Math 406, Spring 2021
HW05, due Wednesday, March 3 [1]

Reading: Read Chapter 6 of Crisman's text.

Graded Problems: Work the following problems for a grade. Turn them in on Gradescope.

Some problems are taken from the Online Version of Crisman's text:

`http://math.gordon.edu/ntic/`

Each problem is worth 20 points.

1. Suppose p is a prime number and $a_1, \ldots, a_r$ are integers (with $r \geq 1$). Assuming that $p | a_1 a_2 \cdots a_r$, show that there exists i such that $p | a_i$.

Hint: Induct on r and use the case $r = 2$, which was proved in class.

2. (Crisman 6.6.22) Show that $\log_{10} 5$ is irrational.

3. (Crisman 6.6.15) Find the gcd of $2^2 \cdot 3^5 \cdot 7^2 \cdot 13 \cdot 37$ and $2^3 \cdot 3^4 \cdot 11 \cdot 31^2$ by hand.

4. (Based on Crisman 6.6.1) According to Crisman, a *repunit* is one of the numbers in the infinite sequence 11, 111, 1111, ... Write r_n for the repunit with n 1s. So that $r_2 = 11$, $r_3 = 111$, ...

(a) (**20 points**) Suppose $n > 2$ and r_n is prime. Show that $n \equiv \pm 1 \pmod 6$.

(b) (**5 point bonus**) Obviously, $r_2 = 11$ is prime. Find two more prime repunits. Say how you found them and checked that they are prime. Don't just say, "on the web." But it's ok to use a computer or to write a short computer program. (The book promotes Sage, but I think using Python in Google Colab is more convenient.) Just include your source, whatever program or language you use.

5. For each $k = 1, 2, 3, \cdots$, let p_k denote the kth prime number. So $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... Set $N(k) = p_1 p_2 \cdots p_k + 1$.

(a) (**10 points**) Show that $N(k)$ is not divisible by any prime number less than or equal to p_k.

(b) (**5 point bonus**) Is $N(k)$ always prime? Prove it or give a counterexample.

(c) (**10 points**) Now let q_k denote the kth prime number congruent to 5 modulo 6. So, for example, $q_1 = 5$, $q_2 = 11$ and $q_3 = 17$. Set $M(k) := 6 q_1 q_2 \cdots q_k - 1$. Show that $M(k)$ is congruent to 5 modulo 6 and that $M(k)$ is not divisible by any q_i with $1 \leq i \leq k$.

(d) (**5 point bonus**) Show that there are infinitely many primes congruent to 5 modulo 6.

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[1] This version created Wednesday 24$^{\text{th}}$ March, 2021 at 19:37.