Real Analysis HW 5 Solutions

Problem 7: Let $f$ be an increasing real-valued function on $[0, 1]$. For a natural number $n$, define $P_n$ to be the partition of $[0, 1]$ into $n$ subintervals on length $1/n$. Show that $U(f, P_n) - L(f, P_n) \leq 1/n[f(1) - f(0)]$. Use Problem 5 to show that $f$ is Riemann integrable over $[0, 1]$.

Proof: Define the interval $I_{k,n} = ((k-1)/n, k/n)$ for all natural numbers $k, n$. From the definition of the upper and lower Darboux sums, and the fact that $f$ is increasing,

$$U(f, P_n) = \frac{1}{n} \sum_{k=1}^{n} \sup_{x \in I_{k,n}} f(x) = \frac{1}{n} \sum_{k=1}^{n} f(k/n)$$

and

$$L(f, P_n) = \frac{1}{n} \sum_{k=1}^{n} \inf_{x \in I_{k,n}} f(x) = \frac{1}{n} \sum_{k=1}^{n} f((k-1)/n)$$

Therefore

$$U(f, P_n) - L(f, P_n) = \frac{1}{n} \sum_{k=1}^{n} (f(k/n) - f((k-1)/n)) = \frac{1}{n} (f(1) - f(0)),$$

since the above summation is telescoping. Now using the result of Problem 5, $P_n$ is a sequence of partitions such that

$$\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0$$

Therefore $f$ is Riemann integrable over $[0, 1]$. ■

Problem 11: Does the Bounded Convergence Theorem hold for the Riemann integral?

Solution: No. Consider the sequence of functions

$$f_n = \sum_{k=1}^{n} \chi_{\{q_k\}},$$

where $\{q_k\}_{k=1}^{\infty}$ is an enumeration of the rationals in $[0, 1]$. It is easy to see for any natural number $n$, $f_n$ is Riemann integrable and $\int_{[0,1]} f_n = 0$, since for any partition $P_k$ of size $1/k$ of $[0, 1]$, we have

$$U(f_n, P_k) \leq \frac{n}{k}, \quad \text{and} \quad L(f_n, P_k) = 0.$$
Therefore \( \lim_{k \to \infty} (U(f_n, P_k) - L(f_n, P_k)) = 0 \). However, \( f_n \) converges pointwise on \([0, 1]\) to \( f = \chi_Q \), which is not Riemann integrable on \([0, 1]\).

**Problem 15:** Verify the assertions in the last Remark of this section.

**Solution:** Yup, the Remark is true

**Problem 16:** Let \( f \) be a nonnegative bounded measurable function on a set of finite measure \( E \). Assume \( \int_E f = 0 \). Show that \( f = 0 \) a.e. on \( E \).

**Solution:** By Chebychev’s inequality, we have

\[
m(\{ E : f > 1/n \}) \leq n \int_E f = 0, \quad \forall n \geq 1.
\]

We conclude that \( m(\{ E : f > 0 \}) = 0 \), since \( \{ E : f > 0 \} = \bigcup_{n=1}^{\infty} \{ E : f > 1/n \} \) is a countable union of measure 0 sets. Therefore \( f = 0 \) a.e. on \( E \).

**Problem 19:** For a number \( \alpha \), define \( f(x) = x^\alpha \) for \( 0 < x \leq 1 \), and \( f(0) = 0 \). Compute \( \int_0^1 f \).

**Solution:** To do this, first consider the sequence of truncated functions \( \{ f_n \} \) given by \( f_n = f \chi_{[1/n, 1]} \). Clearly this sequence is increasing, non-negative, and converges to \( f \) pointwise. Therefore by the monotone convergence theorem

\[
\int_{[0,1]} f = \lim_{n \to \infty} \int_{[1/n, 1]} f.
\]

Since \( f \) is continuous and bounded on \([1/n, 1]\) for every \( n \geq 1 \), the Lebesgue and Riemann integrals coincide,

\[
\int_{[1/n, 1]} f = (R) \int_{1/n}^1 x^\alpha \, dx = \begin{cases} \frac{1}{1+\alpha} (1 - n^{-\alpha-1}) & \text{if } \alpha \neq -1 \\ \log n & \text{if } \alpha = -1 \end{cases}.
\]

Taking the limit we obtain

\[
\int_{[0,1]} f = \begin{cases} \frac{1}{1+\alpha} & \text{if } \alpha > -1 \\ \infty & \text{if } \alpha \leq -1 \end{cases}.
\]

**Problem 21:** Let the function \( f \) be nonnegative and integrable over \( E \) and \( \epsilon > 0 \). Show there is a simple function \( \eta \) on \( E \) that has finite support, \( 0 \leq \eta \leq f \) on \( E \) and \( \int_E |f - \eta| < \epsilon \). If \( E \) is a closed, bounded interval, show there is a step function \( h \) on \( E \) that has finite support and \( \int_E |f - h| < \epsilon \).

**Solution:** Let \( \epsilon > 0 \). We know by the definition of the integral for non-negative measurable functions that there exists a bounded, measurable function of finite support \( h \leq f \), such that

\[
\int_E h > \int_E f - \epsilon/2.
\]
We also know that for any given bounded, measurable function \( h \) with finite support, there exists a simple function \( \eta \leq h \) with the same support as \( h \) such that
\[
\int_E \eta > \int_E h - \epsilon/2.
\]
Therefore we see that there is a simple function \( \eta \) with finite support such that \( \eta \leq f \), and
\[
\int_E (f - \eta) < \epsilon
\]
In the case that \( E \) is a closed and bounded interval, let \( \eta = \sum_{k=1}^{n} c_k \chi_{E_k} \) be a simple function such that
\[
\int_E (f - \eta) < \epsilon/2.
\]
For each \( k \), since \( m(E_k) < \infty \), there exists a finite collection of disjoint open intervals \( \{I^k_i\}_{i=1}^{n_k} \) such that
\[
\int_{\mathbb{R}} \left| \chi_{E_k} - \sum_{i=1}^{n_k} \chi_{I^k_i} \right| = m \left( E_k \Delta \bigcup_{i=1}^{n_k} I^k_i \right) < \epsilon/2n.
\]
Moreover, since \( E \) is a closed bounded interval, we may assume that each \( I^k_i \) is contained inside \( E \), but may no longer be open. Define a function \( \psi \) by
\[
\psi = \sum_{k=1}^{n} \sum_{i=1}^{n_k} c_k \chi_{I^k_i}.
\]
Note that this function is a step function even though some of the intervals may overlap, since any finite collection of intervals that cover \( E \) define a partition of \( E \) by ordering their endpoints. It follows that
\[
\int_E |\psi - \eta| \leq \sum_{k=1}^{n} c_k \int_E \left| \chi_{E_k} - \sum_{i=1}^{n_k} \chi_{I^k_i} \right| \leq \epsilon/2.
\]
Therefore \( \psi \) is a step function such that,
\[
\int_E |\psi - f| < \epsilon
\]

**Problem 22:** Let \( \{f_n\} \) be a sequence of nonnegative measurable functions of \( \mathbb{R} \) that converges pointwise on \( \mathbb{R} \) to \( f \) integrable. Show that if
\[
\int_\mathbb{R} f = \lim_{n \to \infty} \int_\mathbb{R} f_n, \quad \text{then} \quad \int_E f = \lim_{n \to \infty} \int_E f_n
\]
for any measurable set \( E \).
Solution: Let $E$ be a measurable set. Using the fact that $\int_{\mathbb{R}} f = \int_{E} f + \int_{\mathbb{R} \sim E} f$, Fatou’s lemma on $\mathbb{R} \sim E$ implies

$$\int_{\mathbb{R} \sim E} f \leq \liminf_{n} \int_{\mathbb{R} \sim E} f_{n} = \int_{\mathbb{R}} f - \limsup_{n} \int_{E} f_{n}.$$ 

Therefore

$$\int_{E} f \geq \limsup_{n} \int_{E} f_{n}.$$ 

However Fatou’s Lemma on $E$ gives

$$\int_{E} f \leq \liminf_{n} \int_{E} f_{n}.$$ 

These two inequalities conclude the proof. ■

Problem 25: Let $\{f_{n}\}$ be a sequence of nonnegative measurable functions on $E$ that converge pointwise on $E$ to $f$. Suppose $f_{n} \leq f$ on $E$ for each $n$. Show that

$$\lim_{n \to \infty} \int_{E} f_{n} = \int_{E} f.$$ 

Solution: Clearly for each $n$, $\int_{E} f_{n} \leq \int_{E} f$, therefore

$$\limsup_{n} \int_{E} f_{n} \leq \int_{E} f.$$ 

However, since $\{f_{n}\}$ are nonnegative, Fatou’s Lemma implies

$$\int_{E} f \leq \liminf_{n} \int_{E} f_{n}.$$ 

The conclusion follows from these two inequalities. ■