Problem 44: Let $f$ be integrable over $\mathbb{R}$ and $\epsilon > 0$. Establish the following three approximation properties.

(i) There is a simple function $\eta$ on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f - \eta| < \epsilon$

(ii) There is a step function $s$ on $\mathbb{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \epsilon$.

(iii) There is a continuous function $g$ on $\mathbb{R}$ which vanishes outside a bounded set and $\int_{\mathbb{R}} |f - g| < \epsilon$.

Solution:

(i) As shown in Problem 24, if $f$ is non-negative we may find an increasing sequence of non-negative simple functions $\{\phi_n\}$ with finite support such that $\phi_n \rightarrow f$ pointwise. It follows by monotone convergence that we can find a $\phi$ such that

$$\int_{\mathbb{R}} |f - \phi| = \int_{\mathbb{R}} f - \phi < \epsilon.$$

For general $f$ we write $f = f^+ - f^-$, and find $\eta_1$ and $\eta_2$ simple and of finite support such that $\int_{\mathbb{R}} |f^+ - \eta_1| < \epsilon/2$ and $\int_{\mathbb{R}} |f^- - \eta_2| < \epsilon/2$. Since $f^+$ and $f^-$ have disjoint support, we see that $\eta_1$ and $\eta_2$ must also have disjoint support, therefore $\eta = \eta_1 - \eta_2$ is also simple with finite support and it follows that

$$\int_{\mathbb{R}} |f - \eta| \leq \int_{\mathbb{R}} |f^+ - \eta_1| + \int_{\mathbb{R}} |f^- - \eta_2| < \epsilon.$$

(ii) By part (i), since we can approximate by simple functions, by the triangle inequality it suffices to show that the characteristic function $\chi_E$ of a bounded measurable set $E$ can be approximated by step functions. Note that since $E$ is measurable, we can find a disjoint collection of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $E = \bigcup_{k=1}^{\infty} I_k$, and $m(O \sim E) < \epsilon/2$. Since $O$ must have finite measure we can find an $N$ large enough such that $m(\bigcup_{k=N+1}^{\infty} I_k) < \epsilon/2$. Therefore

$$s = \sum_{k=1}^{N} \chi_{I_k}$$
is a step function and

\[ \int_{\mathbb{R}} |\chi_E - s| \leq \sum_{k=1}^{N} \int_{\mathbb{R}} |\chi_{E \cap I_k} - \chi_{I_k}| + \sum_{k=N+1}^{\infty} \int_{\mathbb{R}} \chi_{E \cap I_k} \]

\[ \leq m \left( \bigcup_{k=1}^{N} I_k \sim E \right) + m \left( \bigcup_{k=N+1}^{\infty} I_k \cap E \right) \]

\[ \leq m(\mathcal{O} \sim E) + m \left( \bigcup_{k=N+1}^{\infty} I_k \right) < \epsilon. \]

(iii) Using part (ii), once again we see by the triangle inequality that it suffices to show that any characteristic function of a bounded interval \( \chi_{[a,b]} \) can be approximated by a continuous function. Let \( g \) be the continuous function which is 1 on \([a + \epsilon/2, b - \epsilon/2]\) and linearly interpolated to 0 outside of \([a, b]\), then

\[ \int_{\mathbb{R}} |\chi_{[a,b]} - g| < m([a, a + \epsilon/2) \cup (b - \epsilon/2, b]) = \epsilon. \]

Problem 46: (Riemann-Lebesgue) Let \( f \) be integrable over \((-\infty, \infty)\). Show that

\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \cos nx dx = 0. \]

Solution: Using the result from problem 44, we know there exists a step function \( s \), vanishing outside of a closed bounded interval such that \( \int_{\mathbb{R}} |f - s| < \epsilon/2 \). Let \( s \) be the step function which we write in canonical form as

\[ s = \sum_{k=1}^{K} s_k \chi(a_k, b_k), \]

where \( \{(a_k, b_k)\} \) are a disjoint collection of bounded open intervals and \( \{s_k\} \) are distinct. Note that it doesn’t matter that we don’t define \( s \) at the end points of the intervals since they are a set of measure 0. We see that

\[ \left| \int_{-\infty}^{\infty} s(x) \cos nx dx \right| \leq \sum_{k=1}^{K} |s_k| \int_{a_k}^{b_k} \cos nx dx \]

\[ = \sum_{k=1}^{K} \frac{|s_k|}{n} |\sin nb_k - \sin na_k| \]

\[ \leq \frac{2K \max\{|s_i|\}}{n}. \]
Therefore if \( n > N \equiv 4K \max \{ s_i \} / \epsilon \), we conclude

\[
\left| \int_{-\infty}^{\infty} f(x) \cos nx \, dx \right| \leq \int_{-\infty}^{\infty} |f(x) - s(x)| \, dx + \left| \int_{-\infty}^{\infty} s(x) \cos nx \, dx \right| < \epsilon/2 + \epsilon/2 = \epsilon. \]

\[ \blacksquare \]

**Problem 47:** Let \( f \) be integrable over \((-\infty, \infty)\).

(i) Show that for each \( t \),

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} f(x + t) \, dx.
\]

(ii) Let \( g \) be a bounded measurable function on \( \mathbb{R} \). Show that

\[
\lim_{t \to 0} \int_{-\infty}^{\infty} g(x) \cdot [f(x) - f(x + t)] \, dx = 0.
\]

**Solution:**

(i) Note that this result is true for simple functions. Let \( \varphi = \sum_{k=1}^{n} c_k \chi_{E_k} \) be simple, then since \( \chi_{E + t} = \chi_{E-t}(x) \), we have by the translation invariance of Lebesgue

\[
\int_{\mathbb{R}} \varphi(x + t) \, dx = \sum_{k=1}^{n} c_k m(E_k - t) = \sum_{k=1}^{n} c_k m(E_k) = \int_{\mathbb{R}} \varphi(x) \, dx.
\]

For the case of integrable \( f \), we may restrict ourselves to \( f \) non-negative, since we may always write any integrable \( f \) as \( f = f^+ - f^- \), where \( f^+(x) = \max\{f(x), 0\} \) and \( f^-(x) = \max\{-f(x), 0\} \). For such a non-negative \( f \), we know that there is a sequence of simple functions with finite support \( \{\varphi_n\}_{n=1}^{\infty} \) such that \( \varphi_n \leq f \) with \( \varphi_n \) converging to \( f \) pointwise and monotonically. In particular \( \varphi(\cdot + t) \to f(\cdot + t) \) monotonically for every fixed \( t \). It follows from the monotone convergence theorem that

\[
\int_{\mathbb{R}} f(x + t) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}} \varphi_n(x + t) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}} \varphi_n(x) \, dx = \int_{\mathbb{R}} f(x) \, dx.
\]

(ii) Let \( M < \infty \) be a constant such that \( |g| \leq M \). Since \( f \) is integrable, we know by Problem 44 that for every \( \epsilon > 0 \) there exists a \( h \), continuous and of bounded support such that \( \int_{\mathbb{R}} |f - h| < \epsilon/2M \). It follows that \( h \) uniformly continuous and bounded by some constant \( N \). Let \( \{t_n\} \) be any sequence \( t_n \to 0 \) and define \( h_n(x) = |h(x) - h(x+t_n)| \).

Then \( h_n \to 0 \) pointwise and \( 0 \leq h_n(x) \leq 2N \). It follows by the bounded convergence theorem that

\[
\lim_{n \to \infty} \int_{\mathbb{R}} |f(x) - f(x + t_n)| = 0.
\]
Therefore

\[
\left| \int_{\mathbb{R}} g(x) \cdot [f(x) - f(x + t_n)] \, dx \right| \leq M \int_{\mathbb{R}} |f(x) - h(x)| \, dx
\]

\[
+ M \int_{\mathbb{R}} |f(x + t_n) - h(x) + t_n| \, dx + M \int_{\mathbb{R}} |f(x) - f(x + t_n)| \, dx
\]

By part i) and the way that \( h \) was chosen, the sum of the first two terms on the right-hand side above are bounded by \( \epsilon \). Taking the lim sup of both sides, we obtain

\[
\limsup_{n} \left| \int_{\mathbb{R}} g(x) \cdot [f(x) - f(x + t)] \, dx \right| < \epsilon + M \lim_{n \to \infty} \int_{\mathbb{R}} |f(x) - f(x + t_n)| \, dx = \epsilon.
\]

Since this is true for every \( \epsilon > 0 \) and every \( \{t_n\}, t_n \to 0 \), we conclude

\[
\lim_{t \to 0} \int_{\mathbb{R}} g(x) \cdot [f(x) - f(x + t)] \, dx.
\]

Problem 50: Let \( \mathcal{F} \) be a family of functions, each of which is integrable over \( E \) if and only if for each \( \epsilon > 0 \), there is a \( \delta > 0 \) such that for each \( f \in \mathcal{F} \),

if \( A \subseteq E \) is measurable and \( m(A) < \delta \), then \( \left| \int_{A} f \right| < \epsilon \).

Solution: Clearly by the triangle inequality if \( \int_{A} |f| < \epsilon \), then \( \left| \int_{A} f \right| < \epsilon \) and so uniform integrability easily implies this condition. However, for any \( A \) measurable we may write \( \int_{A} |f| = \int_{A^+} f + \int_{A^-} f \) where \( A^+ = \{x \in A : f \geq 0\} \) and \( A^- = \{x \in A : f < 0\} \) are disjoint. It follows that there exists \( \delta^+ \) and \( \delta^- \) such that if \( m(A^+) < \delta^+ \) and \( m(A^-) < \delta^- \), then

\[
\left| \int_{A^+} f \right| < \epsilon/2 \quad \text{and} \quad \left| \int_{A^-} f \right| < \epsilon/2.
\]

Choosing \( \delta = \min\{\delta^+, \delta^-\} \), we conclude that if \( m(A) < \delta \), then

\[
\int_{A} |f| < \epsilon.
\]

Therefore this condition implies uniform integrability.

Problem 51: Let \( \mathcal{F} \) be a family of functions, each of which is integrable over \( E \). Show that \( \mathcal{F} \) is uniformly integrable over \( E \) if and only if for each \( \epsilon > 0 \), there is a \( \delta > 0 \) such that for each \( f \in \mathcal{F} \), if \( U \) is open and \( m(E \cap U) < \delta \),

\[
\int_{E \cap U} |f| < \epsilon.
\]
**Proof:** Clearly if $\mathcal{F}$ is uniformly integrable, it follows trivially that there exists a $\delta > 0$ such that if $U$ is open and $m(E \cap U) < \delta$, then

$$\int_{E \cap U} |f| < \epsilon.$$ 

Now assume the other property, and let $\delta > 0$ be such that if $U$ is open and $m(E \cap U) < \delta$, then $\int_{E \cap U} |f| < \epsilon$. If $A \subseteq E$ is measurable such that $m(A) < \delta$, then we know that there is an open set $O$ containing $A$ such that $m(O \sim A) < \delta - m(A)$. Therefore

$$m(E \cap O) \leq m(O) = m(A) + m(O \sim A) < \delta,$$

and since $A \subseteq E \cap O$, it follows

$$\int_A |f| \leq \int_{E \cap O} |f| < \epsilon.$$

\[\Box\]