Assignment #3, due Wednesday, May 9

1. We consider the heat equation $u_t - 2u_{xx} = 0$ on the interval $[0, 1]$ with boundary conditions

$$u(0, t) = 0, \quad u_x(1, t) = 0$$

and initial condition $u(x, 0) = 1$. In the series for the solution $u(x, t)$ find the first two terms.

Hint: $\int_0^{k\pi/2} \sin^2(z) dz = \frac{k\pi}{4}$ for integer $k$.

For the heat equation we first want to find special solutions of the form $g(t)v(x)$ with separated variables. Plugging this into the PDE and separating the variables gives

$$-\frac{v''(x)}{v(x)} = -\frac{g'(t)}{2g(t)} = \lambda$$

Solve eigenvalue problem $-v''(x) = \lambda v(x)$ on $[0, 1]$ with boundary conditions $v(0) = 0$, $v'(1) = 0$: We have $\lambda > 0$ and

$$v(x) = C_1 \cos(\lambda^{1/2}x) + C_2 \sin(\lambda^{1/2}x)$$

Now $v(0) = 0$ gives $C_1 = 0$. We must have $C_2 \neq 0$, so $v'(1) = 0$ gives $\cos(\lambda^{1/2}) = 0$ which implies $\lambda^{1/2} = (j - \frac{1}{2})\pi$, $j = 1, 2, 3, \ldots$. Therefore we have the eigenvalues $\lambda_j$ and eigenfunctions $v_j(x)$

$$\lambda_j = \left[(j - \frac{1}{2})\pi\right]^2, \quad v_j(x) = \sin \left[(j - \frac{1}{2})\pi x\right], \quad j = 1, 2, 3, \ldots$$

Now we solve the problem $-g'(t) = 2\lambda_j g(t)$ and obtain $g(t) = c_j e^{-2\lambda_j t}$. Therefore the special solutions are $c_j v_j(x) e^{-2\lambda_j t}$. The solution of the initial value problem is

$$u(x, t) = \sum_{j=1}^{\infty} c_j v_j(x) e^{-2\lambda_j t} \quad c_j = \frac{\langle u_0, v_j \rangle}{\langle v_j, v_j \rangle}$$

where $u_0(x) = 1$. Here we have

$$\lambda_1 = \frac{1}{4}\pi^2, \quad v_1(x) = \sin\left(\frac{\pi}{2}x\right), \quad c_1 = \frac{\int_0^{\pi/2} 1 \cdot \sin(\pi/2) dx}{\int_0^{\pi/2} \sin^2(\pi/2) dx} = \frac{2/\pi}{\pi/2} = \frac{4}{\pi}$$

$$\lambda_2 = \frac{9}{4}\pi^2, \quad v_2(x) = \sin\left(\frac{3\pi}{2}x\right), \quad c_2 = \frac{\int_0^{3\pi/2} 1 \cdot \sin(3\pi/2) dx}{\int_0^{3\pi/2} \sin^2(3\pi/2) dx} = \frac{2/(3\pi)}{3\pi/2} = \frac{4}{3\pi}$$

so that we have

$$u(x, t) = \frac{4}{\pi} \sin\left(\frac{\pi}{2}x\right) \exp\left(-\frac{\pi^2}{2}t\right) + \frac{4}{3\pi} \sin\left(\frac{3\pi}{2}x\right) \exp\left(-\frac{9\pi^2}{2}t\right) + R(x, t), \quad |R(x, t)| \leq C \exp\left(-\frac{25\pi^2}{2}t\right).$$

2. We consider the wave equation $u_{tt} - 4u_{xx} = 0$ on the interval $[0, 1]$ with boundary conditions

$$u(0, t) = 0, \quad u'(1, t) = 0$$

and initial conditions $u(x, 0) = u_0(x) = 0$, $u_t(x, 0) = u_1(x) = 1$.

(a) Use the appropriate extension to define the function $\tilde{u}_1(x)$ for all $x \in \mathbb{R}$ and sketch the graph of this function. Hint: use an even extension at Neumann boundary, odd extension at Dirichlet boundary.

We obtain a function $\tilde{u}_1(x)$ on the real axis which is periodic with period 4 and

$$u_1(x) = \begin{cases} 1 & \text{for } x \in \cdots \cup [-4, -2] \cup [0, 2] \cup [4, 6] \cup \cdots \\ -1 & \text{for } x \in \cdots \cup [-2, 0] \cup [2, 4] \cup [6, 8] \cup \cdots \end{cases}$$
(b) Write down the D’Alembert formula for the extended solution \( \tilde{u}(x,t) \). Use this to find \( u(\frac{1}{2}, \frac{1}{2}) \): mark the interval over which you have to integrate \( \tilde{u}_1(x) \) on your graph of \( \tilde{u}_1(x) \); then find the value of \( u(\frac{1}{2}, \frac{1}{2}) \). Evaluate \( u(x, \frac{1}{2}) \) for \( x \in [0, 1] \).

Here \( c = 2, u_0(x) = 0 \) and the D’Alembert formula gives for \( x \in [0, 1] \)

\[
\tilde{u}(x,t) = \frac{1}{2c} \int_{y=-ct}^{y=ct} \tilde{u}_1(y) dy = \frac{1}{4} \int_{x-2t}^{x+2t} \tilde{u}_1(y) dy \\
u(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} \int_{\frac{1}{2}-2(\frac{1}{2})}^{\frac{1}{2}+2(\frac{1}{2})} \tilde{u}_1(y) dy = \frac{1}{4} \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{u}_1(y) dy = \frac{1}{4} \left( \int_{\frac{1}{2}}^{0} (-1)dy + \int_{0}^{\frac{3}{2}} 1dy \right) = \frac{1}{4} \left( -\frac{1}{2} + \frac{3}{2} \right) = \frac{1}{4}
\]

3. Consider a square metal plate \( G = [0, 1] \times [0, 1] \). At three sides it is cooled to temperature 0 (Dirichlet condition), at the remaining side it is insulated (Neumann condition). The temperature \( u(x,y,t) \) satisfies the heat equation \( u_t - 2\Delta u = 0 \). We start with the initial temperature \( u_0(x,y) = 1 \). At what rate \( \lambda \) will the temperature decay, i.e., \( |u(x,y,t)| \leq ce^{-\lambda t} \)? For large \( t \) give an approximation to \( u(x,y,t) \). \( \text{Hint:} \) Find a solution of the form \( e^{-\lambda t}v(x,y) \) with the smallest possible \( \lambda \) and find the coefficient \( C \) so that \( u(x,y,t) = Ce^{-\lambda t}v(x,y) + \text{faster decaying terms} \).

Assume we have Dirichlet conditions on the left, bottom and top side of the square and Neumann conditions on the right side of the square:

\[
u(0,y,t) = 0, \quad u_x(1,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,1,t) = 0
\]

Separation of variables: Find special solutions \( v(x,y)g(t) \), plugging this into the PDE gives

\[
-\frac{\Delta v(x,y)}{v(x,y)} = -\frac{g'(t)}{2g(t)} = \lambda
\]

Solve eigenvalue problem \(-v_{xx}(x,y) - v_{yy}(x,y) = \lambda v(x,y)\) on \([0,1] \times [0,1]\) with boundary conditions \( v(0) = 0, v'(1) = 0 \): Try to find eigenfunctions \( v(x,y) = p(x)q(y) \) with separated variables. Plugging this into the eigenvalue equation gives

\[
\begin{align*}
-\frac{p''(x)}{p(x)} + &\frac{-q''(y)}{q(y)} = \lambda \\
\end{align*}
\]

with constants \( \mu, \bar{\mu} \). This means for \( p(x) \) and \( \mu \)

\[
-\mu = \mu p(x), \quad p(0) = 0, \quad p'(1) = 0
\]

so we obtain from problem 1 that

\[
\mu_j = \left( j - \frac{1}{2} \right)^2, \quad p_j(x) = \sin \left( (j - \frac{1}{2}) \pi x \right), \quad j = 1, 2, 3, \ldots
\]

We have for \( q(y) \) and \( \bar{\mu} \) that

\[
-q''(y) = \nu q(y), \quad q(0) = 0, \quad q(1) = 0
\]

so we obtain

\[
\nu_k = [k\pi]^2, \quad q_k(y) = \sin [k\pi y], \quad k = 1, 2, 3, \ldots
\]

Therefore the eigenvalues \( \lambda_{jk} \) and eigenfunctions \( v_{jk}(x,y) \) for the eigenvalue problem in the square are

\[
\lambda_{jk} = \mu_j + \nu_k = \left( j - \frac{1}{2} \right)^2 + [k\pi]^2, \quad v_{jk}(x,y) = \sin \left( (j - \frac{1}{2}) \pi x \right) \sin [k\pi y], \quad j = 1, 2, \ldots, \quad k = 1, 2, \ldots
\]
The smallest eigenvalue is \( \lambda_{11} = \left( \frac{\pi}{2} \right)^2 + \pi^2 = \frac{5}{4} \pi^2 \) with the eigenfunction \( v_{11}(x, y) = \sin \left( \frac{\pi}{2} x \right) \sin(\pi y) \). As in problem 1 we obtain \( g(t) = c_{jk} e^{-2\lambda_{jk} t} \). Therefore the solution of the initial value problem is

\[
 u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{jk} v_{jk}(x, y) e^{-2\lambda_{jk} t}, \quad c_{jk} = \frac{\langle u_0, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle}
\]

Here

\[
 \langle u_0, v_{jk} \rangle = \langle 1, p_j(x) q_k(y) \rangle = \left( \int_0^1 p_j(x) dx \right) \left( \int_0^1 q_k(y) dy \right)
\]

\[
 \langle v_{jk}, v_{jk} \rangle = \langle p_j(x) q_k(y), p_j(x) q_k(y) \rangle = \left( \int_0^1 p_j(x) dx \right) \left( \int_0^1 q_k(y) dy \right)
\]

so we have

\[
 c_{11} = \frac{\left( \int_0^1 \sin \left( \frac{\pi}{2} x \right) dx \right) \left( \int_0^1 \sin(\pi y) dy \right)}{\left( \int_0^1 \sin^2 \left( \frac{\pi}{2} x \right) dx \right) \left( \int_0^1 \sin^2(\pi y) dy \right)} = \frac{\left( \frac{2}{\pi} \right) \left( \frac{2}{\pi} \right)}{\left( \frac{1}{2} \right) \left( \frac{1}{2} \right)} = \frac{16}{\pi^2}
\]

and

\[
 u(x, y, t) = \frac{16}{\pi^2} \sin \left( \frac{\pi}{2} x \right) \sin(\pi y) \exp \left( -\frac{5\pi^2}{2} t \right) + R(x, y, t), \quad |R(x, y, t)| \leq C' \exp(-\lambda_{21} t) = C' \exp \left( -\frac{13\pi^2}{2} t \right)
\]

and \( |u(x, y, t)| \leq C \exp(-\lambda_{11} t) = C \exp(-\frac{5\pi^2}{2} t) \).

4. Consider a square membrane \( G = [0, 1] \times [0, 1] \) which is fixed at three sides (Dirichlet conditions) and free at the remaining side (Neumann conditions). The displacement \( u(x, y, t) \) satisfies the wave equation \( u_{tt} - 4\Delta u = 0 \). What is the lowest frequency \( \omega \) which the membrane can generate? \( \text{Hint:} \) Find a solution of the form \( u(x, y, t) = \cos(\omega t) v(x, y) \) with the smallest possible \( \omega \).

Separation of variables: Find special solutions \( v(x, y) g(t) \). For \( v(x, y) \) we obtain the same eigenvalue problem as in problem 3. For \( g(t) \) we have

\[
 \frac{-g''(t)}{4g(t)} = \lambda_{jk}, \quad -g''(t) = 4\lambda_{jk} g(t).
\]

This ODE has the general solution

\[
 g(t) = A_{jk} \cos(\omega_{jk} t) + B_{jk} \sin(\omega_{jk} t) \quad \omega_{jk} = 2\lambda_{jk}^{1/2}.
\]

Therefore we can write the solution of the initial value problem as

\[
 u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} v_{jk}(x, y) \left[ A_{jk} \cos(\omega_{jk} t) + B_{jk} \sin(\omega_{jk} t) \right]
\]

where the coefficients \( A_{jk} \) and \( B_{jk} \) are determined from the initial conditions \( u_0(x, y) \) and \( u_1(x, y) \). We see that the possible frequencies of the membrane are

\[
 \omega_{jk} = 2\lambda_{jk}^{1/2} = 2 \left[ (j - \frac{1}{2})^2 + k^2 \right]^{1/2} \pi, \quad j = 1, 2, \ldots, \quad k = 1, 2, \ldots.
\]

The lowest possible frequency of the membrane is therefore

\[
 \omega_{11} = 2\lambda_{11}^{1/2} = 2 \left[ \frac{5}{4} \right]^{1/2} \pi = \pi \sqrt{5}
\]