Me



Math 403 Fall 2011 UMD

$\boxed{\text{Lecture 1} \quad \text{Functions, Sets, Notations}}$

## §0 Sets of numbers and notation for them

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ = set of natural numbers
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$ = set of integers
- $\mathbb{Q} = \{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \} = \{\text{rational nos}\}$
- $\mathbb{R} = \{\text{set of real nos}\}$

If $S$ is a set,
We write $a \in S$ to mean that $a$ is an element of $S$,
$a \notin S$ means that $a$ is not an element of $S$

Ex $-1 \in \mathbb{Z}$, $-1 \notin \mathbb{N}$, $\pi \in \mathbb{R}$, $\pi \notin \mathbb{Z}$

We write $A \subseteq B$ to mean that every element of $A$ is in $B$. If $A \subseteq B$ then $A$ is said to be a sub-set of $B$.

Ex $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

## §1 Defining sets by properties

If $S$ is a set and $P$ is a property that an element of a set may ~~of~~ or may not have then we can use $P$ to define a subset $T$ consisting of all elements of $S$ satisfying $P$. We use set-builder notation to write this set

$$T = \{x \in S : P(x)\}$$

Ex $E := \{n \in \mathbb{Z} : n \text{ is even}\} = \{\text{even integers}\}$

$O := \{n \in \mathbb{Z} : n \text{ is odd}\} = \{\text{odd integers}\}$

$[0,\infty) := \{x \in \mathbb{R} : x \geq 0\}$

$\mathbb{Z}_+ := \{n \in \mathbb{Z} : n > 0\}$

§2 Logical symbols and notation If $P, Q$ properties, then

$P \Rightarrow Q$ means $P$ implies $Q$

$P \Leftarrow Q$ mean $Q$ implies $P$

$P \Leftrightarrow Q$ means $P$ iff $Q$.

$P \vee Q$ means either $P$ or $Q$ hold

$P \wedge Q$ means both $P$ and $Q$ hold

$\sim P$ means $P$ doesn't hold.

leave off

$\exists$ mean there exists

~~Ex $x \in [0,\infty) \Rightarrow x = y^2$ for some~~

$\forall$ means $\forall$

Ex $x \in [0,\infty) \Rightarrow \exists\, y \in \mathbb{R},\ x = y^2$.

That's a shorthand for saying any non-negative real no is a square.

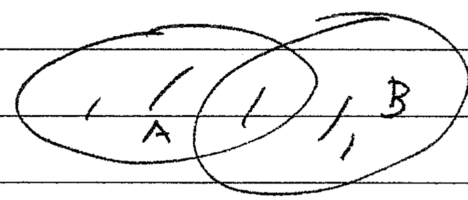
## §3 Operations on Sets

$A \cup B = \{x : x \in A \text{ or } x \in B\}$, union

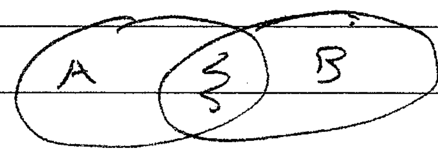
$A \cap B = \{x : x \in A \text{ and } x \in B\}$, intersection

$A \setminus B = \{x \in A : x \notin B\}$, ~~complement~~ difference

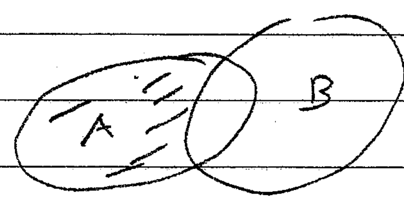
If $B \subseteq A$ then $A \setminus B$ is called the complement of $B$ in $A$.




$A \cup B$




$A \cap B$




$A \setminus B$

~~Ex: $A \setminus B =$~~

There is a special set called the empty set which has no elements. We write it as $\emptyset$.
We say $A$ and $B$ are disjoint if $A \cap B = \emptyset$.

Ex $E \cup O = \mathbb{Z}$ $E \cap O = \emptyset$
Complement of $E$ in $\mathbb{Z}$ is $O$

§4 Other operations

Power Set If $S$ is a set

$P(S) = \{$ all subsets of $S\}$.

Ex $S = \{1\}$, $P(S) = \{\emptyset, \{1\}\}$
$S = \{1,2\}$, $P(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
$S = \{1, 2, 3\}$

Rmk If $S$ is finite with $n$ elts, then $P(S)$ has $2^n$ elts

Cartesian ~~[illegible]~~ product If $S, T$ are sets then $S \times T$ is set of all ordered pairs $(s,t)$ where $s \in S$, $t \in T$.

Ex $S = \{1,2\}$, $T = \{4,5,6\}$

$S \times T = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6)\}$.

If $S, T$ finite with $n, m$ elts resp. then $S \times T$ has $nm$ elts.

Rmks ① $(s,t) = (s',t') \Leftrightarrow s = s',\ t = t'$.

② In math officially everything is a set. (But we don't really like to think that way.)
If $s, t$ are sets then by df

$$(s,t) := \{\{s\}, \{s,t\}\}.$$

Ie it is ~~the set~~ a set with two elts:
- $s$
- the set $\{s,t\}$ having two elts: $s$ and $t$.

This is confusing enough that we try not to bring it up but you can check that, with this df,
$(s,t) = (s',t') \Leftrightarrow s = s',\ t = t'$.

③ Even the elts of $\mathbb{N}$ are formally sets. So the way it's officially set up

$$0 = \emptyset \quad \text{and if } n \in \mathbb{N} \text{ is defined}$$

$$n+1 = n \cup \{n\}.$$

So $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$

$$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

This is super confusing so we never think of it this way. Don't write $\emptyset$ for $0$ when you mean $0$

as a natural number!

But notice $0$ has $0$ elts
$1$ has $1$ elt
$2$ has $2$ elts

~~Mappings / suppose $S$, $T$ are sets. They informally speaking a function from $S$ to $T$ is~~

## §5 Relations and mappings

Suppose $S, T$ are sets. Then a relation between $S$ and $T$ is a subset $R \subseteq S \times T$. If $s \in S$, $t \in T$ then $t$ is *related to* $s$ if $(s,t) \in R$.

A *mapping* from $S$ to $T$ is a relation ~~$R$~~ $f \subseteq S \times T$ st for each $s \in S$ there exists a unique $t \in T$ st $(s,t) \in f$. If $(s,t) \in f$ then we write ~~$f(s)$~~ $f(s) = t$. This defines a rule associating to each element of $s$ a unique element of $t$.

Mappings are also called functions. Often we define the function by giving the rule.

Ex $S = \mathbb{R}$, $T = \mathbb{R}$
$f(x) = x^2$
This is a rule taking any real number to ~~its~~ its

square. Officially $f = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = x^2\}$.
But we like thinking of $f$ better as a rule, more than as a subset of $\mathbb{R}\times\mathbb{R}$. ~~Sometimes we call~~

We write $f: S \to T$ to indicate that $f$ is a mapping from $S$ to $T$. Sometimes we give the rule by writing $s \mapsto f(s)$ ~~or~~ $f(s) =$ "some rule".

__Ex.__ $f: \mathbb{Z} \to \mathbb{Z}$ given by
$n \mapsto 2n$

This is just another ~~word~~ way of saying $f(n) = 2n$.

· $f: \mathbb{N} \to \mathbb{N}$ given by
$n \mapsto n!$

etc.

Obs. If $f: S \to T$, $f': S \to T$ are two functs and $f(s) = f'(s)$ for all $s \in S$. Then $f = f'$. / Intuitively, f'n def by rule

__A special Function__ If $S$ is a ~~set~~ then $id_S : S \to S$ is stupid f'n given by $s \mapsto s$.

__Composition of Functions__ Suppose $f: S \to T$, $g: T \to U$ are f'ns. We can define a new function
$h = g \circ f$ called the composition of $g$ with $f$ or $g$ composed with $f$ by the rule

$$h(s) = g(f(s)).$$

Thm Suppose $f: S \to T$, $g: T \to \mathcal{U}$, $h: \mathcal{U} \to V$ are f'ns. Then

(i) $g \circ id_T = id_{\mathcal{U}} \circ g = g$

(ii) $h \circ (g \circ f) = (h \circ g) \circ f$

Pf (a) $(g \circ id_T)(t) = g(id_T(t)) = g(t)$

So $g \circ id_T = g$. Similarly $id_{\mathcal{U}} \circ g = g$.

(b) $(h \circ (g \circ f))(t) = h((g \circ f)(t))$

$= h(g(f(t))) = (h \circ g)(f(t))$

$= ((h \circ g) \circ f)(t)$.

Since this holds for all $t \in T$, $h \circ (g \circ f) = (h \circ g) \circ f$.

## §6 Range, onto, into

Suppose $f: S \to T$ is a function and $A \subseteq S$. We write

$$f(A) = \{ f(a) : a \in A \}.$$

This is called the <u>image</u> of $A$, $f(S)$ is ~~calle~~ called the <u>range</u> of $f$., $f$ is <u>onto</u> if $f(S) = T$.

$f$ is said to be 1-1 if $f(s) = f(s') \Rightarrow s = s'$.

<u>Ex</u> • The function $f: \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$ is not onto, not 1-1. $f(\mathbb{R}) = [0, \infty)$.
Not 1-1 b/c $f(1) = f(-1)$.

• The function $f: \mathbb{R} \to [0, \infty)$ given by same rule $x \mapsto x^2$ is onto but not 1-1

• The f'n $f: [0, \infty) \to \mathbb{R}$ given by $x \mapsto x^2$ is not onto but is 1-1.

<u>Def</u> A function $f: S \to T$ is said to be a <u>1-1 correspondence</u> if it is both 1-1 and onto. In this case we can define a function $g: T \to S$ given by

$$g(t) = \text{the unique } s \in S \text{ st } f(s) = t.$$

We thus have $g \circ f = id_S$

$f \circ g = id_T$.

$g$ is sometimes called the <u>inverse</u> of $f$.

<u>Ex</u> $f: [0, \infty) \to [0, \infty)$ given by
$x \mapsto x^2$

is 1-1, onto.

Inverse is $g$ $f^{-1}(y) = \sqrt{y}$.