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# MATROIDS, MOTIVES AND A CONJECTURE OF KONTSEVICH

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Abstract. We show that a certain class of varieties with origin in Physics, generates (additively) the Denef-Loeser ring of Motives. In particular, this disproves a conjecture of Kontsevich on the number of points of these varieties over finite fields.

## Introduction

In this paper, we show that a certain class of varieties with origin in the physics of Feynman amplitudes additively generates the Denef-Loeser ring of motives. This disproves a conjecture of Kontsevich on the number of points of these varieties over a finite field. It also enables us to investigate period integrals on these varieties, and to show that the class of integrals obtained is quite general.

0.1. **Kontsevich's conjecture.** Let $G$ be a finite graph with vertex set $V = V(G)$, edge set $E = E(G)$ and betti numbers $b_0(G)$ and $b_1(G)$. Recall that a graph $T$ is called a tree if $b_0(T) = 1$ and $b_1(T) = 0$. A subgraph $T \subset G$ is called a spanning tree if $T$ is a tree and $V(T) = V(G)$.

For each edge $e$, let $x_e$ denote a formal variable. Consider the polynomial

$$P_G = \sum_T \prod_{e \notin T} x_e \qquad (0.1)$$

where the sum runs through all spanning trees of $G$. If $G$ is not connected, $P_G = 0$ because the sum is empty. Otherwise, $P_G$ is a homogeneous polynomial of degree $b_1(G)$.

The polynomial $P_G$ and other related polynomials appear in the analysis of electrical circuits. In the 19th century, these polynomials were studied by Kirchhoff, Maxwell, Borchardt and Sylvester and, for this reason, they are sometimes called *Kirchhoff* polynomials. An important property of Kirchhoff polynomials is that they have an expression in terms of determinants through the Matrix-Tree theorem [24]. In the combinatorics literature, Kirchhoff polynomials are also occasionally called *unsignants* because, while determinantal expressions usually involve minus signs, minus signs are conspicuously absent from $P_G$.

Kirchhoff polynomials also play a role in the evaluation of Feynman amplitudes. Let $V(P_G)$ denote the scheme of zeros of $P_G$ over $\mathbf{Z}$, a hypersurface in $\mathbf{A}^E$, and let $Y_G$ denote its complement. Feynman amplitudes and their counterterms are then related to period integrals on the $Y_G$. (We refer the reader to [27] pp. 13–21, [2] for this relationship.) Motivated by computer calculations of the counterterms appearing in the renormalization of Feynman integrals [4, 14], M. Kontsevich speculated

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that the periods of $Y_G$ are multiple zeta values (MZVs). Under this assumption on the periods, it is natural to expect that the zeta functions associated to the $Y_G$ are the zeta functions of motives of mixed Tate type [31].

Based on this hypothesis and the Weil conjectures, M. Kontsevich made a conjecture about the number of points of $Y_G$ over a finite field [12]. To describe his conjecture, we first make a notational convention: For any scheme $X$ of finite type over $\mathbf{Z}$, let $|X|$ denote the function $q \mapsto \#X(\mathbf{F}_q)$. Thus $|X|$ is a function from the set $\mathcal{Q}$ of prime powers to $\mathbf{Z}$. Clearly, $|X|$ determines the zeta function of $X$. We say that $X$ is *polynomially countable* if $|X|$ is a polynomial in $\mathbf{Z}[q]$.

**Conjecture 0.1** (Kontsevich)**.** For all graphs $G$, $Y_G$ is polynomially countable.

Since $|V(P_G)| + |Y_G| = q^{\#E}$, this conjecture is equivalent to the conjecture that $V(P_G)$ is polynomially countable.

Stembridge [25] verified this conjecture for all graphs with fewer than 12 edges. For certain graphs it is relatively easy to see that the conjecture holds. For example, for $G$ a cycle of length $n$, $V(P_G)$ is isomorphic to $\mathbf{A}^{n-1}$ and, thus, $|Y_G| = q^n - q^{n-1}$.

We will show, however, that Conjecture 0.1 is false. In fact, contrary to the extremely strong restrictions on the arithmetic nature of the schemes $Y_G$ claimed by the conjecture, they are, from the standpoint of their zeta functions, the most general schemes possible.

0.2. **Combinatorial Motives and the Main Theorem.** To make this last statement precise we introduce some notation. Let $\mathsf{CMot}^+$ denote the group generated by all functions of the form $|X|$ for $X$ a scheme of finite type over $\mathbf{Z}$. We think of $\mathsf{CMot}^+$ as a coarse version of the ring of motives over $\mathbf{Z}$. We will discuss a finer ring of motives at the end of this introduction. As $|X \times Y| = |X||Y|$, $\mathsf{CMot}^+$ is a ring. And, as $|\mathbf{A}^1| = q$, $\mathsf{CMot}^+$ is a $\mathbf{Z}[q]$ module. We call $\mathsf{CMot}^+$ the ring of *effective combinatorial motives*.

Let $\mathsf{S}$ be the saturated multiplicative system in $\mathbf{Z}[q]$ generated by the functions $q^n - q$ for $n > 1$. Set $\mathsf{CMot} = \mathsf{S}^{-1}\,\mathsf{CMot}^+$. We remark that, since the functions in $\mathsf{S}$ are nonvanishing on $\mathcal{Q}$, elements of $\mathsf{CMot}$ give everywhere-defined functions from $\mathcal{Q}$ to $\mathbf{Q}$. We call $\mathsf{CMot}$ the ring of *combinatorial motives*.

Let $\mathsf{R} = \mathsf{S}^{-1}\mathbf{Z}[q]$. (We remark that $\mathsf{R}$ is a principal ideal domain [10].) Let $\mathsf{CGraphs}$ denote the $\mathsf{R}$-module generated by all functions of the form $|Y_G|$. We can now state our main theorem.

**Theorem 0.2.** $\mathsf{CGraphs} = \mathsf{CMot}$.

The theorem immediately implies that Conjecture 0.1 is false. For, if the conjecture were true, all functions of the form $|X|$ would be in $\mathsf{R}$. In particular, they would be rational functions. However, if we let $X$ be the closed subscheme of $\mathbf{A}^1_{\mathbf{Z}}$ defined by $px = 0$ for $p$ a given prime, then $|X|(q) = q$ if $p|q$ and 1 otherwise. Thus $|X|$ cannot be a rational function. Of course, other more interesting examples of $X$ such that $|X|$ is not rational exist. For example let $E/\mathbf{Z}$ be an integral model of a smooth elliptic curve over $\mathbf{Q}$. It is well known that $|E|$ is not a polynomial, even if we restrict it to any "large" subset of $\mathcal{Q}$. In particular , this gives a counterexample to the question on p363 of [23] which asks if $|Y_G|$ is always a quasi-polynomial.

0.3. **Stanley's Reformulation of Conjecture 0.1.** The proof of Theorem 0.2 is based on Stanley's reformulation of Kontsevich's conjecture in terms of a polynomial

$Q_G$ which is, roughly speaking, dual to $P_G$. In [23], Stanley sets

$$Q_G = \sum_T \prod_{e \in T} x_e. \tag{0.2}$$

where the sum again runs through all spanning trees. For $G$ connected, $Q_G$ is homogeneous of degree $\#E(G) - b_1(G)$. Let $X_G = \mathbf{A}^E - V(Q_G)$. Stanley showed that Kontsevich's conjecture is equivalent to the following analogous conjecture:

**Conjecture 0.3.** For all graphs $G$, $|X_G| \in \mathbf{Z}[q]$.

In fact, we will see in Theorem 1.2 that the $\mathsf{R}$-submodule of $\mathsf{CMot}$ generated by the $|X_G|$ is exactly the same as the one generated by the $|Y_G|$. Thus, by Theorem 0.2, the $|X_G|$ also generate $\mathsf{CMot}$.

The schemes $X_G$ are, however, more tractable than the $Y_G$ — particularly when the graph $G$ is simple (i.e., has neither loops nor multiple edges) and has an apex. This is because, when $G$ is simple, the polynomial $Q_G$ has an uncomplicated expression as a determinant via the Matrix-Tree theorem (see section 3.) This expression simplifies even further when $G$ has an apex. (There is also an expression for $P_G$ as a determinant, but this expression seems unmanageable for our purposes.)

We remind the reader that a vertex $v$ is said to be an apex if there is an edge from $v$ to every other vertex in $G$. Suppose that $G$ is an arbitrary simple graph with vertex set $V = \{v_1, \dots, v_n\}$. Then we form a graph $G^*$ with apex by simply adding a vertex $v_0$ and connecting it by an edge to all other vertices. All graphs with apex can be obtained through this process.

Using the Matrix-Tree theorem, Stanley showed that, for any field $K$, $X_{G*}(K)$ is isomorphic to the set of $n \times n$ nondegenerate, symmetric matrices $M$ satisfying the condition that

$$M_{ij} = 0 \text{ if } i \neq j \text{ and there is no edge from } v_i \text{ to } v_j. \tag{0.3}$$

Here $i, j \in [1, n]$.

We then let $Z_G^o$ be the scheme of all $n \times n$ nondegenerate, symmetric matrices $M$ satisfying condition 0.3. (See section 3.) Stanley's observation essentially shows that $Z_G^o \cong X_{G^*}$. Thus, the following conjecture, stated by Stembridge as Conjecture 7.1 [25], would follow from Conjecture 0.3.

**Conjecture 0.4.** For every simple graph $G$, $Z_G^o$ is polynomially countable.

Note that, while Conjectures 0.1 and 0.3 are trivial when $G$ is disconnected, Conjecture 0.4 is not. This is related to the fact that the operation $G \mapsto G^*$ always produces a connected graph.

However, we will see that Conjecture 0.4 is also false.

For any subgraph $H$ of $G$, let $G - H$ be the graph obtained by removing the edges in $H$ but leaving all vertices. Note that $(G - H)^* = G^* - H$. If $G$ is a simple graph with $n$ vertices, then $G$ is contained in the complete graph $K_n$. We define the *complement* $G^o$ of $G$ to be the graph $K_n - G$. Note that $(G^o)^* = (DG)^o$ where $D$ is the operation of adding a disjoint vertex.

It becomes convenient at this point to shift attention from $G$ to its complement. We therefore define $Z_G = Z_{G^o}^o$. When $G$ has vertices $\{v_1, \dots, v_n\}$ as above, $Z_G$ is then the scheme of all $n \times n$ matrices $M$ satisfying the condition

$$M_{ij} = 0 \text{ if there is an edge from } v_i \text{ to } v_j. \tag{0.4}$$

We mention that many of the results obtained thus far on Conjecture 0.3 are most easily stated in terms of the $|Z_G|$. For example, in Theorem 5.4 of [23], Stanley showed that Conjecture 0.3 holds when $G = K_n - K_{1,s}$ where $K_{1,s}$ is a star (one vertex connected by edges to $s$ other vertices) and $s \leq n-2$. In the case $n = s+2$, $G = \Gamma^*$ with $\Gamma = K_{s+1} - K_{1,s}$. Thus $\Gamma = K^o_{1,s}$, and $X_G = Z^o_\Gamma = Z_{K_{1,s}}$. It follows that Stanley's Theorem 5.4 is equivalent to the statement that $|Z_{K_{1,s}}| \in \mathbf{Z}[q]$.

0.4. **Overview.** Let $\mathsf{CGraphs}_*$ be the $\mathsf{R}$-module generated by all functions of the form $|Z_G|$ for $G$ a simple graph. Since $|Z_G| = |Z^o_{G^o}| = |X_{(G^o)^*}|$, it is clear that $\mathsf{CGraphs}_* \subset \mathsf{CGraphs}$. Therefore the following theorem implies Theorem 0.2:

**Theorem 0.5.** $\mathsf{CGraphs}_* = \mathsf{CMot}$.

The proof of Theorem 0.5 involves two steps. In the first, we study certain incidence schemes $A_G(s,r,k)$. These schemes are defined so that, when $K$ is a field, the $K$ points of $A_G(s,r,k)$ are the sets of pairs $(Q,f)$ with $Q$ a symmetric bilinear form on $K^s$ of rank $r$ and $f$ a function from $V(G)$ to $K^s$ whose span is of dimension $k$. The pair $(Q,f)$ is also subject to the incidence condition that

$$Q(f(v_i), f(v_j)) = 0 \text{ if there is an edge from } v_i \text{ to } v_j. \tag{0.5}$$

If $G$ has $n$ vertices, then $|A_G(n,n,n)| = |Z_G||\operatorname{GL}_n|$. Since $|\operatorname{GL}_n| \in \mathsf{R}$, this implies that $|A_G(n,n,n)| \in \mathsf{CGraphs}_*$. Moreover, there are important relations between the $A_G(s,r,k)$ for varying $s,r$ and $k$, and between the $A_G(s,r,k)$ for varying $G$. By exploiting these relations, we will see that the $\mathsf{R}$-module generated by the $|A_G(s,r,k)|$ is exactly $\mathsf{CGraphs}_*$.

This fact allows us to shift our focus from the symmetric form $Q$ to the function $f$. In particular, for each $s$ we consider the scheme, $J_G(s) = \cup_k A_G(s,s,k)$. Again, it turns out that the $\mathsf{R}$-module generated by the $J_G(s)$ is exactly $\mathsf{CGraphs}_*$. Moreover, the $J_G(s)$ are quite manageable schemes because the dimension of the span of $f$ is allowed to vary.

The second step in our proof of Theorem 0.5 involves comparing the $J_G(s)$ to the representation spaces of matroids. For any matroid $M$, we define a scheme $X(M,s)$. For $K$ a field, $X(M,s)(K)$ is the set of all possible representations of $M$ in $K^s$. We then let $\mathsf{CMatroids}$ denote the $\mathsf{R}$-module generated by all functions $|X(M,s)|$. As we will see in Section 10, it follows from Mnëv's Universality Theorem [19] that $\mathsf{CMatroids} = \mathsf{CMot}$. On the other hand, we prove that, for each matroid $M$, there is a finite set of graphs $\{G_i\}$ and rational functions $a_i \in \mathsf{R}$ such that

$$|X(M,s)| = \sum a_i |J_{G_i}(s)|. \tag{0.6}$$

This equation proves that $\mathsf{CMatroids} \subset \mathsf{CGraphs}_*$ and, thus, it proves Theorem 0.5. Moreover, as we will see, (0.6) can be used even without Mnëv Universality to produce a contradiction to Conjecture 0.4. This is because there are matroids $M$, for example the Fano matroid, which are representable only over fields of characteristic 2. Thus, for such matroids, $|X(M,r)|$ (with $r$ equal to the rank of $M$) could not possibly be a rational function as Conjecture 0.4 and (0.6) would demand. As Conjecture 0.1 implies Conjecture 0.4, this shows that Conjecture 0.1 is false.

0.5. **Forest Complements.** A considerable amount of work has been done to find examples of graphs for which Conjecture 0.1 (resp. Conjecture 0.3, Conjecture 0.4) holds and to compute the functions $|Y_G|$ (resp. $|X_G|$, $|Z^o_G|$) explicitly [5, 23, 25, 30]. It remains an interesting question to determine the largest classes of graphs for which these conjectures are valid.

The class of graphs for which Conjecture 0.3 holds is already known to include to include several important examples. Stanley showed that $X_{K_n - K_m}$ is polynomially countable. Chung and Yang then computed the polynomial $|X_{K_n - K_m}|$ explicitly [5]. Yang showed that $X_G$ is polynomially countable when $G$ is an outplanar graph. And, as mentioned above, a consequence of Theorem 5.4 of [23] is that $Z_{K_{1,s}}$ is polynomially countable.

Recall that a *forest* is a graph with no cycles. In section 11, we show that $Z_F$ is polynomially countable whenever $F$ is a forest. This generalizes Stanley's Theorem 5.4 and implies that Conjecture 0.4 holds for forest complements. The result is essentially a consequence of the manageability of the schemes $J_F(s)$ which allows us to compute $|J_F(s)|$ inductively in terms of the $|J_{F'}(s)|$ for smaller forests $F'$.

0.6. **Geometric Motives.** We have written the majority of this paper in terms of combinatorial motives because they suffice for the proof that Kontsevich's conjecture is false. However, the reader who is familiar with the Kontsevich-Denef-Loeser theory of motivic integration will see that the statements in the paper are valid in a finer setting once the combinatorial process of counting points is replaced with the algebraic process of stratifying a scheme into disjoint subschemes. (This is, in fact, the process used by Stembridge's Maple program [25] to verify Kontsevich's conjecture for graphs with less than 12 edges.)

Following Denef, Loeser and Craw [8, 6], we define the ring of motives as follows: Write $\mathsf{GeoMot}^+$ for the abelian group generated by the symbols $[X]$ for $X$ a scheme of finite type over $\mathbf{Z}$ modulo the relations:

(a) $[X] = [Y]$ if $X \cong Y$,
(b) $[X] = [X - V] + [V]$ if $V$ is closed in $X$.

The group $\mathsf{GeoMot}^+$ becomes a ring once we check that it is consistent to define $[X][Y] = [X \times Y]$. This ring, which we call the ring of *effective geometric motives* over $\mathbf{Z}$ has $[\mathrm{Spec}\,\mathbf{Z}]$ as its unit. In Section 12, we will define the ring of effective geometric motives over an arbitrary base, and give several results concerning $\mathsf{GeoMot}^+$ which we hope will be of independent interest.

There is an obvious surjection $\mathrm{ev} : \mathsf{GeoMot} \twoheadrightarrow \mathsf{CMot}$ given by sending $[X]$ to $|X|$. Following tradition, we write $\mathbf{L}$ for $[\mathbf{A}^1]$ and note that $\mathrm{ev}(\mathbf{L}) = q$. ($\mathbf{L}$ is known as the *Tate motive.*) Clearly the evaluation map restricted to $\mathbf{Z}[\mathbf{L}]$ is an isomorphism onto its image which is $\mathbf{Z}[q]$. We therefore write $\mathsf{S}$ for the saturated multiplicative subset of $\mathbf{Z}[\mathbf{L}]$ generated by $\mathbf{L}^n - \mathbf{L}$ for $n > 1$ and $\mathsf{R}$ for the localization $\mathsf{S}^{-1}\mathbf{Z}[\mathbf{L}]$. That is, in this paper, we will use the symbols $\mathsf{S}$ and $\mathsf{R}$ in the context of $\mathsf{CMot}$ and in the context of $\mathsf{GeoMot}$. We hope that this slight abuse of notation will not lead to confusion.

We write $\mathsf{GeoMot} = \mathsf{S}^{-1}\mathsf{GeoMot}^+$. This ring, which we call the ring of *geometric motives*, essentially appears in the work of Denef, Loeser and Craw on motivic integration. (See, for example, [8] Corollary 6.3.4 or [6] 1.18.) When $n > 1$, $\mathbf{L}^n - \mathbf{L}$ is invertible in the completion of the ring of motives where Kontsevich's motivic measure takes its values.

We now state the main results of our paper in the context of geometric motives.

**Theorem 0.6.** *Let* $\mathsf{Graphs}$ *be the sub-*$\mathsf{R}$*-module of* $\mathsf{GeoMot}$ *generated by the* $[Y_G]$ *and let* $\mathsf{Graphs}_*$ *be the sub-*$\mathsf{R}$*-module of* $\mathsf{GeoMot}$ *generated by the* $[Z_G]$ *where* $G$ *runs over all simple graphs. Let* $\mathsf{Forests}$ *denote the sub-*$\mathsf{R}$*-module of* $\mathsf{GeoMot}$ *generated by the* $[J_F(s)]$ *for all integers* $s$ *and all forests* $F$*.*

(a) $\mathsf{Graphs}_* = \mathsf{Graphs} = \mathsf{GeoMot}$.
(b) $\mathsf{Forests} = \mathsf{R}$.

The advantage of having these statements in the context of geometric motives is that we can use them to investigate geometric invariants of the graph schemes. In particular, we can study the original motivation for Kontsevich's conjecture, namely the periods of the varieties defined by Kirchhoff polynomials. In the last section, using the "stringy" E-polynomials of motivic integration [6], we show that there is a graph $G$ and two integers $p \neq q$ such that the Hodge-Deligne number $h^{p,q}(Y_G)$ is nonzero. If one accepts a recent conjecture of Kontsevich and Zagier [13] concerning the nature of periods, this implies that the periods of the $Y_G$ are, in fact, not always multiple zeta values.

The verification of Theorem 0.6 is left to the end of the paper where we point out the modifications needed to turn counting arguments into algebro-geometric ones. Much of this is routine and left to the reader. However, the burden of working with motives over $\operatorname{Spec} \mathbf{Z}$ is daunting enough that stating everything the first time around in terms of geometric motives would obscure the logic of the arguments significantly.

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## Contents

## 1. Preliminary Results

In this section we carry out two minor adjustments to two theorems of Stanley.

1.1. **The Module of all Graphs.** The first adjustment is an amplification of Proposition 2.1 of [23]. It concerns the relation between the schemes $Y_G$ and the schemes $X_G$. Here we work in $\mathsf{GeoMot}^+$.

**Proposition 1.1.** *The subgroup of* $\mathsf{GeoMot}^+$ *generated by the* $[X_G]$ *is equal to the subgroup generated by the* $[Y_G]$.

We remark that the proof of this proposition is completely contained in Stanley's proof of his Proposition 2.1. However, for the convenience of the reader, we translate Stanley's proof into our own setting.

*Proof.* Let $S$ be a subset of $E = E(G)$. Let $\mathbf{A}^S$ be the image of the obvious inclusion $i^S : \mathbf{A}^{\#S} \to A^E$. Let $\mathbf{G}_m^S = i^S(\mathbf{G}_m^{\#S})$. Note that, as $S$ varies over all subsets of $E$, the subschemes $\mathbf{G}_m^S$ stratify $\mathbf{A}^E$.

For any subscheme $X \subset \mathbf{A}^E$, let $X_S = X \cap \mathbf{A}^{E-S}$ (resp. $X_S^+ = X \cap \mathbf{G}_m^{E-S}$). Thus $X_S$ is the intersection of $X$ with the hyperplanes defined by the equations $x_e$ for $e \in S$. Note that $X_\emptyset = X$, and, as $S$ varies over the subsets of $E$, the subschemes $X_S^+$ stratify $X$. We therefore have,

$$[X_S] = \sum_{T \supset S} [X_T^+] \tag{1.1}$$

and, by the Inclusion-Exclusion Principle,

$$[X_S^+] = \sum_{T \supset S} (-1)^{\#(T-S)} [X_T]. \tag{1.2}$$

By inspecting the $Q_G$, it is easy to see that $X_{G,S} \cong X_{G-S}$ and $X_{G,S}^+ \cong X_{G-S}^+$. Dually, if $S$ is a forest, $Y_{G,S} \cong Y_{G/S}$ (resp. $Y_{G,S}^+ \cong Y_{G/S}^+$) where $G/S$ is the graph obtained by contracting each component of $S$ to a point. On the other hand, if $S$ is not a forest, it is easy to see that $Y_{G,S}$ is empty.

Now, as Stanley notes, $Q_G(x) = P_G(1/x) \prod_{e \in E} x_e$. Thus

$$X_{G,\emptyset}^+ \cong Y_{G,\emptyset}^+ \tag{1.3}$$

through the map $x \mapsto 1/x$. That is, by sending the point $x = (x_e)_{e \in E}$ of $X_{G,\emptyset}^+$ to the point in $Y_{G,\emptyset}^+$ with coordinates $(1/x_e)_{e \in E}$.

Putting our equations together we obtain the following:

$$[Y_G] = \sum_{\substack{S \subset E \\ b_1(S)=0}} \sum_{T \subset G/S} (-1)^{\#T} [X_{(G/S)-T}], \tag{1.4}$$

$$[X_G] = \sum_{S \subset E} \sum_{\substack{T \subset E-S \\ b_1(T)=0}} (-1)^{\#T} [Y_{(G-S)/T}]. \tag{1.5}$$

Together, these two equations, the first of which appears (in a different notation) as Proposition 4.1 of [25], prove the proposition. □

The proposition implies the following theorem as a corollary.

**Theorem 1.2.** $\mathsf{Graphs}$ *is equal to the* $\mathsf{R}$*-submodule of* $\mathsf{GeoMot}$ *spanned by the* $[X_G]$. $\mathsf{CGraphs}$ *is equal to the* $\mathsf{R}$*-submodule of* $\mathsf{CMot}$ *spanned by the* $|X_G|$.

We remark that, as $[Y_G] = \mathbf{L}^n - \mathbf{L}^{n-1}$ for $G$ a cycle of length $n$, $\mathsf{R}$ is itself a submodule of $\mathsf{Graphs}$.

1.2. **An Observation on Polynomial Countability.** Our second adjustment to Stanley's results is to Proposition 2.2 of [23]. This proposition, which Stanley deduces from the Weil conjectures, essentially states that, if $X$ is a scheme of finite type over $\mathbf{Z}$, then the knowledge that $|X| \in \mathbf{Q}[q]$ implies that, in fact, $|X| \in \mathbf{Z}[q]$.

In Section 11, we require a result which is analogous to Stanley's Proposition 2.2 but easier to prove. While the result is not strictly weaker than Stanley's proposition, it does not require the Weil conjectures. Rather, it is a consequence of the Euclidean algorithm.

**Proposition 1.3.** *If $f \in \mathsf{R}$ and $f(q) \in \mathbf{Z}$ for all $q \in \mathcal{Q}$, then $f \in \mathbf{Z}[q]$.*

We will use the proposition in the case $f = |X|$ for $X$ a scheme of finite type over $\mathbf{Z}$.

*Proof.* Write $f = a/s$ with $a \in \mathbf{Z}[q]$ and $s \in \mathsf{S}$. Since $s$ is monic, we can write $f = d + r/s$ with $d, r \in \mathbf{Z}[q]$ and $\deg(r) < \deg(s)$. But this implies that $r(q)/s(q) \in \mathbf{Z}$ for all $q$ which implies that $r = 0$. Thus $f = d$. □

## 2. Determinantal Schemes

In this section, we collect certain basic properties of determinantal schemes which are necessary for the definition of the incidence schemes $A_G(s, r, k)$. We first describe the general theory of determinantal schemes in functorial language and then restrict to the specific case of determinantal schemes over $\mathbf{Z}$ that are the focus of the paper. These results are necessary for Theorem 0.6, but they are not strictly necessary for the proof that Conjecture 0.1 is false. The reader only interested in Kontsevich's conjecture may, therefore, skim the section until the end where we state formulas for the motives of four important types of determinantal schemes.

2.1. **Degeneracy Loci.** Let $S$ be a scheme and let $E$ and $F$ be two locally free $\mathcal{O}_S$-modules of ranks $e$ and $f$ respectively. Let $\phi : E \to F$ be a morphism. The $r$-th degeneracy locus $D_r(\phi)$ of $\phi$ is the closed subset consisting of all points $s \in S$ such that $\phi \otimes k(s)$ has rank less than or equal to $r$. We put a structure of a closed subscheme on $D_r(\phi)$ by writing it as the scheme of zeros of the morphism

$$\wedge^{r+1}\phi : \wedge^{r+1}E \to \wedge^{r+1}F. \tag{2.1}$$

Equivalently, $D_r(\phi)$ is the closed subscheme of $S$ corresponding to the ideal generated by the $(r+1) \times (r+1)$-minors of $\phi$. Note that the subschemes $Z_r(\phi) = D_r(\phi) - D_{r-1}(\phi)$ partition $S$ into a disjoint union of locally closed subschemes. (See Chapter 14 of [9] for more details.)

2.2. **Determinantal Schemes.** With $S$, $E$ and $F$ as above, we write $\mathrm{Hom}_{\mathcal{O}_S}(E, F)$ for the abelian group of all homomorphisms from $E$ to $F$. The scheme of homomorphisms $\mathrm{Hom}(E, F)$ is then an abelian group scheme over $S$ representing the functor

$$T \rightsquigarrow Hom_{\mathcal{O}_T}(E_T, F_T.) \tag{2.2}$$

Write $\pi : \mathrm{Hom}(E, F) \to S$ for the structure map. $\mathrm{Hom}(E, F)$ is then equipped with a universal map $\phi : \pi^*E \to \pi^*F$. The fact that $\mathrm{Hom}(E, F)$ represents the homomorphism functor can be expressed by saying that, for any $S$-scheme $T$ and any map $\psi : E_T \to F_T$, there is a unique map $T \to \mathrm{Hom}(E, F)$ such that $\psi \cong \phi_T$.

We write $\mathrm{Hom}_{\leq r}(E,F)$ for the degeneracy locus $D_r(\phi)$, and we write $\mathrm{Hom}_r(E,F)$ for the locally closed subscheme $Z_r(\phi)$. We call both types of scheme *determinantal schemes.* The $\mathrm{Hom}_r(E,F)$ are important in this paper as they stratify $\mathrm{Hom}(E,F)$ into a disjoint union of locally closed schemes.

They also represent a natural functor, and the functorial description is useful as a language for describing other schemes in terms of the $\mathrm{Hom}_r(E,F)$ and for defining maps from the $\mathrm{Hom}_r(E,F)$ to other schemes. Let us say that the rank of a morphism $\psi : E \to F$ is $r$ if the cokernel of $\psi$ is a locally free sheaf on $S$ of rank $f - r$.

**Proposition 2.1.** *Let $S$ be a noetherian scheme, then $\mathrm{Hom}_r(E,F)$ represents the functor*

$$T \rightsquigarrow \{\psi : E_T \to F_T |\, \mathrm{rk}(\psi) = r\}. \tag{2.3}$$

*Proof.* Suppose we are given a scheme $T$ and a morphism $\psi : E_T \to F_T$. By the universal property of $\mathrm{Hom}(E,F)$, this information gives us a unique morphism $\sigma : T \to S$ such that $\psi = \phi_T$ where $\phi$ is the universal morphism. We need to show that this morphism $\sigma$ factors through $\mathrm{Hom}_r(E,F)$ if and only if $\mathrm{rk}(\psi) = r$.

Now $\sigma$ will factor through $\mathrm{Hom}_r(E,F)$ if and only if the pull-back of the sheaf of ideals defining $\mathrm{Hom}_{\leq r}(E,F)$ is 0 on $T$ and $D_{r-1}(\psi) = \emptyset$. This will be the case if and only if the $(r+1) \times (r+1)$-minors of $\psi$ are 0 while some $r \times r$-minor is invertible in $\mathcal{O}_T$. These two conditions are local on $T$. The proof of the proposition will therefore follow from the following:

**Lemma 2.2.** *Suppose $A$ is a noetherian local ring and $\psi : A^e \to A^f$ is a morphism. Then the following are equivalent:*

(a) $\mathrm{coker}\,\psi$ *is free of rank $f - r$.*
(b) *Every $(r+1) \times (r+1)$-minor of $\psi$ is 0, but some $r \times r$-minor of $\psi$ is invertible.*

See Prop 20.8, page 495 in [7]. □

It will be useful to have an explicit local description of our determinantal schemes in terms of coordinates and ideals: Let $\{y_{ij}\}_{i=1}^{e}{}_{j=1}^{f}$ be a set of formal variables, and consider each $y_{ij}$ as an entry in an $e \times f$ matrix. Let $A[y]$ be the polynomial ring in all variables $y_{ij}$. For each $k$, let $m_l^k \in \mathbf{Z}[y_{ij}]$ be a complete list of the $k \times k$ minors, and let $I_k$ be the ideal generated by the $m_l^k$. In this notation, $Hom_r(\mathcal{O}_S^e, \mathcal{O}_S^f)$ is the locally closed subscheme of $Hom(\mathcal{O}_S^e, \mathcal{O}_S^f)$ given by the union of the affine schemes

$$\cup_i \operatorname{Spec}(A[y]/I_{r+1})_{(m_i^r)}. \tag{2.4}$$

It follows that the set of points associated to $Hom_r(\mathcal{O}_S^e, \mathcal{O}_S^f)$ is simply $\cap_i V(m_i^{r+1}) - \cap_i V(m_i^r)$.

2.2.1. *Maps to the Grassmanian.* Write $\mathrm{Gr}(r,E)$ for the Grassmanian of $r$ planes in $E$. This is defined to be the scheme representing the functor

$$T \rightsquigarrow \{K \subset E | E/K \text{ is locally free of rank } e - r\}. \tag{2.5}$$

$\mathrm{Hom}_r(E,F)$ is equipped with two maps to Grassmanians. We have a map $p : \mathrm{Hom}_r(E,F) \to \mathrm{Gr}(r,F)$ given by sending a map $\phi$ to its image. And we have a map $q : \mathrm{Hom}_r(E,F) \to \mathrm{Gr}(e-r,F)$ given by sending $\phi$ to its kernel.

2.2.2. *Function Spaces.* When $V$ is a finite set we write $\mathrm{Fun}(V,E)$ for $\mathrm{Hom}(\mathcal{O}_S^V, E)$ (resp. $\mathrm{Fun}_r(V,E)$) for $\mathrm{Hom}_r(\mathcal{O}_S^V, E)$.

2.2.3. *Symmetric Bilinear Forms.* Let $E^\vee$ denote the dual of $E$. There is a natural transpose automorphism

$$t : \mathrm{Hom}(E, E^\vee) \to \mathrm{Hom}(E, E^\vee) \tag{2.6}$$

and we define $\mathrm{Sym}\, E$ to be the subscheme fixed by $t$. We then write $\mathrm{Sym}_r E$ (resp. $\mathrm{Sym}_{\le r} E$) for the scheme-theoretic intersection of $\mathrm{Sym}\, E$ with $\mathrm{Hom}_r(E, E^\vee)$ (resp. $\mathrm{Hom}_{\le r}(E, E^\vee)$).

2.3. **A Specific Case.** We will be primarily interested in the case $S = \mathrm{Spec}\,\mathbf{Z}$, $E = \mathcal{O}_S^e$ and $F = \mathcal{O}_S^f$. In this case, $\mathrm{Hom}(E,F)$ is $\mathrm{Spec}\,\mathbf{Z}[y]$. $\mathrm{Hom}_{\le r}(E,F)$ is the closed subscheme in $\mathrm{Hom}(E,F)$ defined by the $(r+1)\times(r+1)$ minors. And $\mathrm{Hom}_r(E,F)$ is the Zariski open subset of $\mathrm{Hom}_{\le r}(E,F)$ defined by requiring at least one $r\times r$ minor to be invertible.

These equalities can be used without reference to the preceding general theory to define the $\mathrm{Hom}_r(E,F)$. It follows directly that for any field $K$, $\mathrm{Hom}_r(E,F)(K)$ is the set of maps from $K^e$ to $K^f$ of rank $r$.

Similarly, when a $E = \mathcal{O}_S^e$ with $S = \mathrm{Spec}\,\mathbf{Z}$, $\mathrm{Sym}\, E$ can be viewed as the closed subscheme of $\mathbf{Z}[y]$ defined by the equations $y_{ij} = y_{ji}$. $\mathrm{Sym}_{\le r} E$ is then the closed subscheme of $\mathrm{Sym}\, E$ defined by the $(r+1)\times(r+1)$ minors. And $\mathrm{Sym}_r E$ is the Zariski open subset of $\mathrm{Sym}_{\le r} E$ defined by requiring at least one $r\times r$ minor to be invertible. The $K$ points of $\mathrm{Sym}_r E$ are the bilinear forms on $K^e$ of rank $r$.

2.4. **Polynomial Countability.** When $E = \mathcal{O}_S^e$, $F = \mathcal{O}_S^f$ and $S = \mathrm{Spec}\,\mathbf{Z}$, we write $\mathrm{Hom}_r(e,f)$ for $\mathrm{Hom}_r(E,F)$, $\mathrm{GL}_e$ for $\mathrm{Hom}_e(e,e)$, $\mathrm{Gr}(r,e)$ for $\mathrm{Gr}(r,E)$, and $\mathrm{Sym}_r^e$ for $\mathrm{Sym}_r E$.

We now list a few results concerning the polynomial countability of the schemes just discussed.

$$\begin{aligned} |\mathrm{GL}_n| &= (q^n-1)(q^n-q)\cdots(q^n-q^{n-1}) && (2.7)\\ |\mathrm{Gr}(a,b)| &= \frac{|\mathrm{GL}_b|}{|\mathrm{GL}_a||\mathrm{GL}_{b-a}|q^{a(b-a)}} && (2.8)\\ |\mathrm{Hom}_r(e,f)| &= |\mathrm{Gr}(r,e)||\mathrm{Gr}(r,f)||\mathrm{GL}_r| && (2.9)\end{aligned}$$

The first two of the above equalities are well known and the last is easy. Note that each of the functions given is a polynomial lying in the multiplicative set $\mathsf{S}$.

The following formula of MacWilliams [17] is more difficult.

$$|\mathrm{Sym}_r^n| = \begin{cases} \prod_{i=1}^s \frac{q^{2i}}{q^{2i}-1}\cdot\prod_{i=0}^{2s-1}(q^{n-i}-1), & 0\le r=2s\le n,\\ \prod_{i=1}^s \frac{q^{2i}}{q^{2i}-1}\cdot\prod_{i=0}^{2s}(q^{n-i}-1), & 0\le r=2s+1\le n\end{cases} \tag{2.10}$$

Note again that $|\mathrm{Sym}_r^n| \in \mathsf{S}$.

All four of these equalities remain valid in the ring $\mathsf{GeoMot}$ once $q$ is replaced with $\mathbf{L}$.

## 3. The Matrix Tree Theorem

Stanley's positive results mentioned in the introduction were mainly consequences of the *Matrix-Tree Theorem* of Kirchhoff, Borchardt and Sylvester, which gives an expression of the polynomial $Q_G$ as the determinant of a symmetric matrix. As this theorem is also basic to our results, we describe it in this section after fixing some useful notation.

3.1. **Notation.** When $G$ is a simple graph, an assumption we will make for the remainder of this paper, $E$ can be considered as a subset of $\mathrm{Sym}^2 V$. For $v, w \in V$, we write $e_{vw}$ for the set $\{v, w\}$. Thus the statement $e_{vw} \in E$ means that there is an edge in $G$ connecting $v$ to $w$.

It is convenient to pick an ordering $V = \{v_1, \dots, v_{n_G}\}$ of the vertices $V$, where $n_G = \#V(G)$. We write $n$ for $n_G$ when there is only one graph under consideration.

Set $e_{ij} = e_{v_i v_j}$. We write $x_{ij}$ for the variable $x_{e_{ij}}$ when $e_{ij} \in E$, and we extend this notation by setting $x_{ij} = 0$ when $e_{ij} \notin E$.

3.2. **The Laplacian.** Let $L = L_{ij}$ be the $n \times n$ matrix defined by

$$L_{ij} = \begin{cases} \sum_{k=1}^n x_{ik} & \text{if } i = j \\ -x_{ij} & \text{if } i \neq j \end{cases}$$

Let $L_0$ be $L$ with first row and the first column removed. $L$ is called the *generic Laplacian matrix of* $G$ and $L_0$ the *reduced generic Laplacian.* The following theorem can be found in the work of Cayley, Kirchhoff, Maxwell and Sylvester. For a proof, see [24].

**Theorem 3.1** (The Matrix-Tree Theorem)**.** $Q_G = \det L_0$.

Now, as in the introduction, let $Z_G^0$ be the scheme of all $n \times n$ symmetric, non-degenerate bilinear forms $M_{ij}$ such that $M_{ij} = 0$ whenever $i \neq j$ and $e_{ij} \notin E$. In the notation of section 2, $Z_G^0$ is simply the closed subscheme of $\mathrm{Sym}_n^n$ defined by the equations $y_{ij} = 0$ for all $i \neq j$ with $e_{ij} \notin E$.

Our use of Theorem 3.1, is based on the following important consequence, recognized by Stanley.

**Theorem 3.2.** $X_{G^*} \cong Z_G^o$.

*Proof.* Let $\mathbf{Z}[x]$ be the ring generated by the variables $x_{ij}$ for $0 \leq i < j \leq n$. Let $I$ be the ideal generated by the variables $x_{ij}$ for all pairs $i < j$ with $e_{ij} \notin E$. Then $X_{G^*} = \mathrm{Spec}\, A$ with $A = (\mathbf{Z}[x]/I)_{Q_G}$.

On the other hand, let $\mathbf{Z}[y]$ be the ring generated by all $y_{ij}$ for $i, j \in \{1, \dots, n\}$, and let $J$ be the ideal generated by all expressions of the form $y_{ij} - y_{ji}$ for $i \neq j$ and $y_{ij}$ for $i \neq j$ and $e_{ij} \notin E$. Then, letting $D$ be the determinant of the matrix of $y_{ij}$'s, $Z_G^0 = \mathrm{Spec}\, B$ with $B = (\mathbf{Z}[y]/J)_D$.

Let $p : \mathbf{Z}[y] \to \mathbf{Z}[x]$ be the map

$$y_{ij} \mapsto \begin{cases} \sum_{k<i} x_{ki} + \sum_{i<k} x_{ik} & i = j \\ -x_{ij} & i < j \\ -x_{ji} & j < i \end{cases} \tag{3.1}$$

Let $q : \mathbf{Z}[x] \to \mathbf{Z}[y]$ be the map

$$x_{ij} \mapsto \begin{cases} \sum_k y_{jk} & i = 0 \\ -y_{ij} & i > 0 \end{cases} \tag{3.2}$$

It is easy to verify that $p(I) \subset J$, that $q(J) \subset I$, and that $p$ and $q$ give inverse isomorphisms between the rings $\mathbf{Z}[x]/I$ and $\mathbf{Z}[y]/J$. It then follows from the Matrix-Tree theorem that $p(Q_G) = D$. Thus $p$ and $q$ give inverse isomorphisms between the rings $A$ and $B$. □

As mentioned in the introduction, it is convenient to shift our attention from the simple graph $G$ to its complement. We therefore set $Z_G = Z^o_{G^o}$. Thus $Z_G$ is the subscheme of $\mathrm{Sym}^n_n$ defined by the equations $y_{ij} = 0$ for every pair $i, j$ with $e_{ij} \in E$, and $Z_G = X_{(G^o)^*} = X_{(DG)^o}$ where $D$ is the operation of adding a disjoint vertex.

**Example 3.3.** Let $G$ be a graph with $n$ vertices and no edges. Then $Z_G \cong \mathrm{Sym}^n_n$. This is recognized in [23]. By Equation 2.10, it follows that $|Z_G| \in \mathbf{Z}[q]$. In fact, $|Z_G| \in \mathsf{S}$, and this shows that $\mathsf{R} \subset \mathsf{CGraphs}_*$.

## 4. Incidence Schemes

We now introduce the incidence schemes mentioned in the introduction. At first, we work in full generality over a base scheme $S$. But our main interest is the case $S = \mathrm{Spec}\,\mathbf{Z}$.

**Definition 4.1.** Let $W$ be a locally free $\mathcal{O}_S$-module, and let $G$ be a graph. We write $A_G(W)$ for the closed subscheme of

$$\mathrm{Sym}\, W \times_S \mathrm{Fun}(V, W)$$

representing the functor

$$\begin{aligned} T \rightsquigarrow \{(Q, f) \in &\mathrm{Sym}\, W_T \times \mathrm{Fun}(V, W_T)| \\ &Q(f(v), f(w)) = 0 \text{ if } e_{vw} \in E(G)\}. \end{aligned} \tag{4.1}$$

If $r$ and $k$ are integers, we write $A_G(W, r, k)$ for

$$A_G(W) \cap (\mathrm{Sym}_r(W) \times_S \mathrm{Fun}_k(V, W))$$

That is, $A_G(W, r, k)(T)$ consists of pairs $(Q, f) \in A_G(W)$ such that $Q$ has rank $r$ and $f$ has rank $k$. When $S = \mathrm{Spec}\,\mathbf{Z}$ and $W = \mathcal{O}^s_S$, we write $A_G(s)$ for $A_G(W)$ and $A_G(s, r, k)$ for $A_G(W, r, k)$.

The $A_G(W, r, k)$ form a stratification of $A_G(W)$ by locally closed subschemes. Note that $A_G(s, r, k)$ is empty unless $0 \le k < n$ and $0 \le r, k \le s$. Also note that $A_G(s, r, 0) = \mathrm{Sym}^s_r$, and $A_G(W, 0, k) = \mathrm{Fun}_k(V, W)$. Thus $|A_G(s, r, 0)|$ and $|A_G(s, 0, k)|$ are both in $\mathbf{Z}[q]$.

Now assume that $V(G) = \{v_1, \dots, v_n\}$ as in paragraph 3.1. Recall from the introduction that $\mathsf{CGraphs}_*$ is the $\mathsf{R}$-submodule of $\mathsf{CMot}$ spanned by the functions $Z_G$.

**Theorem 4.2.** (a) $A_G(n, n, n) \cong Z_G \times \mathrm{GL}_n$.

(b) $\mathsf{CGraphs}_*$ *is exactly equal to the* $\mathsf{R}$*-module generated by the functions* $|A_G(n, n, n)|$.

*Proof.* We first remark that (b) follows directly from (a) and the fact that $|\,\mathrm{GL}_n\,| \in \mathsf{S}$.

To prove (a) we let $W = \mathcal{O}^n_S$ with $S = \mathrm{Spec}\,\mathbf{Z}$. Then $\mathrm{Fun}_n(V, W) = \mathrm{GL}_n$. The map $(Q, f) \mapsto (f^t Q f, f)$ then identifies $A_G(n, n, n)$ with $Z_G \times \mathrm{GL}_n$. (Here $f^t$ denotes the transpose of $f$.) □

*Remark* 4.3. Let $Z_G(r)$ be the scheme consisting of all $n \times n$ symmetric bilinear forms of rank $r$ such that $M_{ij} = 0$ whenever $e_{ij} \in E$. These schemes have been studied implicitly in [5, 23]. In Stanley's notation, $h(G, r) = |Z_{G^o}(r)|$, and Chung and Yang call a graph $G$ *strongly admissible* if $Z_{G^o}(r)$ is polynomially countable for all $r$. An easy modification of the proof above shows that $A_G(n, r, n) \cong Z_G(r) \times \mathrm{GL}_n$.

## 5. Extensions of Bilinear Forms

In this section, we review a result of MacWilliams [17] counting the number of ways to extend a bilinear form of rank $r_1$ to a bilinear form of rank $r_2$. This count will be important in the next section for finding relations among the $A_G(s, r, k)$.

Let $Q$ be a fixed bilinear form on $\mathbf{F}_q^{d_1}$ with rank $r_1$. Let $C_Q(d_2, r_2, d_1, r_1)$ be the number of ways to extend $Q$ to a form on $\mathbf{F}_q^{d_2}$ of rank $r_2$. The following result is Lemma 4 of [17].

**Theorem 5.1.**

$$C_Q(d_1+1, r_2, d_1, r_1) = \begin{cases} q^{r_1} & r_2 = r_1 \\ q^{r_1+1} - q^{r_1} & r_2 = r_1 + 1 \\ q^{d_1+1} - q^{r_1+1} & r_2 = r_1 + 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Note that $C_Q(d_1 + 1, r_2, d_1, r_1)$ only depends on $d_1, r_2$ and $r_1$. By induction on $d_2 - d_1$, we can show that $C_Q(d_2, r_2, d_1, r_1)$ only depends on the integer parameters $d_2, r_2, d_1$ and $r_1$. Thus we simply write $C(d_2, r_2, d_1, r_1)$ for this number. We can also see by induction that the following recursion is satisfied

$$\begin{aligned} C(d_2, &r_2, d_1, r_1) = \\ &\sum_{j=0}^{2} C(d_2, r_2, d_1+1, r_1+j) C(d_1+1, r_1+j, d_1, r_1.) \end{aligned} \tag{5.1}$$

**Corollary 5.2.**

(a) $C(d_2, r_2, d_1, r_1)$ *is a polynomial in* $q$.
(b) $C(d_2, r_2, d_1, r_1) \neq 0$ *iff* $d_2 \geq r_2$, $d_1 \geq r_1$, *and* $0 \leq r_1 \leq r_2 \leq r_1 + 2(d_2 - d_1)$.

*Proof.* (a) follows directly from the recursion formula 5.1.

The necessity of the first two inequalities of (b) are obvious for dimension reasons (the rank of a bilinear form cannot be greater than the dimension of the ambient space.) Necessity of the third inequality follows from formula 5.1 by induction.

We prove the sufficiency of the the inequalities in (b) by induction on $i = d_2 - d_1$ using formula 5.1. We do not actually need this for the rest of the paper so the reader may safely skip the proof.

For $i = 0$ sufficiency is obvious. For $i = 1$ the sufficiency results from the fact that $C(d_1+1, r_2, d_1, r_1) \neq 0$ iff $r_2 \in [r_1, r_1+2]$ when $d_1 \neq r_1$ and iff $r_2 \in [r_1, r_1+1]$ when $d_1 = r_1$.

Now suppose sufficiency is known for $d_2 - d_1 < i$ and assume that $(d_2, r_2, d_1, r_1)$ satisfies the conditions in (b) with $d_2 = d_1 + i$ and $r_2 = r_1 + k$. By formula 5.1, $C(d_2, r_2, d_1, r_1) \neq 0$ if there is a $j$ such that both

(1) $C(d_1 + i, r_1 + k, d_1 + 1, r_1 + j) \neq 0$ and
(2) $C(d_1 + 1, r_1 + j, d_1, r_1) \neq 0$.

One computes that (2) is satisfied whenever $j \leq d_1 - r_1 + 1$. Using the the induction hypothesis, we see that (1) is satisfied for

$$k - 2i + 2 \leq j \leq \min(k, d_1 - r_1 + 1.)$$

So we need only show that $k - 2i + 2 \leq \min(k, d_1 - r_1 + 1)$. That $k - 2i + 2 \leq k$ only says that $i \geq 1$ which we are of course assuming. And $k - 2i + 2 \leq d_1 - r_1 + 1$ iff $(d_2 - d_1) + (d_2 - r_2) \geq 1$ which then follows from the fact that $d_2 \geq r_2$. □

## 6. Reduction Formulas

In this section we give three formulas which allow us to reduce questions about $A_G(s, r, k)$ for given $s, r$ or $k$ to questions where $s, r$ or $k$ is smaller. We also give a formula that allows us to connect the $A_G(s, r, k)$ to the $A_{DG}(s, r, k)$ where $DG$, as in the introduction, is the graph obtained from $G$ by adding a disjoint vertex.

In the proof of the following theorem and the next one we will pick a base field $\mathbf{F}_q$ at the beginning and then, for any scheme $X$ we encounter, write $X$ instead of $X(\mathbf{F}_q)$.

**Theorem 6.1.**

$$|A_G(s, r, k)| = |\operatorname{Gr}(k, s)| \sum_j C(s, r, k, j)|A_G(k, j, k)|.$$

*Proof.* Write $W = \mathbf{F}_q^s$. For every map $f \in W^V$ let $\langle f \rangle$ denote the span of the $f(v_i)$. The map $(Q, f) \mapsto \langle f \rangle$ fibers the set $A_G(s, r, k)$ over $\operatorname{Gr}(k, s)$. The fiber over a subspace $U \subset W$ is then the set $A_G(s, r, U)$ of $(Q, f) \in A_G(s, r, k)$ such that $\langle f \rangle = U$. The transitivity of the $\mathrm{GL}_s$ action on $\operatorname{Gr}(k, s)$ shows that the fibers all have the same number of points. Thus for any given $U$

$$\# A_G(s, r, k) = \# \operatorname{Gr}(k, s) \cdot \# A_G(s, r, U). \tag{6.1}$$

Now let $A_G(s, r, U, j)$ be the set of $(Q, f) \in A_G(s, r, U)$ such that $Q|_U$ has rank $j$. This decomposes $A_G(s, r, U)$ into disjoint subsets. Consider the map

$$\begin{aligned} p_U : A_G(s, r, U, j) &\rightarrow A_G(U, j, k) \text{ given by} \\ (Q, f) &\mapsto (Q|_U, f.) \end{aligned}$$

The fiber of $p_U$ above a given $(\overline{Q}, f) \in A_G(U, j, k)$ is $C_{\overline{Q}}(s, r, k, j)$. Thus

$$\# A_G(s, r, U, j) = C(s, r, k, j) \cdot \# A_G(k, j, k) \tag{6.2}$$

Summing over all the $j$ in Equation 6.2 and substituting the result into Equation 6.1 we obtain the desired result. □

**Theorem 6.2.**

$$|A_G(s, r, k)| = |Gr(r, s)| \sum_{l=0}^{r} q^{l(s-r)} |Hom_{k-l}(n - l, s - r)||A_G(r, r, l)|. \tag{6.3}$$

*Proof.* Write $W = \mathbf{F}_p^s$ and let $\Psi : A_G(W, r, k) \to \operatorname{Gr}(s - r, W)$ be the map associating to every $(Q, f)$ the kernel $\ker Q$ of $Q$. The fiber of $\Psi$ over a subspace $U \subset W$ is the set $A_G(W, r, k)_U$ consisting of all $(Q, f) \in A_G(W, r, k)$ with $Q|_U = 0$. The transitivity of the action of $\mathrm{GL}(W)$ on $\operatorname{Gr}(s - r, W)$ shows then that

$$\# A_G(W, r, k) = \# Gr(s - r, W) \cdot \# A_G(W, r, k)_U \tag{6.4}$$

Let $T = W/U$ and let $\pi : W \to T$ be the quotient map. $Q$ reduces in an obvious way to a form $\overline{Q}$ on $T$. In fact $Q \mapsto \overline{Q}$ is a one-one correspondence between bilinear forms on $W$ with kernel $U$ and non-degenerate bilinear forms on $T$.

Now stratify $A_G(W, r, k)_U$ by the dimension of $\langle \pi \circ f \rangle$. The stratum corresponding to span $= l$ maps to $A_G(T, r, l)$ by sending $(Q, f)$ to $(\overline{Q}, \pi \circ f)$. The fiber above a pair $(\overline{Q}, g)$ is identified with the set of maps $f : V \to W$ such that $\pi \circ f = g$ and $\langle f \rangle$ is of dimension $k$.

It is now elementary linear algebra to verify that the number of such $f$'s is given by

$$q^{l(s-r)} \operatorname{Hom}_{k-l}(n-l, s-r)(\mathbf{F}_q.) \tag{6.5}$$

To see this, pick a splitting $W = T \oplus U$ and visualize any $f$ as an $n \times s$-matrix whose upper $n \times r$-rectangle agrees with $g$. This upper rectangle is then a matrix of rank $l$, and without loss of generality we can assume that the first $l$ columns are linearly independent. It is then easy to check that $f$ has rank $k$ if and only if the bottom right $(n-l) \times (s-r)$-rectangle has rank $k-l$. Thus this bottom right rectangle can be chosen to be an arbitrary matrix in $\operatorname{Hom}_{k-l}(n-l, s-r)$. The bottom left $l \times (s-r)$-rectangle can then be arbitrarily chosen among $q^{l(s-r)}$ possible matrices.

Now putting (6.4) and (6.5) together we obtain the desired result. □

It is worth recording an important special case of Theorem 6.2.

**Corollary 6.3.** $A_G(s, r, s) = q^{l(s-r)} |\operatorname{Gr}(r, s)| |\operatorname{Hom}_{s-l}(n-r, s-r)|$.

*Proof.* To get a non-zero contribution corresponding to $l$ in the previous theorem we need

(1) $r \le s$.
(2) $l \le r$.
(3) $l \le k$.
(4) $l \ge k + r - s$.

In the case of the corollary $s = k$, so we get $l \le r$ and $l \ge r$. Hence $l = r$, and the formula reduces to exactly the above. □

We now give a reduction theorem relating the incidence schemes of $DG$ to those of $G$.

**Theorem 6.4.**

$$|A_{DG}(s, r, k)| = q^k |A_G(s, r, k)| + (q^s - q^{k-1}) |A_G(s, r, k-1)|.$$

*Proof.* Let $W = \mathbf{F}_q^s$. Let $f(V(DG)) \to W$ with $\langle f \rangle$ a $k$-dimensional subspace. The span of $f|_{V(G)}$ is either a $k$ or a $k-1$ dimensional subspace. If $\{v\} = V(DG) - V(G)$, counting the possibilities for $f(v)$ proves the theorem. □

## 7. The Module of a Graph

For a simple graph $G$ with $n$ vertices, let $M(G)$ be the $\mathsf{R}$-submodule of $\mathsf{CMot}$ generated by the $|A_G(s, r, k)|$. Let $M(G)_t$ be the submodule of $M(G)$ generated by the $|A_G(s, r, k)|$ for $s \le t$. Theorem 6.1 shows that $|A_G(s, r, k)| \in M(G)_k$. Thus we have a finite filtration

$$M(G) = M(G)_n \supset M(G)_{n-1} \supset \ldots \supset M(G)_0 = \mathsf{R}.$$

The goals of this section are to compute the structure of $M(G)$ and to show that, in fact, $M(G) \subset \mathsf{CGraphs}_*$. To do this we introduce three special schemes: $K_G(s) = A_G(s,s,s)$, $J_G(s) = \cup_k A_G(s,s,k)$ and $H_G(s) = A_G(n,s,n)$. Note that $J_G(s)$ consists of the scheme of all pairs $(Q,f) \in A_G(s)$ with $Q \in \mathrm{Sym}^s_s$; that is, there is no restriction on the rank of $f$. Note also that $K_G(n) = H_G(n)$.

**Theorem 7.1.** (a) $|A_G(s,r,k)| \in M(G)_d$ *for* $d = \min(s,r,k)$.
(b) $M(G)_t$ *is spanned as an* R*-module by the* $|K_G(s)|$ *for* $s \le t$.
(c) $M(G)_t$ *is spanned as an* R*-module by the* $|J_G(s)|$ *(resp. by* $H_G(s)$*) for* $s \le t$.

*Proof.* (a) and (b.) Apply Theorem 6.1 to obtain an expression for $|A_G(s,r,k)|$ as a $\mathbf{Z}[q]$-linear combination of terms of the form $|A_G(k,j,k)|$ with $j \le \min(r,k) \le s$. Then apply Corollary 6.3 to obtain an expression for each $|A_G(k,j,k)|$ as a $\mathbf{Z}[q]$-linear combination of terms of the form $|K_G(j)|$.

(c) To see that the $|J_G(s)|$ span, note that (a) implies that $|J_G(s)| \equiv |K_G(s)|$ modulo $M(G)_{s-1}$. To see that the the $|H_G(s)|$ span, use the fact that $|H_G(s)| = \sigma|K_G(s)|$ for $\sigma \in \mathsf{S}$, a consequence of Corollary 6.3. □

Our interest in the $H_G(s)$ is based on the following lemma, which allows us to compare the $|H_G(s)|$ to the $|H_{DG}(s)|$. The lemma is essentially a translation of Theorem 5.1 of [23] into our language. As there are two graphs involved in the lemma, we write $n_G$ for the cardinality of $V(G)$.

**Lemma 7.2.** *For* $r \le n_G + 1$,

$$|H_{DG}(r)| = a_G(r)|H_G(r)| + b_G(r)|H_G(r-1)| + c_G(r)|H_G(r-2)| \tag{7.1}$$

*with*

$$\begin{aligned} a_G(r) &= q^{n_G+r}(q^{n_G+1}-1) \\ b_G(r) &= q^{n_G+r-1}(q^{n_G+1}-1)(q-1) \\ c_G(r) &= q^{n_G}(q^{n_G+1}-1)(q^{n_G+1}-q^{r-1}) \end{aligned}$$

*all polynomials in* S.

*Proof.* By the Theorem 6.4,

$$|A_{DG}(n_G+1, r, n_G+1)| = (q^{n_G+1} - q^{n_G})|A_G(n_G+1, r, n_G)|.$$

Now applying Theorem 6.1 to $|A_G(n_G+1,r,n_G)|$ and expanding out $|\operatorname{Gr}(n_G, n_G+1)|$ in terms of $q$ gives the result.

The polynomials $a_G, b_G$ and $c_G$ in the theorem are clearly in S as long as they are nonzero. Inspection shows that this is the case under the assumption that $r \le n_G + 1$. □

There is a simpler identity relating $J_G(s)$ to $J_{DG}(s)$.

**Proposition 7.3.** $|J_{DG}(s)| = q^s|J_G(s)|$.

*Proof.* The obvious map $J_{DG}(s)(\mathbf{F}_q) \to J_G(s)(\mathbf{F}_q)$ restricting $f$ from $V(DG)$ to $V(G)$ has fiber $\mathbf{F}_q^s$. □

A direct consequence of Proposition 7.3 and Theorem 7.1 (c) is the following.

**Theorem 7.4.** $M(DG) = M(G)$.

We are now ready to prove the main theorem of this section.

**Theorem 7.5.** $M(G)$ *is equal to the* R*-module spanned by the functions* $|Z_{D^kG}|$ *for* $k \ge 0$. *In particular,* $M(G) \subset \mathsf{CGraphs}_*$.

*Proof.* For the proof, let $N(G)$ be the $\mathsf{R}$-module spanned by the functions $|Z_{D^kG}|$ for $k \geq 0$. Since $|K_G(n_G)| = |Z_G||\operatorname{GL}_{n_G}|$, $|Z_G| \in M(G)$. Thus it follows from Theorem 7.4 that $N(G) \subset M(G)$. To prove that $M(G) \subset N(G)$ we use Lemma 7.2 and an inductive argument.

By Theorem 7.1 (c), it will be enough to show that $|H_G(s)| \in N(G)$ for all $s$. Since $|H_G(n_G)| = |K_G(n_G)|$ this is obvious for $s = n_G$. Now by Lemma 7.2

$$|H_{DG}(n_G + 1)| = b_G|H_G(n_G)| + c_G|H_G(n_G - 1)| \tag{7.2}$$

with $b_G, c_G \in \mathsf{S}$. (The first term on the right hand side of (7.1) vanishes because $H_G(n_G + 1)$ is empty.) We know that $|H_{DG}(n_G + 1)|$ and $|H_G(n_G)|$ are in $N(G)$. Thus $|H_G(n_G - 1)| \in \mathsf{S}$.

We then assume inductively that $|H_G(n_G - i)| \in N(G)$ for all $i \leq a$ and for all graphs $G$. Another application of Lemma 7.2 shows us that

$$\begin{aligned} |H_{DG}(n_G - (a-1))| =& a_G|H_G(n_G - (a-1))| + b_G|H_G(n_G - a)| \\ &+ c_G|H_G(n_G - (a+1))|. \end{aligned} \tag{7.3}$$

By induction, the left-hand side and the two first terms on the right hand side are in $N(G)$. Thus, as $c_G \in \mathsf{S}$, $|H_G(n_G - (a+1))| \in N(G)$ as well. □

## 8. Matroid Theory

A matroid $M$ consists of a finite set $E$ called the edges of the matroid and a rank function $\rho : 2^E \to \mathbf{N}$ satisfying the following axioms

(1) For $X \subset E$, $\rho(X) \leq \#X$.
(2) For $X \subset Y \subset E$, $\rho(X) \leq \rho(Y)$.
(3) For any $X, Y \subset E$,

$$\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y.) \tag{8.1}$$

The integer $\rho(E)$ is said to be the *rank* of the matroid.

Matroids were introduced by H. Whitney [29] as a simultaneous generalization of matrices and graphs. An excellent modern reference for matroid theory is [21].

8.0.1. *Representability.* A matroid $M$ of rank $r$ is said to be representable over a field $K$ if there is a function $f : E \to K^r$ such that the dimension of the span of the set $f(X)$ is equal to $\rho(X)$ for all $X \subset E$.

8.0.2. *Matroids from matrices.* To every subset $E \subset K^s$ there is naturally a matroid $M$ representable over $K$ given by setting $\rho(X) = \dim\langle X\rangle$ for every $X \subset E$.

Let $K^{n+1} - \{0\} \overset{\pi}{\mapsto} \mathbf{P}^n(K)$ be the natural map taking a nonzero vector in $v \in K^{n+1}$ to the line $Kv$. Suppose $E \subset \mathbf{P}^n(K)$. Then any set-theoretic splitting $\sigma : \mathbf{P}^n(K) \mapsto K^{n+1}$ gives a subset $\sigma(E)$ of $K^{n+1}$ and, thus, defines a matroid. It is easy to see that this matroid is independent of the splitting $\sigma$. Thus, since such splittings always exist, $E$ defines a matroid.

8.0.3. *Representation schemes.* For any matroid $M$ and a locally free sheaf $W$ over a base $S$, let $X(M, W)$ be the subscheme of $\operatorname{Fun}(E, W)$ consisting of all $f$ whose restrictions to $\operatorname{Fun}(X, W)$ lie in $\operatorname{Fun}_{\rho(X)}(X, W)$ for all $X \subset E$. This is the scheme of representations of $M$ in $W$. When $S = \operatorname{Spec} \mathbf{Z}$ and $W = \mathcal{O}_S^s$, we write $X(M, s)$ for $X(M, W)$ as in the introduction. For a field $K$, $X(M, s)(K)$ is the set of all maps $f : E \to K^s$ such that $\dim\langle f(X)\rangle = \rho(X)$ for all $X \subset E$. That is, $X(M, s)(K)$ is the set of all representations of $M$ in $K^s$. When $r$ is the rank of $M$, we write $X(M)$

for $X(M,r)$. $X(M)(K)$ is non-empty if and only if $M$ is representable over $K$. Clearly $X(M,s)(K)$ is non-empty if and only if $s \geq r$ and $X(M)(K)$ is non-empty.

**Definition 8.1.** Let $\mathsf{CMatroids}$ be the $\mathsf{R}$-module generated by all functions of the form $|X(M)|$.

*Remark* 8.2. It is easy to see that $|X(M,s)| = |\operatorname{Gr}(r,s)||X(M)|$. Thus $\mathsf{CMatroids}$ is the same as the $\mathsf{R}$-module generated by all functions of the form $|X(M,s)|$.

In the next section we show that $\mathsf{CMatroids} \subset \mathsf{CGraphs}_*$.

## 9. A Counterexample to Kontsevich's Conjecture

Let $G$ be a graph, $V$ the set of its vertices, $U \subset 2^V$. A function $\pi : U \mapsto \mathbf{N}$ will be called a *partially defined rank function for $V$*. Notice that the data of a partially defined rank function $\pi$ determines $U = \operatorname{dom}(\pi)$. Associated to every such function we have a scheme defined as follows:

**Definition 9.1.** $J_G(s,\pi)$ is the scheme of all of all $(Q,f) \in J_G(s)$ such that $f_{|H}$ has rank $\rho(H)$ for all $H \in \operatorname{dom}\pi$.

**Theorem 9.2.** *For every $G$ and every partially defined rank function $\pi$ for $V(G)$, $|J(s,\pi)| \in \mathsf{CGraphs}_*$.*

*Proof.* The proof is by induction on the cardinality of $\operatorname{dom}(\pi)$. If $\operatorname{dom}(\pi)$ is empty, $J_G(s,\pi) = J_G(s)$. Thus the result follows from Theorem 7.5.

Now assume the result holds for all graphs $G$ and all $\pi$ such that $\#\operatorname{dom}\pi \leq a$. Let $W \subset 2^V$ be a set of subsets with $a+1$ elements, let $H \in W$ and let $U = W - \{H\}$. Let $\pi : U \to \mathbf{N}$ be a partially defined rank function, and let $\pi_i : W \to \mathbf{N}$ be the extension of $\pi$ to $W$ such that $\pi_i(H) = s - i$. Clearly, any partially defined rank function with domain $W$ is of the form $\pi_i$ for some $\pi : U \to \mathbf{N}$ and some $i \in [0,s]$.

Now for each $t \in \mathbf{N}$ we define a graph $G_t$ as follows: $G_t$ is the graph obtained from $G$ by adjoining $t$ disjoint vertices $y_1,\dots,y_t$ and connecting each of the $y_i$ by edges only to the vertices in $H$. Thus $V(G_t) = V(G) \cup Y$ where $Y = \{y_1,\dots,y_t\}$, and

$$E(G_t) = E(G) \cup \{e_{hy}\}_{\substack{h\in H\\ y\in Y}}.$$

Since $V(G) \subset V(G_t)$, $U \subset 2^{V(G_t)}$, we can consider $\pi$ as a partially defined rank function for $V(G_t)$.

The result will follow from the following equation:

$$|J_{G_t}(s,\pi)| = \sum_{i=0}^{s} q^{ti}|J_G(s,\pi_i)| \tag{9.1}$$

To see that the equation holds, note that we can stratify the $\mathbf{F}_q$ points of $J_{G_t}(s,\pi)$ according to the dimension of the span of $f(H)$. Let $J_{G_t}(s,\pi)_i$ be the stratum where this dimension is $s-i$. This stratum maps to $J_G(s,\pi_i)$ by restricting $f$ from $V(G_t)$ to $V(G)$. The fiber of map above any point $(Q,f)$ is an affine space $\mathbf{A}^{ti}$. This is because the only condition on the $f(y_i)$ is that they be orthogonal to the span of $f(H)$. Thus, as the bilinear form $Q$ is always nondegenerate, they must lie in a linear subspace of dimension $i$.

To complete the proof, note that by varying the $t$ from 0 to $s$ we obtain a system of equations for the $|J_G(s,\pi_i)|$ in terms of the $|J_{G_t}(s,\pi)|$. Solving this system for the $J_G(s,\pi_i)$ using Cramer's rule, we have to invert a Vandermonde determinant which lies in $\mathsf{S}$. Thus, as we assumed by induction that $|J_{G_t}(s,\pi)|$ lies in $\mathsf{CGraphs}_*$, it follows that each $|J_G(s,\pi_i)|$ lies in $\mathsf{CGraphs}_*$ as well. □

This leads to the following theorem:

**Theorem 9.3.** $\mathsf{CMatroids} \subset \mathsf{CGraphs}_*$.

*Proof.* Let G be a discrete graph (that is $E(G)$ is empty.) In this case, if $\pi$ is a partially defined rank function then

$$|J(s,\pi)| = |\operatorname{Sym}^s_s||L(s,\pi)| \tag{9.2}$$

where $L(s,\pi)$ is the scheme consisting of all $f \in \operatorname{Fun}(V, \mathcal{O}^s_{\operatorname{Spec}\mathbf{Z}})$ such that $f$ restricts to $\operatorname{Fun}_{\pi(H)}(H, \mathcal{O}^s_{\operatorname{Spec}\mathbf{Z}})$ for all $H \in \operatorname{dom}(\pi)$. To see this, note that the definition of $J(s,\pi)$ makes it clear that Q does not enter in the definition of the $J$'s for the discrete graph. And only the vertex set $V$ is needed for the definition of the $L$'s since $G$ is discrete.

As $\operatorname{Sym}^s_s \in \mathsf{S}$, it follows that the $L$'s are all in $\mathsf{CGraphs}_*$. Now note that, if $M$ is a matroid, with rank function $\rho : 2^E \to \mathbf{N}$, then $X(M,s) = L(s,\rho)$.

□

It is now possible to see directly that Conjecture 0.4 and thus Conjecture 0.1 are false. Let $M$ be the Fano matroid. This is a rank 3 matroid whose edge set $E$ is the set $\mathbf{P}^2(\mathbf{F}_2)$. This matroid is representable over a field $\mathbf{F}_q$ if and only if $2|q$ (see [28] Chapter 9.)

Thus the function $|X(M)|$ is supported on the set of $q$ such that $2|q$. It follows that $|X(M)|$ cannot be a rational function. And this contradicts Conjecture 0.4 by Theorem 9.3.

*Remark* 9.4. By unravelling the induction used in the proof Theorem 9.3, we can be a bit more specific about the nature of the counterexamples arising from the Fano matroid. The first step is to use the Fano to find a graph $G$ such that $J_G(3)$ is not polynomially countable. Let $V = \{1,\dots,7\}$, and view $V$ as the vertex set of a disjoint graph. Let $\mathcal{F}$ be the set of functions from $2^V$ to $\{0,1,2,3\}$. For each such function $\phi \in \mathcal{F}$ we construct a bipartite graph $G_\phi$ as follows: For every $H \subset V$, we add $\phi(H)$ new vertices to $V$, and we connect each of these new vertices by an edge to the vertices of $H$. In the end, we have a bipartite graph with $\sum_{H\subset V}\phi(H) + 7$ vertices and $\sum_{H\subset V}\phi(H)|H|$ edges. An inspection of the induction from the proof of the theorem shows that, if we range over all $\phi \in \mathcal{F}$, we are guaranteed to produce a graph $G_\phi$ such that $J_{G_\phi}(3)$ is not polynomially countable. Unfortunately, $|\mathcal{F}| = 4^{2^{|V|}} = 2^{256}$, thus, we are far from having an explicit graph. Also note that the set of graphs produced from a given matroid depends only on the order of the matroid. Thus, we would obtain the same set of graphs from any matroid of order 7.

To produce an explicit counterexample to Kontsevich's conjecture we must work even harder. Once we find a $G$ such that $J_G(3)$ is not polynomially countable, we know that $Z_{D^kG}$ is not polynomially countable for some $k$. The induction used to prove Theorem 7.5 allows $k$ to range from 0 to the number of vertices of $G$. Thus, if we use a $G_\phi$ as above, we could have as many as $3\cdot 2^7 + 7$ graphs to search through. However, once we do have a graph $G$ with $Z_G$ not polynomially countable, $G^o$ will be a counterexample to Conjecture 0.4. Thus we learn that there is a bipartite graph whose complement is a counterexample to Conjecture 0.4.

Set $C = (G^o)^*$. $C$ is then a counterexample to Stanley's version of Kontsevich's conjecture, Conjecture 0.3. To find a counterexample to Kontsevich's original conjecture we have one more step which, following Stanley's proof of the equivalence of the two conjectures, involves searching through all graphs of the form $(C/S) - T$ with $S \subset E(C)$ and $T \subset E(C/S)$. Clearly, this is also a very large number of graphs. Thus, the interesting problem of finding an explicit counterexample to Kontsevich's conjecture is totally open.

## 10. Mnëv-Sturmfels Universality

Our objective in this section is to show that $\mathsf{Matroids} = \mathsf{Mot}$ and thus that $\mathsf{CGraphs}_* = \mathsf{CMot}$ completing the proof of Theorem 0.5. This will follow from the known results on the Matroid representation problem.

We saw in the previous section that $|L(k,\pi)|$ were in $\mathsf{CGraphs}_*$ even if $\pi$ is only partially defined. It suffices therefore to show that the $\mathsf{R}$-module generated by all functions of the form $|L(k,\pi)|$ is all of $\mathsf{CMot}$. This was in essence proved by Mnëv [19, 20] as the unoriented matroid component of a more difficult theorem concerning the representation spaces of oriented matroids [1]. (See also [11, 22].) It was independently proved by Bokowski and Sturmfels [3, 26]. Moreover, the idea of the proof using von Staudt's "algebra of throws" goes back at least to [16] (see [15] for an enlightening explication.) However, as we have been unable to extract a proof of the exact statement we need from the literature, we give a sketch of the proof in our context.

**Theorem 10.1** (Mnëv,Sturmfels)**.** *If $X$ is a quasi-projective scheme of finite type over $\mathbf{Z}$, then there is a set $V$, a set of subsets $W$ of $V$, a function $\pi : W \to \mathbf{Z}$, and an element $\sigma \in \mathsf{S}$ so that*

$$\sigma|X| = |L(3,\pi)|.$$

*Remark* 10.2.

1. The theorems in Matroid theory are not in such a direct form because, in Matroid theory we are committed to declare the rank of all the subsets of $V$. Our partially defined $\pi$ does not have this problem. By inclusion-exclusion principles the $\mathsf{R}$-module generated by all functions of the form $|L(k,\pi)|$ where $\pi$ may only be partially defined is same as the $\mathsf{R}$-module generated by all functions of the form $|L(k,\pi)|$ where $\pi$ is defined on all subsets of $V$.
2. Note that any scheme of finite type/$\mathbf{Z}$ is a finite disjoint union of quasiprojective schemes/$\mathbf{Z}$.

*Proof.* The proof follows essentially from the following observations

1. 4 elements in $P^2$ such that any 3 are linearly independent can by a unique automorphism of $P^2$ in PGL(2), be assumed to be $(1,0,0), (0,1,0), (0,0,1)$ and $(1,1,1)$.
2. if given two points $(x,0,1)$ and $(x',0,1)$ on the $X - axis$, then by drawing lines alone through the 4 points above and these two points, we can locate $(x+x',0,1)$,$(xx',0,1)$,$(-x,0,1)$. The intersection of two lines is a point which lies on both lines. We can code this using Matroids by introducing

[1] Mnëv showed that any variety over a prime field $k = \mathbf{Q}$ or $\mathbf{F}_p$ has a nonempty open subvariety which is isomorphic to the variety of projective representations of a rank 3 matroid with 4 fixed edges in the matroid mapping to 4 fixed points in $\mathbf{P}^2(k)$. Using this statement, Theorem 10.1 can be proved by noetherian induction.

a new point and adding linear dependence conditions on this point and points on the two lines. These constructions can be found for example in the proof of Theorem 2.2 of [3] .

(3) Iterating these constructions, given $(x_1, x_2, \ldots, x_n)$ we can determine the points $(f(x_1, x_2, \ldots, x_n), 0, 1)$, where $f$ is a polynomial with integer coefficients by just drawing lines starting from the configuration of the four given points and the points $(x_i, 0, 1)$. Setting $f(x_1, \ldots, x_n)$ either equal to zero or not equal to zero is just another spanning condition: A condition on whether
$$(f(x_1, x_2, \ldots, x_n), 0, 1), (0, 0, 1)$$
is linearly independent or not.

(4) The cone over any quasi-projective scheme/$\mathbf{Z}$ can be written as a set of equalities and a set of nonequalities in a finite set of variables $(x_1, \ldots, x_n)$. Note that we can also have conditions of the form $n = 0$ in the list.

(5) The cone over any quasi-projective scheme/$\mathbf{Z}$ can be written as a set of equalities and a set of nonequalities in a finite set of variables $(x_1, \ldots, x_n)$. Note that we can also have conditions of the form $n = 0$ in the list.

(6) From the previous considerations we obtain a set $S = \{P_1, P_2, P_3, P_4, Q_1, \ldots Q_n, T_1, \ldots T_l\}$ of cardinality (say) $c$, where the $P_i$ correspond to $(1,0,0), (0,1,0), (0,0,1)$ and $(1,1,1)$, The $Q_j$ correspond to $(x_j, 0, 1)$, and the $T$'s intermediate points in the constructions. The preceding discussion also gives a partially defined rank function $\pi$ on this set (this function, and the set $S$ is determined by the equations and nonequalities defining the scheme $X$), and a map $p : L(k,\pi) \mapsto (P^2_{\mathbf{Z}})^4$ by the image of the $P_i$, so that the fiber over $((1,0,0),(0,1,0),(0,0,1),(1,1,1))$, is equal to $\mathbf{G}_m^c \times_{\mathbf{Z}} X$ as a scheme over $\mathbf{Z}$. We can identify the image of $p$ with PGL(3). Moreover, over this image, $p$ is a Zariski locally trivial fibration. Thus we have
$$[L(k,\pi)] = [\mathrm{PGL}(3)][\mathbf{G}_m^c][X]. \tag{10.1}$$

□

## 11. Forests

In this section we prove that $|J_G(s)| \in \mathbf{Z}[q]$ whenever $G$ is a forest. It follows that $M(G) = \mathbf{Z}[q]$ for such graphs. To do so we need to introduce a two operations on graphs.

Let $v \in V(G)$. We obtain a graph $I_v(G)$ by adding one edge $e$ connected to $v$ and one new vertex $w$ connected to $e$. That is, we insert an edge at $v$. Clearly, a graph is tree if and only if it can be obtained from the graph with one vertex by successive applications of $I_v$ for various $v$. A graph is a forest if and only if it can be obtained from the empty graph $\emptyset$ by successive applications of $I_v$ and the operation $D$. We write $D_n$ for the graph $D^n\emptyset$.

We define $R_v$ to be the graph obtained from $G$ by deleting $v$ and all edges meeting it. Note that if $G$ is a forest and $v$ is any vertex in $G$, $R_vG$ is also a forest.

**Theorem 11.1.** *Let $G$ be a graph with $v \in V(G)$.*

(a) $|J_{D_n}(s)| = q^{ns} |\operatorname{Sym}^s_s|$.
(b) $|J_{DG}(s)| = q^s J_G(s.)$
(c) $|J_{I_v}(s)| = q^{s-1} \left(J_G(s) + (q-1) J_{R_vG}(s)\right)$

*Proof.* (b) is a restatement of Proposition 7.3. For (a) assume first that $n = 0$. Then tracing through the definitions one sees that $J_\emptyset(s) = \mathrm{Sym}^s_s$. The rest of (a) follows by induction from (b).

For (c) we work over $\mathbf{F}_q$ and consider the map $\pi : J_{I_v(G)}(s) \to J_G(s)$ given by $(Q, f) \mapsto (Q, f|_{V(G)})$. The fiber of $\pi$ above a point $(Q, g) \in J_G(s)$ depends on whether $g(v)$ is 0 or not. Let $J^0_G(s)$ (resp. $J^\times_G(s)$) be the set where $g(v) = 0$ (resp. $g(v) \neq 0$.) Above a point $(Q, g) \in J^0_G(s)$ the fiber of $\pi$ will have $q^s$ points. Above a point $(Q, g) \in J^\times_G(s)$ the fiber will have $q^{s-1}$ points since $Q$ is non-degenerate. Thus

$$|J_{I_v(G)}(s)| = q^{s-1}|J^\times_G(s)| + q^s|J^0_G(s)| \tag{11.1}$$

The result now follows from the observation that $|J^0_G(s)| = |J_{R_vG}(s))|$. □

**Corollary 11.2.** *For $F$ a forest, $|Z_F| \in \mathbf{Z}[q]$.*

*Proof.* An easy induction using Theorem 11.1 shows that $|J_F(s)| \in \mathbf{Z}[q]$ for any $s$. Thus $M(F) = \mathsf{R}$. It follows from Theorem 7.5 that $|Z_F| \in \mathsf{R}$. But this implies that $|Z_F| \in \mathbf{Z}[q]$ by Proposition 1.3. □

The next corollary follows from Theorem 11.1 and the results in Section 3.1.

**Corollary 11.3.** *Let $F$ be a forest with $r$ vertices contained in a complete graph $K_s$. Let $G = K_s - F$.*

(1) $|Z^o_G| \in \mathbf{Z}[q]$.
(2) *If $s > r$, then $|X_G| \in \mathbf{Z}[q]$.*

## 12. Geometric Motives

The purpose of this section is to develop the necessary machinery for showing that the results stated thus far in the context of combinatorial motives continue to hold in the geometric context. The main tool we need is an efficient apparatus for converting fibrations of the type used to prove the results of section 6 into formulas in $\mathsf{GeoMot}$. The fibrations that automatically yield such formulas are called *piecewise Zariski fibrations* in the motivic integration literature [8]. If

$$F \to X \to Y$$

is a piecewise Zariski fibration then one obtains a formula $[X] = [F][Y]$. To write such a formula one must know $F$ as a scheme. It will be useful for us to have a slightly more general notion which we call *motivic fibrations* which allow us to write formulas where the fiber $F$ is only known as a motive.

To say precisely what we mean by a motivic fibration, we must first develop the notion of motives over a general Noetherian base. This is not much more difficult that the notion of a motive over $\mathbf{Z}$, and we will see that it allows more flexibility. We will also see that there are tools for reducing questions about motives over a base to questions about motives over fields.

Let $S$ be a Noetherian scheme. (We make the standing assumption that all base schemes are noetherian.) We write $\mathsf{GeoMot}^+_S$ for the ring of motives over $S$. As an abelian group $\mathsf{GeoMot}^+_S$ is generated by the symbols $[X \xrightarrow{f} S]$ for $X$ a scheme of finite type over $S$ under the relations

(a) $[X \xrightarrow{f} S] = [Y \xrightarrow{g} S]$ if $X \cong Y$ as $S$-schemes.

(b) If $V$ is closed in $X$ and $U = X - V$, then

$$[X \xrightarrow{f} S] = [U \xrightarrow{f_{|U}} S] + [V \xrightarrow{f_{|V}} S].$$

When the map $f$ is clear, we write $[X]_S$ or simply $[X]$ for $[X \xrightarrow{f} S]$. If $S = \operatorname{Spec} R$, we also write $[X]_R$.

It follows from (a) and (b) that $[X]_S = 0$ when $X$ is the empty scheme.

A structure of a commutative ring is induced on $\mathsf{GeoMot}_S^+$ by setting $[X][Y] = [X \times_S Y]$. It is easy to check that this operation is well-defined with respect to the relations (a) and (b.) The unit in this ring is $[S]_S$.

Note that $[X]_S = [X_{red}]_S$. This is because $X_{red}$ is a closed subscheme of $X$ whose complement is empty. Using this fact and noetherian induction, we obtain a well-defined class $[C]_S$ for any constructible subset $C \subset X$ of an $S$-scheme.

12.1. **Base Change.** Suppose $u : S \to T$ is a morphism. Base change then provides a ring homomorphism $u^* : \mathsf{GeoMot}_T^+ \to \mathsf{GeoMot}_S^+$ explicitly given by

$$[X]_T \mapsto [X \times_T S]_S \tag{12.1}$$

Suppose $S$ is of finite type over $T$. Then there is a group homomorphism $u_* : \mathsf{GeoMot}_S^+ \to \mathsf{GeoMot}_T^+$ given by

$$[X \xrightarrow{f} S] \mapsto [X \xrightarrow{u \circ f} T] \tag{12.2}$$

The ring of motives over $S$ is 'topological', that is depends only on the reduced scheme structure of $S$.

**Lemma 12.1.** *The map $i : S_{red} \to S$ induces an isomorphism $i^* : \mathsf{GeoMot}_S^+ \to \mathsf{GeoMot}_{S_{red}}^+$, with inverse given by $i_* : \mathsf{GeoMot}_{S_{red}}^+ \to \mathsf{GeoMot}_S^+$.*

*Proof.* This is a consequence of the fact that

$$(X \times_S S_{red})_{red} = X_{red}.$$

□

12.2. **Motives over finite fields.** Suppose $k$ is a finite field. Then there is a map $\mathsf{GeoMot}_k^+ \to \mathbf{N}$ given by $[X] \mapsto \#X(k)$. If $S = \operatorname{Spec} \mathbf{Z}$, there is an evaluation map from $\mathsf{GeoMot}_S^+$ to the set of functions from $\mathcal{Q}$ to $\mathbf{Z}$.

**Example 12.2.** Let $X$ and $S$ be two copies of the multiplicative group scheme $\mathbf{G}_m$, and let $f : X \to S$ be the squaring map. Then $[X]_{\operatorname{Spec} \mathbf{Z}} = [S]_{\operatorname{Spec} \mathbf{Z}}$ but $[X]_S \neq [S]_S$. To see this, let $k$ be any finite field and let $\eta : \operatorname{Spec} k \to \mathbf{G}_m$ be the map corresponding to the point $1 \in \mathbf{G}_m(k)$. Then $\eta^*[X]_S$ has two points while $\eta^*[S]_S$ has only one.

12.3. **Zariski Fibrations.** A map $X \xrightarrow{f} Y$ of schemes over a base $S$ is said to be a *Zariski fibration* with fiber $F$ if there is a covering of $Y$ by open sets $Y_i$ such that

$$X \times_Y Y_i \cong F \times_S Y_i. \tag{12.3}$$

We remark that the pull-back of a Zariski fibration with fiber $F$ is also a Zariski fibration with fiber $F$. Also, if $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are Zariski fibrations over $S$ with fibers $A$ and $B$ respectively, then $g \circ f$ is a Zariski fibration with fiber $A \times_S B$.

**Proposition 12.3.** *Let $u : Y \to S$ be a structure map and suppose $X \xrightarrow{f} Y$ is a Zariski fibration over $S$ with fiber $F$. Then*

$$[X]_Y = u^*[F]_S. \tag{12.4}$$

*Proof.* Let $Y_i$ be any open cover of $Y$. Set $X_i = X \times_Y Y_i$. As $X_i$ is an open cover of $X$, $[X]_Y = \sum [X_i]_Y$.

Now pick an open cover $\{Y_i\}$ so that $X_i \cong F \times_S Y_i$. Then $[X]_Y = \sum [F \times_S Y_i]_Y$. On the other hand, $[F \times Y]_Y = \sum [F \times_S Y_i]_Y$. Thus $[X]_Y = u^*[F]_S$. □

12.4. **Motivic Fibrations.** For a sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{u} S$, we say that $f$ is a *motivic* fibration over $S$ with fiber $A \in \mathsf{GeoMot}_S$ if $[X \xrightarrow{f} Y]_Y = u^*A$ for $A \in \mathsf{GeoMot}_S$.

A Zariski fibration is a motivic fibration. If $X \xrightarrow{f} Y$ is a morphism and $V$ is a closed subset of $X$ with complement $U$ such that $V \xrightarrow{f_{|V}} Y$ and $U \xrightarrow{f_{|U}} Y$ are motivic fibrations with fibers $A$ and $B$ respectively, then $X \xrightarrow{f} Y$ is a motivic fibration with fiber $[A] + [B]$. The property of being a motivic fibration is invariant under base change. That is, if $X \to Y$ is a motivic fibration over $S$ with fiber $A$ and $Y' \to Y$ is any map, then $X \times_Y Y' \to Y'$ is a motivic fibration with fiber $A$.

**Proposition 12.4.** *If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{u} S$ is a sequence of morphisms and both $f$ and $g$ are motivic fibrations over $S$ with fibers $A$ and $B$ respectively, then $g \circ f$ is a motivic fibration with fiber $A \times B$.*

*Proof.* There are $S$-schemes $A_i$ and $B_j$ such that $\sum [A_i] = A$ and $\sum [B_j] = B$. By assumption, we have $[X]_Y = (u \circ g)^* A$ and $[Y]_Z = u^*[B]$. Thus $[X]_Y = \sum [A_i \times_S Y]_Y$. It follows that $[X]_Z = \sum [A_i \times_S Y]_Z$ and, thus, $[X]_Z = \sum_{ij} [A_i \times_S B_j \times_S Z]$. □

**Lemma 12.5.** *Let $M$ be a motive over a reduced and irreducible base $S$. If $M_{K(S)} = 0$, then there is a non-empty open set $U \subset S$ such that $M_U = 0$ (where $K(S) =$ field of fractions of the coordinate ring of $S$, also called the function field of $S$).*

Set $K = K(S)$. The lemma follows from two considerations:

(1) If $f : X \to Y$ is a map of schemes of finite type over $S$ such that $f_K$ is an isomorphism, then there is a nonempty open set $U \subset S$ such that $f_U$ is an isomorphism.
(2) If $X_K = A \cup B$ with $A$ a closed subscheme and $B$ its complement, then there is a nonempty open subset $U$ of $S$ and two disjoint schemes $A'$ and $B'$ with $A'$ closed and $B'$ its complement such that $A = A'_K$ and $B = B'_K$.

**Theorem 12.6.** *Let $f : X \to Y$ be a map of two schemes of finite type over a base $S$. Suppose there exists a motive $M$ over $S$ such that, for every every field valued point $\eta \in Y(K)$, $f_K : X_K \to \operatorname{Spec} K$ is a motivic fibration with fiber $M_K$. Then $f : X \to Y$ is a motivic fibration with fiber $M$.*

*Proof.* We can assume without loss of generality that $Y$ is reduced and irreducible. Let $K$ be the function field of $Y$. Then the hypothesis of the theorem implies that $[X_K]_K - M_K = 0$. Thus there exists a nonempty open set $U \subset Y$ such that $[X_U]_U - M_U = 0$ Let $V = Y - U$. Then $f_V : X_V \to V$ again satisfies the hypothesis of the theorem, hence, the theorem follows by noetherian induction. □

## 13. Vector Bundles and Extensions

The purpose of this section is to show that the formulas of Section 5 involving extensions of quadratic forms continue to hold in a motivic setting with $q$ replaced with $\mathbf{L}$. Once this is done, we can obtain a motivic version of MacWilliam's formula for $\mathrm{Sym}^n_r$. We also build the groundwork for motivic versions of the formulas in Section 6.

13.1. **Bundle and Cobundle Categories.** A *vector bundle* $M$ of rank $m$ over a base $S$ is a locally free coherent sheaf of rank $m$ on $S$. We consider two categories of vector bundles which are dual to each other. In the first category $\mathsf{CoBun}_S$, the objects are vector bundles over $S$ of arbitrary rank, and the morphisms are surjective morphisms $M \twoheadrightarrow N$ of coherent sheaves. In the second category $\mathsf{Bun}_S$, the objects are the same as in $\mathsf{CoBun}_S$, but the morphisms are injective morphisms $i : N \rightarrowtail M$ such that $M/i(N)$ is locally free of constant rank. The contravariant functor $M \to M^\vee$ sets up an equivalence between the opposite category $\mathsf{Bun}_S^{\mathsf{op}}$ and $\mathsf{CoBun}_S$. We call the morphisms in $\mathsf{Bun}_S$ *bundle maps* and the morphisms in $\mathsf{CoBun}_S$ *cobundle maps*.

13.2. **Flag and Coflag Categories.** A subsheaf $N \subset M$ is a subbundle if the inclusion is a bundle map. Let $d : \{0, \dots, p\} \to \mathbf{N}$ be an increasing function. A flag of type $d$ on $M$ is a strictly increasing sequence $F$ of subbundles of $M$ of the form

$$0 = F_0 \subsetneqq F_1 \subsetneqq F_2 \subsetneqq \cdots \subsetneqq F_p = M. \tag{13.1}$$

with $\operatorname{rk} F_i = d(i)$. We write $d_F$ for the type of an arbitrary flag. The integer $p = p_F$ is the *length* of the flag. Note that $M = F_p$ and is thus determined by $F$. We may thus consider a flag either as an ordered pair $(M, F)$ or simply as the singleton $F$. $M \supsetneqq 0$ is the *trivial* flag on $M$. By abuse of notation, we write $M$ for this flag.

Each of the two category structures on vector bundles induces a category structure on flags. We define a category $\mathsf{CoFlags}_S$ as follows: The objects of of $\mathsf{CoFlags}_S$ are flags on $S$. If $F$ is a flag of length $p$ and and $E$ is a flag of length $q$, a morphism $F \to E$ is a cobundle morphism $\phi : F_p \twoheadrightarrow E_q$ such that $\{\phi^{-1}E_i\}_{i=1}^q \subset \{F_i\}_{i=1}^p$. The second category $\mathsf{Flags}_S$ has the same objects. However, a morphism $E \to F$ in $\mathsf{Flags}_S$ is a bundle morphism $\psi : E_q \rightarrowtail F_p$ such that $\{\psi E_i\}_{i=1}^q \subset \{F_i\}_{i=1}^p$.

There is a contravariant functor $F \to F^\vee$ from $\mathsf{Flags}_S$ to $\mathsf{CoFlags}_S$ defined by sending a flag $F$ of length $p$ to the flag $F_i^\vee = (F_{p-i})^\perp$. The same definition also produces a contravariant functor $F \to F^\vee$ from $\mathsf{CoFlags}_S$ to $\mathsf{Flags}_S$. These functors together form an equivalence of $\mathsf{Flags}_S^{\mathsf{op}}$ with $\mathsf{CoFlags}_S$. Morphisms in $\mathsf{Flags}_S$ are called *flag* morphisms and morphisms in $\mathsf{CoFlags}_S$ are called *coflag* morphisms.

For any flag, $d(i) \geq i$. A flag is complete if $d(p) = p$, and in this case, $d(i) = i$ for all $i$. Suppose $E$ and $F$ are flags of length $p$ and $q$ repsectively and $E_q = F_p$. Then $F$ is a *refinement* of $E$ if the identity morphism from $E_q$ to $F_p$ is a flag morphism or equivalently if the identity morphism from $F_p$ to $E_q$ is a coflag morphism.

13.3. **Rank and Corank Functions on Flags.** We will consider functions $\rho : \{F_i\}_{i=1}^p \to \mathbf{N}$. Such functions pullback under flag morphisms and pushforward under coflag morphisms. That is, if $\phi : F \to E$ is a coflag morphism and $\rho$ is a function on $F$, we set $(\phi_*\rho)(E_i) = \rho(\phi^{-1}E_i)$. If $\phi : E \to F$ is a flag morphism and $\rho$ is a function on $F$, we set $(\phi^*\rho)(E_i) = \rho(\phi E_i)$. We also write $\rho|E$ for $\phi^*\rho$. By abuse of notation, we identify the function $\rho' : \{1, \dots, p\} \to \mathbf{N}$ defined by $\rho'(i) = \rho(F_i)$

with $\rho$ itself. We can thus regard a function $\rho$ on $F$ as a rank function on any flag $G$ of length $p$. In particular, we can regard it as a function on $F^\vee$.

A function on a flag $F$ is called a *rank* function if $\rho(0)=0$ and $\rho(i+1)\geq\rho(i)$. It is called a *corank* function if $\rho(p)=0$ where $p$ is the length of $F$ and $\rho(i+1)\leq\rho(i)$ The property of being a rank function is preserved under pullback through a flag morphism, and the property of being a corank function is preserved by pushforward through a coflag morphism. Given a rank function $\rho$ on $F$ the *dual* corank function is $\rho^\vee(i)=\rho(p-i)$.

**Lemma 13.1.** *If $\phi: E\to F$ is a flag morphism and $\rho$ is a rank function on $F$, then $(\phi^*\rho)^\vee=\phi^\vee_*(\rho^\vee)$. Dually, if $\psi: F\to E$ is a coflag morphism and $\rho$ is a corank function, then $(\psi_*\rho)^\vee=(\psi^\vee)^*\rho^\vee$.*

*Proof.* For the first statement, suppose $(\phi^*\rho)^\vee(i)=r$. Then, by definition, $\phi^*\rho(q-i)=r$ where $q$ is the length of $E$. Now assume that $\phi(E_{q-i})=F_{p-j}$. Then we have that $\rho(p-j)=r$, and, thus, $\rho^\vee(j)=r$. It also follows that $(\phi^\vee)^{-1}(E^\perp_{p-i})=F^\perp_{p-j}$. Thus $(\phi^\vee)^{-1}(E^\vee_i)=F^\vee_j$. Hence, $\phi^\vee_*\rho^\vee(i)=\rho^\vee(j)=r$.

The proof of the second statement is similar. □

13.4. **Motive of Morphisms from a Flag.** Let $M$ and $U$ be vector bundles of rank $m$ and $u$ respectively over a base scheme $S$, and let $F$ be a flag of type $d$ and length $u$ on $M$. For a rank function $\rho$ on $M$ write $Hom_\rho(F,U)$ for the scheme representing the functor

$$T\rightsquigarrow\{\phi\in Hom_T(M_T,U_T)|\,\mathrm{rk}(\phi|F_i)=\rho(i)\}. \tag{13.2}$$

Note that if $i: E\to F$ is a flag morphism, then there is a canonical morphism

$$i^*:\mathrm{Hom}_\rho(F,U)\to\mathrm{Hom}_{\rho|E}(E,U.) \tag{13.3}$$

**Proposition 13.2.** *Let $F$ be the flag $M\supsetneq N\supsetneq 0$, and let $\rho(1)=r$ and $\rho(2)=q$. Then for any vector bundle $U$ of rank $u$,*

$$i^*:\mathrm{Hom}_\rho(F,U)\to\mathrm{Hom}_r(N,U) \tag{13.4}$$

*is a motivic fibration over $\mathbf{Z}$ with fiber*

$$\mathbf{L}^{r(m-n)}\,\mathrm{Hom}_{q-r}(m-n,u-r.) \tag{13.5}$$

*Proof.* Let $k$ be a field with a map $\eta:\mathrm{Spec}\,k\to\mathrm{Hom}_r(N,U)$. The map $\eta$ corresponds uniquely to a morphism $\phi: N_k\to U_k$ of rank $r$. The fiber $X=\mathrm{Hom}_\rho(F,U)_\eta$ is then the scheme of all extensions $\psi: M_k\to U_k$ of rank $q$ such that $\psi|N_k=\phi$. Pick a basis $e_1,\dots,e_n$ of $N_k$ and extend it to a basis $e_1,\dots,e_m$ of $M_k$. Let $C$ be the $k$-vector space spanned by the $e_{n+1},\dots,e_m$. Let $f_1,\dots,f_r$ be a basis for $\phi(N_k)$ and extend it to a basis of $f_1,\dots,f_p$ of $U_k$. Write $U_1$ for $\phi(N_k)$, $U_2$ for the $k$ vector space spanned by the $f_i$ for $i>r$, and $\mathrm{pr}_i: U\to U_i$ for respective projections.

An extension $\psi$ of $\phi$ will have rank $q$ if and only if the restriction of $\mathrm{pr}_2\circ\psi$ to $C$ has rank $q-r$. It follows that there is an isomorphism

$$X\to\mathrm{Hom}(C,U_1)\times\mathrm{Hom}_{q-r}(C,U_2) \tag{13.6}$$

given by

$$\psi\mapsto(\mathrm{pr}_1\circ\psi|C,\mathrm{pr}_2\circ\psi|C.) \tag{13.7}$$

□

13.5. **Motive of Morphisms to a Flag.** Using Proposition 13.2, we can by duality recover the fiber of motives to a flag. Given a flag $F$ of length $p$ on a bundle $M$, let $\mathrm{pr}_i : M \to M/F_i$ be the projection map.

Given a vector bundle $U$ and a corank function $\rho$ let $\mathrm{Hom}_\rho(U, F)$ be the scheme representing the functor

$$T \rightsquigarrow \{\phi \in \mathrm{Hom}(U_T, M_T) |\, \mathrm{rk}\, \mathrm{pr}_i \circ \phi = m - \rho(i)\}. \tag{13.8}$$

If $\pi : F \to E$ is a coflag morphism, there is a map

$$\pi_* : \mathrm{Hom}_\rho(U, F) \to \mathrm{Hom}_{\pi_*\rho}(U, E) \tag{13.9}$$

**Proposition 13.3.** (a) *The transpose map induces a canonical isomorphism* $\mathrm{Hom}_\rho(F, U) \to \mathrm{Hom}_{\rho^\vee}(U^\vee, F^\vee.)$

(b) *If* $\pi : F \to E$ *is a coflag morphism the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\rho^\vee}(F^\vee, U^\vee) & \longrightarrow & \mathrm{Hom}_\rho(U, F) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{(\pi_*\rho)^\vee}(E^\vee, U^\vee) & \longrightarrow & \mathrm{Hom}_{\pi_*\rho}(U, E) \end{array} \tag{13.10}$$

*commutes.*

(c) *Suppose* $F$ *is the flag* $M \supsetneq N \supsetneq 0$ *of type* $d$ *and* $E$ *is the trivial flag on* $M/N$. *Let* $\pi : F \to E$ *be the obvious coflag morphism. Then the morphism*

$$\pi_* : \mathrm{Hom}_\rho(U, F) \to \mathrm{Hom}_{\pi_*\rho}(U, E) \tag{13.11}$$

*is a motivic fibration with fiber*

$$\mathbf{L}^{\rho(2)d(1)}\, \mathrm{Hom}_{\rho(1)-\rho(2)}(d(1), u - \rho(2).) \tag{13.12}$$

*Proof.* (a) and (b) are easily verified by checking through the definitions. (c) is then a consequence of (a), (b) and Proposition 13.2. □

13.6. **Extensions of Bilinear Forms.** For any flag $F$ of length $p$ and rank function $\rho$, we write $\mathrm{Sym}_\rho F$ for the representable functor

$$T \rightsquigarrow \{Q \in \mathrm{Sym}(F_{pT}) |\, \mathrm{rk}(Q_{|F_{iT}}) = \rho(i)\}. \tag{13.13}$$

It is easy to see that $\mathrm{Sym}_\rho F$ is a locally closed subscheme of $\mathrm{Sym}\, F_p$. If $\phi : E \to F$ is a flag morphism, then there is a restriction map

$$\phi^* : \mathrm{Sym}_\rho F \to \mathrm{Sym}_{\rho|E} E. \tag{13.14}$$

**Proposition 13.4.** *Let* $F$ *be a flag and* $\phi : F \to \overline{F}$ *be a refinement. Then* $\mathrm{Sym}_\rho(F)$ *is the disjoint union of the all schemes* $\mathrm{Sym}_\sigma(\overline{F})$ *such that* $\sigma|F = \rho$. *Thus,*

$$[\mathrm{Sym}_\rho F] = \sum_{\sigma|F=\rho} [\mathrm{Sym}_\sigma \overline{F}]. \tag{13.15}$$

*Proof.* Let $p$ and $\overline{p}$ be the lengths of $F$ and $\overline{F}$ respectively, and let $M = F_p$. Since the restriction maps $\mathrm{Sym}(\overline{F}, \sigma) \to \mathrm{Sym}(F, \rho)$ are induced by the identity map on $\mathrm{Sym}\, M$, the $\mathrm{Sym}_\sigma \overline{F}$ are subschemes of $\mathrm{Sym}_\rho F$.

We must now show that the $\mathrm{Sym}_\sigma \overline{F}$ are disjoint and cover $\mathrm{Sym}_\rho F$. Let $T \to \mathrm{Sym}_\rho F$ be an $S$-morphism corresponding to a symmetric bilinear form $Q$ on $M_T$. For each $\sigma$ with $\sigma|F = \rho$, set

$$Y(\sigma) = \mathrm{Sym}_\sigma \overline{F} \times_{\mathrm{Sym}_\rho F} T. \tag{13.16}$$

We must show the $Y(\sigma)$ form a disjoint cover of $T$. A point $y \in Y(\sigma)$ if and only if, for any $i \in \{1, \dots, \overline{p}\}$, the rank of $Q$ restricted to the vector space $\overline{F}_{iT} \otimes k(y)$ is $\sigma(i)$. This shows that the that $Y(\sigma) \cap Y(\sigma') = \emptyset$ for $\sigma' \neq \sigma$. Moreover, for any point $t \in T$ and any $i \in \{1, \dots, p\}$, the rank of $Q$ restricted to $F_{iT} \otimes k(t)$ is $\rho(i)$. Thus, if we set $\sigma_t(i) = \operatorname{rk} \overline{F} \otimes k(t)$, then $t \in Y(\sigma_t)$ and $\sigma | F = \rho$. □

For any three integers $d, r$ and $i$ set

$$\gamma(d, r_2, r_1) = \begin{cases} \mathbf{L}^{r_1}, & r_1 = r_2 \\ \mathbf{L}^{r_2} - \mathbf{L}^{r_1} & r_2 = r_1 + 1 \\ \mathbf{L}^{d+1} - \mathbf{L}^{r_1+1} & r_2 = r_1 + 2 \\ 0 & \text{otherwise.} \end{cases} \tag{13.17}$$

**Lemma 13.5.** *Let $F$ be the flag $M \supsetneq N \supsetneq 0$, and let $\phi : N \to F$ be the inclusion of the trivial flag $N$ in $F$. Suppose* $\operatorname{rk} N = n$ *and* $\operatorname{rk} M = n + 1$. *For any rank function $\rho$ on $F$ with $\rho(1) = r$ and $\rho(2) = q$. The map*

$$\phi^* : \operatorname{Sym}(F, \rho) \to \operatorname{Sym}(E, \rho | F') \tag{13.18}$$

*is a motivic fibration over* $\operatorname{Spec} \mathbf{Z}$ *with fiber* $\gamma(n, q, r)$.

*Proof.* The restriction map $\phi^*$ reduces to the map

$$\operatorname{Sym}_\rho F \to \operatorname{Sym}_r(N.) \tag{13.19}$$

Suppose $\eta : \operatorname{Spec} k \to \operatorname{Sym}_r(N)$ is a map from the spectrum of a field to $\operatorname{Sym}_r N$ corresponding to a symmetric bilinear from $Q$ of rank $r$ on $N_k$.

Set

$$X = \operatorname{Sym}_\rho F \times_{\operatorname{Sym}_r N} \operatorname{Spec} k. \tag{13.20}$$

Then $X$ is the scheme of all bilinear forms $R$ on $M_k$ of rank $q$ such that $R | N_k = Q$.

Let $Y$ be the scheme of all bilinear forms $R$ on $M_k$ such that $R | N_k = Q$, and, for each $i$, let $Y_i$ be the closed subscheme of $Y$ consisting of forms of rank less than or equal to $i$. Then $X = Y_q - Y_{q-1}$.

The theorem follows from the following set of isomorphisms which can each be proven using elementary linear algebra:

(a) $Y = Y_{r+2} \cong \mathbf{A}^{n+1}$;
(b) $Y_{r+1} \cong \mathbf{A}^{r+1}$;
(c) $Y_r \cong \mathbf{A}^r$.

□

Now we define geometric analogues $\Gamma(d_2, r_2, d_1, r_1)$ of the $C(d_2, r_2, d_1, r_1)$ by requiring that

(a) $\Gamma(d_1 + 1, r_2, d_1, r_1) = \gamma(d_1, r_2, r_1)$,
(b) the following recursion is satisfied:

$$\Gamma(d_2,r_2, d_1, r_1) = \sum_{j=0}^{2} \Gamma(d_2, r_2, d_1 + 1, r_1 + j)\Gamma(d_1 + 1, r_1 + j, d_1, r_1.) \tag{13.21}$$

Clearly, $\Gamma(d_1 + 1, r_2, d_1, r_1)$ is a polynomial in $\mathbf{L}$ and

$$\operatorname{ev}(\Gamma(d_1 + 1, r_2, d_1, r_1)) = C(d_1 + 1, r_2, d_1, r_1.) \tag{13.22}$$

**Proposition 13.6.** *Let $F$ be type $d$ flag $M \supsetneq N \supsetneq 0$. Let $\rho$ be a rank function with $\rho(i) = r_i$. Then*

$$\operatorname{Sym}_\rho F \to \operatorname{Sym}_{r_1} N \tag{13.23}$$

*is a motivic fibration with fiber $\Gamma(d_2, r_2, d_1, r_1)$.*

*Proof.* Let $K$ be a field and let $\eta \in \operatorname{Sym}_r N(K)$ be a point corresponding to a bilinear form $Q \in \operatorname{Sym}_r N_K$. Let $X$ be the variety of extensions of $Q$ to symmetric bilinear form on $M_K$ of rank $r_2$. By Theorem 12.6, it suffices to show that $[X]_K = \Gamma(d_2, r_2, d_1, r_1)$.

We prove the proposition by induction on $d_2 - d_1$. If $d_2 \leq d_1 + 1$, the result follows from Lemma 13.5. Otherwise, let $P$ be a subspace of dimension $d_2 - 1$ such that $M_K \supset P \supset N_K$. For a given $q$, let $Y_q$ be the scheme of extensions of $Q$ to a symmetric bilinear form on $P$ of rank $q$. Let $X_q$ be the locally closed subscheme of extensions $Q \in X$ such that $Q|P \in Y_q$. The $X_q$ stratify $X$ into a disjoint union of locally closed subschemes.

By the inductive hypothesis, $[Y_q]_K = \Gamma(d_2 - 1, q, d_1, r_1)$. But then note that the diagram

$$\begin{array}{ccc} X_q & \longrightarrow & Y_q \\ \downarrow & & \downarrow \\ \operatorname{Sym}_\sigma F_K & \longrightarrow & \operatorname{Sym}_q P \end{array} \tag{13.24}$$

is a pullback. Thus $X_q \to Y_q$ is a motivic fibration with fiber $\Gamma(d_2, r_2, d_2 - 1, q)$. The proposition now follows from Proposition 12.4. □

We now obtain a motivic version of MacWilliams' result counting the number of symmetric bilinear forms of a given rank and dimension over $F_q$.

**Theorem 13.7.** *The equations (2.10) hold with $\mathbf{L}$ replacing $q$. That is,*

$$[\operatorname{Sym}^n_r] = \begin{cases} \prod_{i=1}^s \frac{\mathbf{L}^{2i}}{\mathbf{L}^{2i}-1} \cdot \prod_{i=0}^{2s-1} (\mathbf{L}^{n-i} - 1), & 0 \leq r = 2s \leq n, \\ \prod_{i=1}^s \frac{\mathbf{L}^{2i}}{\mathbf{L}^{2i}-1} \cdot \prod_{i=0}^{2s} (\mathbf{L}^{n-i} - 1), & 0 \leq r = 2s+1 \leq n \end{cases} \tag{13.25}$$

*Proof.* MacWilliams' proof (see [17] pp. 154–156) relies only on the recursion:

$$|\operatorname{Sym}^{n+1}_r| = q^r |\operatorname{Sym}^n_r| + (q-1)q^{r-1}|Sym^n_{r-1}| + (q^{n+1} - q^{r-1})|Sym^n_{r-2}| \tag{13.26}$$

Thus the theorem follows from MacWilliams proof once this recursion is known to hold in $\mathsf{GeoMot}^+$ with $\mathbf{L}$ replacing $q$. This follows from Proposition 13.6. □

## 14. Reduction Formulas for GeoMot

We begin with an inventory of what remains to be reproved in the geometric context. The equations (2.7–2.9) were stated without proof in the combinatorial context. As they are easy, we will leave them without proof in the geometric context. The three formulas in Section 6 are more difficult. We will prove the geometric versions of Theorems 6.1 and 6.2. We leave the geometric version of Theorem 6.4 whose proof uses Proposition 13.2 and is similar to the proof of Theorem 6.2 to the reader. The remaining formulas that need to be verified in GeoMot are Proposition 7.3 and Equation 9.1. These we will also leave to the reader. Once these formulas are checked to hold in GeoMot, Theorem 0.6 part (a) follows by pure algebra. To obtain part (b) of the theorem, it must be checked that Theorem 11.1 part (c) holds in GeoMot with $\mathbf{L}$ replacing $q$. Again, we leave this to the reader.

We need some notation concerning Grassmanians. If $W$ is a vector bundle of over $S$, let $\pi : \mathrm{Gr}(r, W) \to S$ be the structure map from the Grassmanian of rank $r$ subbundles of $W$ to $S$. We write $\mathsf{Sub}$ for the universal subbundle and $\mathsf{Quot}$ for the universal quotient. Thus we have an exact sequence

$$\mathsf{Sub} \rightarrowtail \pi^* W \twoheadrightarrow \mathsf{Quot}. \tag{14.1}$$

Suppose that $W = \mathcal{O}_S^s$. Then $A_G(W, r, k) = A_G(s, r, k) \times S$. This can be easily checked by working through the functorial definition of $A_G(W, r, k)$. It follows that, if $W$ now is any locally free sheaf of rank $s$, the map $A_G(W, r, k) \to S$ is a motivic fibration with fiber $A_G(s, r, k)$ for any locally free sheaf $W$ of rank $s$.

**Proposition 14.1.** *For any locally free sheaf $W$ of rank $s$ over a scheme $S$,*

$$[A_G(W, r, k)]_S = [Gr(k, W)]_S \sum_j \Gamma(s, r, k, j)[A_G(k, j, k)] \tag{14.2}$$

*Proof.* For every $j$, let $X_j$ be the subscheme of $A_G(W, r, k)$ representing the functor

$$T \rightsquigarrow \{(Q, f) \in A_G(W, r, k)(T) |\, \mathrm{rk}(Q|\langle f \rangle) = j\}. \tag{14.3}$$

The $X_j$ stratify $A_G(W, r, k)$ into a disjoint union of locally closed subschemes. Let $\pi : \mathrm{Gr}(k, W) \to S$ be the structure map and let $\mathsf{Sub}$ be the universal subbundle. There is then a map

$$p_j : X_j \to A_G(\mathsf{Sub}, r, k) \tag{14.4}$$

given functorially by sending a pair $(Q, f) \in X_j(T)$ to the pair $(Q|\langle f \rangle, f)$.

Let $F$ be the flag $\pi^* W \supsetneqq \mathsf{Sub} \supsetneqq 0$ on $\mathrm{Gr}(k, W)$. Let $\rho$ be the rank function assigning the rank $r$ to $\pi^* W$ and $j$ to $\mathsf{Sub}$.

It is an exercise in chasing the functorial definitions of the various schemes involved to check that there is a pullback diagram as follows:

$$\begin{array}{ccc} X_j & \longrightarrow & A_G(\mathsf{Sub}, j, k) \\ \downarrow & & \downarrow \\ \mathrm{Sym}_\rho F & \longrightarrow & \mathrm{Sym}_j(\mathsf{Sub}.) \end{array} \tag{14.5}$$

From this it follows that

$$[\mathbf{A}_G(W, r, k)] = \sum_{j=0}^{k} \Gamma(s, r, k, j)[A_G(\mathsf{Sub}, r, k)]. \tag{14.6}$$

The proposition follows from fact that $A_G(\mathsf{Sub}, r, k)$ fibers motivically over its base with fiber $A_G(k, j, k)$. □

**Corollary 14.2.** *In* $\mathsf{GeoMot}_{\mathbf{Z}}$

$$[A_G(s, r, k)] = [\mathrm{Gr}(k, s)] \sum_{j=0}^{k} [\Gamma(s, r, k, j)][A_G(k, j, k)]. \tag{14.7}$$

We now give a version of Theorem 6.2 in $\mathsf{GeoMot}$.

**Proposition 14.3.** *Let $W$ be a locally free sheaf of rank $s$ on $S$. Let $\pi : \mathrm{Gr}(r, W) \to S$ be the canonical map, and let* $\mathsf{Quot}$ *be the canonical quotient of $\pi^* W$.*

$$[A_G(W, r, k)]_S = \sum_{l=0}^{r} \mathbf{L}^{l(s-r)} [\mathrm{Hom}_{k-l}(n - l, s - r)][A_G(\mathsf{Quot}, r, l)]_S. \tag{14.8}$$

*Proof.* Let $p : W \to W/(\ker Q)$ be the quotient map, and let $Y_l$ be the subscheme of $A_G(W, r, k)$ representing the functor

$$T \rightsquigarrow \{(Q, f) \in A_G(W, r, k) |\, \mathrm{rk}(p \circ f) = l\}. \tag{14.9}$$

Let $F$ be the flag given by $\pi^* W \supsetneqq \mathsf{Quot} \supsetneqq 0$. Let $\rho$ be the corank function given by $\rho(0) = r$ and $\rho(1) = l$. It can then be checked that there is a pullback diagram:

$$\begin{array}{ccc} Y_l & \longrightarrow & A_G(\mathsf{Quot}, r, l) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_\rho(\mathcal{O}_S^V, F) & \longrightarrow & \mathrm{Hom}_l(\mathcal{O}_S^V, \mathsf{Quot}.) \end{array} \tag{14.10}$$

The proposition then follows from Proposition 13.3. □

## 15. Periods

We want to discuss briefly the implications on periods (integrals) which formed the basis for the original conjecture. The main difficulty is that we are working in the Grothendieck group of Motives, and periods are not defined for these objects.

We first show that there is a graph G so that the scheme $Y_G$ has a cohomology group that is not mixed Tate. Then the periods of that cohomology group cannot be in the ring of multiple zeta values if we assume the conjecture that if periods of a cohomology group (of an algebraic variety over $\mathbf{Q}$) are multiple zeta values, then the cohomology group is mixed Tate. In this discussion by cohomology we mean the pair of betti cohomology with the mixed Hodge structure and algebraic de Rham cohomology with the comparison isomorphism. Note that the de Rham cohomology is defined over $\mathbf{Q}$ for varieties defined over $\mathbf{Q}$.

Consider the $E$-polynomials of Craw [6]. For X a projective scheme over $\mathbf{Z}$,

$$E(X) = \sum_{1 \le p,q \le dim(X)} \sum_{k \ge 0} (-1)^k h^{p,q}(H_c^k(X_{\mathbf{C}}, \mathbf{C})) u^p v^q.$$

where $h^{p,q}$ are the Hodge-Deligne numbers. A mixed Tate cohomology group has $h^{p,q} = 0$ unless $p = q$.

It is shown by Craw that this function factors through $\mathsf{GeoMot}^+_{\mathbf{Z}}$ and can be extended to a function

$$GeoMot_{\mathbf{Z}} \mapsto \mathbf{Z}[u, v, (uv)^{-1}]$$

In $\mathbf{Z}[u, v, (uv)^{-1}]$ there is a distinguished subring namely $\mathbf{Z}[(uv), (uv)^{-1}]$ Now suppose all the Kontsevich schemes $Y_G$'s map to this subring. Then one can see easily from our theorem (that $Y_G$'s generate $\mathsf{GeoMot}_Z$) that the image of $\mathsf{GeoMot}_Z$ is in this ring. But taking $E$ to be an elliptic curve over $\mathbf{Q}$, and taking a model over $\mathbf{Z}$, we reach a contradiction. Therefore suppose that $E(Y_G)$ is not in $\mathbf{Z}[uv, (uv)^{-1}]$. There exist $p \neq q$ and a $k$ so that $h^{p,q}(H_c^k(Y_G, \mathbf{C})) \neq 0$. Since Poincare duality applies for $Y_G$ (they are smooth over $\mathbf{Q}$) we can drop compact supports above (and change $k$.)

Therefore,

**Proposition 15.1.** *There exists a graph $G$, so that a period of $Y_G$ is not a multiple zeta value, if we grant the conjecture that periods of a cohomology group (of a variety defined over $\mathbf{Q}$) are multiple zeta implies the cohomology is mixed Tate.*

This proposition does not however show that Feynman amplitudes coming from Graphs need not be (in the **Q**-span of) multiple zeta values. This is because of two reasons: The precise relation between feynman amplitudes and the periods of $Y_G$ has not been worked out yet and perhaps not all periods of $Y_G$ correspond to Feynman amplitudes.

We will return to the question of the integrals considered by Kontsevich in a later paper [2].

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