Reading Assignment

Begin reading about rings in Chapter 3.1-2.

Problem 1. (40 points) Recall that, if $X$ is a set, $A(X)$ denotes the group of all maps $\phi : X \to X$, which are one-one and onto. If $G$ is a group and $g \in G$, define $L(g) : G \to G$ by $L(g)(h) = gh$.

(a) Show that, if $g_1, g_2 \in G$, then $L(g_1 g_2) = L(g_1) \circ L(g_2)$.

(b) Show that, if $g \in G$, $L(g)$ is one-one and onto. Conclude that $L : G \to A(G)$ given by $g \mapsto L(g)$ is a homomorphism of groups.

(c) Show that $L(g)$ is the identity in $A(G)$ iff $g$ is the identity in $G$. Conclude that $L : G \to A(G)$ is one-to-one.

(d) Draw the following conclusion: If $G$ is a group with $n$ elements, then $G$ is isomorphic to a subgroup of the symmetric group $S_n$. (This is called Cayley’s theorem.)

Problem 2. (40 points) Let $A$ be a ring.

(a) Show that there exists exactly one ring homomorphism $h : \mathbb{Z} \to A$. (Hint: If $h$ is a ring homomorphism, we must have $h(1) = 1$, so $h(2) = 1 + 1, h(-2) = -(1 + 1)$, etc.)

(b) Let $h : \mathbb{Z} \to A$ be as in (a). Then $\ker h = n\mathbb{Z}$ for some (uniquely determined) $n \in \mathbb{N}$. Set $\text{char} A = n$; this is called the characteristic of $A$. Show that, if $A$ is an integral domain, then $\text{char} A$ is either prime or $0$.

(c) Show that any field of characteristic $p > 0$ contains a subfield isomorphic to $\mathbb{Z}/p$.

(d) Show that any field of characteristic $0$ contains a subfield isomorphic to the rationals.

Problem 3. (20 points) Write $F_2 := \mathbb{Z}/2$ for the field with 2 elements. Set $p = x^2 + x + 1 \in F_2[x]$.

(a) Show that $p$ is irreducible.

(b) Show that $K := F_2[x]/(p)$ is a field.

(c) Show $K$ has $4$ elements and write down the addition and multiplication table in $K$. 

1