Problem 1. (15 points) Let \( K = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \) and let \( A = \{a + b\sqrt{2} \in K : a, b \in \mathbb{Z}\} \).
(a) Show that \( K \) is a subring of \( \mathbb{R} \).
(b) Show that \( A \) is a subring of \( K \).
(c) Show that \( K \) is a field.

Problem 2. (10 points) Suppose \( R \) is a commutative ring.
(a) Suppose \( A \) is a subring of \( R \times R \). Assume that \( A \) is also an equivalence relation on \( R \). Show that \( I_A := \{r \in R : (r, 0) \in A\} \) is an ideal in \( R \).
(b) Suppose \( I \) is an ideal in \( R \). Set \( A_I := \{(r, s) \in R \times R : r - s \in I\} \). Show that \( A_I \) is a subring of \( R \times R \) which is also an equivalence relation on \( R \).
(c) (5 point bonus) Show that \( I_{A_I} = I \) for any ideal \( I \subset R \), and that \( A_{I_A} = A \) for any equivalence relation \( A \subset R \times R \).

Problem 3. (10 points) Suppose \( D \) is a division ring and \( I \) is a left ideal in \( D \). Show that either \( I = \{0\} \) or \( I = D \). Then draw the following conclusion: If \( \rho : D \to R \) is a homomorphism of rings where \( D \) is a division ring, then either \( R = 0 \) or \( \rho \) is one-to-one.

Problem 4. (10 points) Suppose \( R \) is a commutative ring.
(a) Suppose \( A \) is a subring of \( R \times R \). Assume that \( A \) is also an equivalence relation on \( R \). Show that \( I_A := \{r \in R : (r, 0) \in A\} \) is an ideal in \( R \).
(b) Suppose \( I \) is an ideal in \( R \). Set \( A_I := \{(r, s) \in R \times R : r - s \in I\} \). Show that \( A_I \) is a subring of \( R \times R \) which is also an equivalence relation on \( R \).
(c) (5 point bonus) Show that \( I_{A_I} = I \) for any ideal \( I \subset R \), and that \( A_{I_A} = A \) for any equivalence relation \( A \subset R \times R \).

Problem 5. (25 points) Suppose \( R \) is a ring.
(a) Show that, if \( A \) and \( B \) are subrings of \( R \), then so is \( A \cap B \).
(b) Generalize (a) in the following way. Suppose \( \{A_i\}_{i \in I} \) is a set of subrings of \( R \). Show that \( A = \bigcap_{i \in I} A_i \) is a subring of \( R \).
(c) Suppose \( S \) is a subset of \( R \). Let \( A \) denote the intersection of all subrings of \( R \) containing \( S \). Show that \( A \) is the smallest subring of \( R \) containing \( S \). It is called the subring of \( R \) generated by \( S \).
(d) Keeping a notation of (c), define a sequence \( A_n \) of subsets of \( R \) inductively as follows: \( A_0 = \{0, 1\} \cup S \), \( A_{n+1} = \{x - y, xy : x, y \in A_n\} \). Show that \( A = \bigcup_{n=0}^{\infty} A_n \) (In other words, \( A \) is the union of the sets \( A_n \)).
(e) Now suppose that \( B \) is a subring of \( R \) and \( S \) is a subset of \( R \). Let \( B[S] \) denote the subring of \( R \) generated by \( B \cup S \). Now set \( R = \mathbb{R} \) (the ring of real numbers), \( B = \mathbb{Q} \) and \( S = \{\sqrt{2}\} \). Set \( \mathbb{Q}[\sqrt{2}] := B[S] \). Show that \( \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \).

Problem 6. (30 points) Let
\[
H = \left\{ \begin{pmatrix} t + x & -y - zi \\ y - zi & t - xi \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}
\]
contained in the ring \( M_2(\mathbb{C}) \) of \( 2 \times 2 \) matrices with complex coefficients. To ease the notation, define
\[
\tilde{a} := \begin{pmatrix} 0 & 1 \\ i & -i \end{pmatrix}, \tilde{b} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tilde{c} := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.
\]
These are elements of \( H \) and the general matrix in \( H \) can then be written (uniquely and more compactly) as \( t + x\tilde{a} + y\tilde{b} + z\tilde{c} \).
(a) Show that \( \tilde{a}^2 = \tilde{b}^2 = \tilde{c}^2 = -1 \) and \( \tilde{a}\tilde{b} = \tilde{c}, \tilde{b}\tilde{c} = \tilde{a}, \tilde{c}\tilde{a} = \tilde{b} \).
(b) Show that \( H \) is a subring of \( M_2(\mathbb{C}) \).
(c) Show that the determinant of any element of \( H \) is a non-negative real number. Show further that \( \det X = 0 \) iff \( X = 0 \) for \( X \in H \).
(d) For \( X = t + x\tilde{a} + y\tilde{b} + z\tilde{c} \in H \), define \( X^* := t - x\tilde{a} - y\tilde{b} - z\tilde{c} \). Show that \( XX^* = \det X \).
(e) Show that, for \( X \neq 0 \), \( X^{-1} = (\det X)^{-1}X^* \). Then conclude that \( H \) is a division ring.
(f) Show that \( \tilde{b}\tilde{a} = -\tilde{c} \) and conclude that \( H \) is not a field.