HW4, due Wednesday, March 5
Math 403, Spring 2014
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1. Let $\mathbf{O}_2 = \mathbf{O}_2(\mathbb{R})$ denote the second orthogonal group. Suppose $T \in \mathbf{O}_2$ and $\det T = -1$.

(1) Show that $T^2 = \mathrm{id}$.
(2) Recall that $v(\theta) := (\cos\theta, \sin\theta) \in \mathbb{R}^2$. Show that there exists a unique $\theta \in [0,\pi)$ such that $Tv(\theta) = v(\theta)$.
(3) Suppose $S \in \mathbf{O}_2$ is an element such that $\det S = -1$ and $Sv(\theta) = v(\theta)$. Show that $S = T$.

We call the element T such that $\det T = -1$ and $Tv(\theta) = v(\theta)$ the *reflection* through the line spanned by $v(\theta)$. By (3), this reflection is unique so we write it as $H(\theta)$. Write $H(\theta)$ in the form $R(\eta)H^i$ for some $\eta \in [0,2\pi), i \in \{0,1\}$.

2. Suppose $z \in \mathbb{R}$. The *floor* of z is the greatest integer $\lfloor z \rfloor$ which is less than or equal to z. Another way to say this is to say that $\lfloor z \rfloor$ is the unique integer such that

$$\lfloor z \rfloor \le z < \lfloor z \rfloor + 1.$$

For example, $\lfloor 3.1 \rfloor = \lfloor 3 \rfloor = 3$, while $\lfloor -3.1 \rfloor = \lfloor -4 \rfloor = -4$. Suppose $x, y \in \mathbb{R}$ and $y > 0$. Show that

$$0 \le x - \lfloor \frac{x}{y} \rfloor y < y.$$

3. Suppose G is a finite subgroup of $\mathbf{SO}_2$ with $|G| = n$. Show that G is cyclic. In fact, show that $G = \langle R(2\pi/n) \rangle$.

Hint: If $|G| > 1$, show that there is a smallest $\theta \in (0, 2\pi)$ such that $R(\theta) \in G$. Use the previous problem to show that $G = \langle R(\theta) \rangle$.

4. Suppose G is a group and $x, y \in G$. We say that x is *conjugate to* y if there exists $g \in G$ such that $gxg^{-1} = y$. The *conjugacy class* of x is the set $\mathrm{cl}(x) = \{z \in G : z \text{ is conjugate to } x\}$.

(1) Suppose x and y are conjugate. Show that the order of x is equal to the order of y.
(2) Show that $P = \{\mathrm{cl}(x) : x \in G\}$ is a partition of G.

5 (20 points)**.** For each non-negative integer n, write $X_n := \{x \in \mathbb{N} : x < n\} = \{0, 1, \cdots, n-1\}$. Let p_n denote the number of partitions of X_n. Show that $p_0 = p_1 = 1$ and, for $n \ge 1$,

$$p_{n+1} = \sum_{k=0}^{n} \binom{n}{k} p_k.$$

Use the formula to compute p_n for $n \le 5$.

Hint: Remember that $\binom{n}{k}$ is the number of k-element subsets of X_n.