HW1, due Wednesday, April 30
Math 403, Spring 2014
Patrick Brosnan, Instructor

1. Suppose R is a commutative ring. Let $M_2(R)$ denote the set of 2×2 matrices with coordinates in R. Define operations of addition and multiplication on $M_2(R)$ in the usual way:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}.$$

Show that with the above operations $M_2(R)$ is a ring.

2. Suppose p is a prime number, then I write $\mathbb{F}_p$ for the field $\mathbb{Z}/p\mathbb{Z}$. Let G denote the group of units $M_2(\mathbb{F}_2)^\times$ of the ring $M_2(\mathbb{F}_2)$. Show that G is isomorphic to the dihedral group $\mathbf{D}_3$.

3. Suppose R is a ring, and $r \in R$. We say that r is

(a) *idempotent* if $r^2 = r$;
(b) *nilpotent* if $r^k = 0$ for positive integer k.

Show that, in an integral domain 1 and 0 are the only idempotents and 0 is the only nilpotent element.

4. Suppose R is a commutative ring and $r \in R$. Set $\operatorname{Ann}(r) = \{x \in R : xr = 0\}$.

(1) Show $\operatorname{Ann}(r)$ is an ideal in R.
(2) In $R = \mathbb{Z}/6$ what is $\operatorname{Ann}(2)$?

5. Set

$$\mathbb{F}_4 := \left\{ \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \in M_2(\mathbb{F}_2) : a, b \in \mathbb{F}_2 \right\}.$$

Show that $\mathbb{F}_4$ is a field with 4 elements.

6 (20 point Bonus)**.** Suppose that R is a finite ring. For each element $x \in R$ write $o(x)$ of the order of x in the group $(R, +)$.

(1) Suppose $x, y \in R$. Show that $o(xy) | (o(x), o(y))$.
(2) Use (1) and Cauchy's theorem to show that a finite integral domain has order p^n for some prime p.