

Theorem 252. *Suppose $\mathcal{H}$ is a torsion free variation of pure Hodge structure on $(\Delta^*)^r$ and $(\alpha, \beta) \in \mathrm{IH}^1(\mathcal{H}) \times \mathrm{IH}^1(\mathcal{H}^\vee)$. Let $(t_1, \ldots, t_r) \in \mathbb{Z}_{\geq 0}^r$, and write $\bar{\varphi} : \Delta \to \Delta^r$ for the test curve $s \mapsto (s^{t_1}, \ldots, s^{t_r})$. Then*

$$h(\alpha, \beta)(t) = -j(\alpha, \beta, \bar{\varphi}_t).$$

Proof. Apply Proposition 251 to $X = X(\alpha, \beta)$ using the fact that $\tilde{\tau}_{\epsilon_i} X = 0$ for all i. ☐

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