Assignment #3, due Wednesday, May 9

1. We consider the heat equation \( u_t - 2u_{xx} = 0 \) on the interval \([0, 1]\) with boundary conditions

\[
u(0, t) = 0, \quad u_x(1, t) = 0
\]

and initial condition \( u(x, 0) = 1 \). In the series for the solution \( u(x, t) \) find the first two terms. 

Hint: \( \int_0^{k\pi/2} \sin^2(z) dz = \frac{2k\pi}{4} \) for integer \( k \).

2. We consider the wave equation \( u_{tt} - 4u_{xx} = 0 \) on the interval \([0, 1]\) with boundary conditions

\[
u(0, t) = 0, \quad u'(1, t) = 0\]

and initial conditions \( u(x, 0) = u_0(x) = 0, \ u_t(x, 0) = u_1(x) = 1 \).

(a) Use the appropriate extension to define the function \( \tilde{u}_1(x) \) for all \( x \in \mathbb{R} \) and sketch the graph of this function. Hint: use an even extension at Neumann boundary, odd extension at Dirichlet boundary.

(b) Write down the D’Alembert formula for the extended solution \( \tilde{u}(x, t) \). Use this to find \( u(\frac{1}{2}, \frac{1}{2}) \): mark the interval over which you have to integrate \( \tilde{u}_1(x) \) on your graph of \( \tilde{u}_1(x) \); then find the value of \( u(\frac{1}{2}, \frac{1}{2}) \). Evaluate \( u(x, \frac{1}{2}) \) for \( x \in [0, 1] \).

3. Consider a square metal plate \( G = [0, 1] \times [0, 1] \). At three sides it is cooled to temperature 0 (Dirichlet condition), at the remaining side it is insulated (Neumann condition). The temperature \( u(x, y, t) \) satisfies the heat equation \( u_t - 2\Delta u = 0 \). We start with the initial temperature \( u_0(x, y) = 1 \). At what rate \( \lambda \) will the temperature decay, i.e., \( |u(x, y, t)| \leq ce^{-\lambda t} \)? For large \( t \) give an approximation to \( u(x, y, t) \). Hint: Find a solution of the form \( e^{-\lambda t}v(x, y) \) with the smallest possible \( \lambda \) and find the coefficient \( C \) so that \( u(x, y, t) = Ce^{-\lambda t}v(x, y) + \) faster decaying terms.

4. Consider a square membrane \( G = [0, 1] \times [0, 1] \) which is fixed at three sides (Dirichlet conditions) and free at the remaining side (Neumann conditions). The displacement \( u(x, y, t) \) satisfies the wave equation \( u_{tt} - 4\Delta u = 0 \). What is the lowest frequency \( \omega \) which the membrane can generate? Hint: Find a solution of the form \( u(x, y, t) = \cos(\omega t)v(x, y) \) with the smallest possible \( \omega \).