HW5, due Wednesday, March 25
Math 404, Spring 2015
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1. Show that the splitting field E of the polynomial x^4+1 in $\mathbb{Q}[x]$ has degree 4.

2. Let E be the splitting field of the polynomial $x^3-2 \in \mathbb{Q}[x]$. How many embeddings are there of E in itself?

3. Let p be a prime and let F be a field of characteristic p. Let $f(x) = x^p - x - 1$.

(1) Show that $f(x) = f(x+1)$.
(2) Suppose α is a root of f in a field extension E/F. Show that $\alpha+1$ is also a root of f.
(3) Suppose $g \in F[x]$ is an irreducible factor of f. Show that every other irreducible factor of f is of the form $g(x+a)$ for some $a \in \mathbb{F}_p$.
(4) Show that either f is irreducible in $F[x]$ or f splits in F.

4. Using Problem 3, show that, for each prime p, the polynomial $x^p - x - 1$ is irreducible over $\mathbb{Q}$.

5. Let $\alpha = e^{2\pi i/5}$ and let $\beta = e^{2\pi i/25}$.

(a) What is $[\mathbb{Q}(\alpha):\mathbb{Q}]$?
(c) Show that β is a root of the polynomial $f(x) = x^{20}+x^{15}+x^{10}+x^5+1$.
(d) Show that f is irreducible in $\mathbb{Q}[x]$, and conclude that $[\mathbb{Q}(\beta):\mathbb{Q}] = 20$.