Problem 33: Let \( \{f_n\} \) be a sequence of functions on \([a, b]\) converging pointwise to \( f \). Then \( \text{TV}(f) \leq \lim \inf_n \text{TV}(f_n) \).

Solution: Let \( P \) be any partition of \([a, b]\), then since \( V(f_n, P) \) only depends on \( f_n \) at a finite number of points,

\[
V(f, P) = \lim_n V(f_n, P).
\]

Furthermore \( V(f_n, P) \leq TV(f_n) \) and therefore

\[
V(f, P) = \lim_n V(f_n, P) \leq \lim \inf_n TV(f_n).
\]

Taking the \( \sup \) over all partitions \( P \) gives

\[
\text{TV}(f) \leq \lim \inf_n \text{TV}(f_n).
\]

Problem 35: For \( \alpha \) and \( \beta \) positive numbers, define the function \( f \) on \([0, 1]\) by

\[
f(x) = \begin{cases} 
x^\alpha \sin \left(1/x^\beta\right) & \text{for } 0 < x \leq 1 \\
0 & \text{for } x = 0.
\end{cases}
\]

Show that if \( \alpha > \beta \), then \( f \) is of bounded variation on \([0, 1]\), by showing that \( f' \) is integrable over \([0, 1]\). Then show that if \( \alpha \leq \beta \), then \( f \) is not of bounded variation on \([0, 1]\).

Solution: If \( \alpha > \beta \), then

\[
f'(x) = \alpha x^{\alpha-1} \sin \left(1/x^\beta\right) - \beta x^{\alpha-\beta-1} \cos \left(1/x^\beta\right).
\]

Since \( f \) is \( C^1 \) and bounded on \((0, 1)\) we can use the fundamental theorem of calculus for Riemann integrals to conclude that for any partition \( P \)

\[
V(f, P) = \sum_{k=0}^{n} |f(x_k) - f(x_{k-1})| \leq \int_0^1 |f'| 
\]

and therefore

\[
\text{TV}(f) \leq \int_0^1 |f'| \leq \int_0^1 \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1} < \infty,
\]
since $\alpha > 0$ and $\alpha - \beta > 0$.

If $\alpha \leq \beta$, choose a partition $P_n = \{0, a_n, a_{n-1}, \ldots, a_1\}$, where

$$a_n = \left(\frac{n\pi}{2}\right)^{-1/\beta}$$

then we see that

$$V(f, P_n) = \sum_{k=1}^{n} \left(\frac{k\pi}{2}\right)^{-\alpha/\beta}.$$ 

Note that this series diverges as $n \to \infty$ since $\alpha/\beta \leq 1$. Therefore

$$\lim_{n \to \infty} V(f, P_n) \leq TV(f) = \infty.$$ 

Problem 37: Let $f$ be a continuous function on $[0, 1]$ that is absolutely continuous on $[\epsilon, 1]$ for each $0 < \epsilon < 1$.

(i) Show that $f$ may not be absolutely continuous on $[0, 1]$.

(ii) Show that $f$ is absolutely continuous on $[0, 1]$ if it is increasing.

(iii) Show that the function $f$ on $[0, 1]$, defined by $f(x) = \sqrt{x}$ for $0 \leq x \leq 1$, is absolutely continuous, but not Lipschitz, on $[0, 1]$.

Solution:

(i) Consider

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Note that on $[\epsilon, 1]$

$$|f'(x)| = \left| \sin(1/x) - \frac{1}{x} \cos(1/x) \right| \leq 1 + \frac{1}{\epsilon}$$

Therefore $f$ is Lipschitz on $[\epsilon, 1]$ and hence absolutely continuous on $[\epsilon, 1]$. However, we know from Problem 35 that $f$ is not BV on $[0, 1]$ and therefore not absolutely continuous on $[0, 1]$.

(ii) Suppose $f$ is increasing, let $\eta > 0$ and choose $\epsilon$ so that

$$f(\epsilon) - f(0) < \eta/2.$$ 

Since $f$ is absolutely continuous on $[\epsilon, 1]$ choose $\delta > 0$ in response to $\eta/2$ in the absolute continuity condition on $[\epsilon, 1]$. Now suppose that $\{(a_k, b_k)\}_{k=1}^{N}$ is a collection of disjoint open intervals such that $\sum_{k=1}^{N} |b_k - a_k| < \delta$. We may assume that $\epsilon$ is not contained
in any of the intervals since we may always split any such interval \((a_{k_0}, b_{k_0})\) into two consecutive intervals \((a_{k_0}, \epsilon) \cup (\epsilon, b_{k_0})\) such that, by the fact that \(f\) is increasing,

\[
|f(b_{k_0}) - f(\epsilon)| + |f(\epsilon) - f(a_{k_0})| = |f(b_{k_0}) - f(a_{k_0})|.
\]

It follows that we may divide the set of intervals into \(n^-\) intervals to the left of \(\epsilon\), \(\{(a^-_k, b^-_k)\}_{k=1}^{n^-}\) and \(n^+\) intervals to the right of \(\epsilon\), \(\{(a^+_k, b^+_k)\}_{k=1}^{n^+}\). We observe that

\[
\sum_{k=1}^{n^+} |b^+_k - a^-_k| < \delta
\]

and so by uniform integrability on \([\epsilon, 1]\),

\[
\sum_{k=1}^{n^+} |f(b^+_k) - f(a^+_k)| < \eta/2.
\]

Also, since \(f\) is increasing,

\[
\sum_{k=1}^{n^-} |f(b^-_k) - f(a^-_k)| \leq f(\epsilon) - f(0) < \eta/2.
\]

Therefore

\[
\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \eta.
\]

(iii) Clearly \(\sqrt{x}\) is not Lipschitz on \([0, 1]\) since its derivative is unbounded as \(x \to 0\). However, \(\sqrt{x}\) is increasing and is Lipschitz on \([\epsilon, 1]\) for any \(\epsilon > 0\). Therefore by (ii), \(f\) is absolutely continuous on \([0, 1]\).

\[\blacksquare\]

**Problem 39:** Use the preceding problem to show that if \(f\) is continuous and increasing on \([a, b]\), then \(f\) is absolutely continuous on \([a, b]\) if and only if for each \(\epsilon\), there is a \(\delta > 0\) such that for a measurable subset \(E\) of \([a, b]\),

\[
m^*(f(E)) < \epsilon \text{ if } m(E) < \delta.
\]

**Solution:** Suppose that \(f\) is absolutely continuous, and let \(\delta > 0\) be chosen in response to \(\epsilon > 0\) in the absolute continuity condition as generalized in Problem 38. Suppose that \(E \subseteq [a, b]\) is measurable and \(m(E) < \delta/2\). We can find a countable cover of disjoint open intervals \(\{(a_k, b_k)\}_{k=1}^{\infty}\) of \(E\) so that

\[
m \left( \bigcup_{k=1}^{\infty} I_k \right) < \delta/2 + m(E) < \delta
\]
and therefore

\[ \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon. \]

Since \( f \) is increasing and continuous \( \{f(I_k)\}_{k=1}^{\infty} \) is an open cover of \( f(E) \) and \( m(f(I_k)) = f(b_k) - f(a_k) \). By the definition of outer measure we conclude that

\[ m^*(f(E)) \leq \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon. \]

For the converse, let \( \delta > 0 \) be chosen to satisfy the converse condition with \( \epsilon > 0 \) and let \( \{(a_k, b_k)\}_{k=1}^{n} \) be a finite collection of disjoint open interval such that \( E = \bigcup_{k=1}^{n}(a_k, b_k) \) and \( m(E) < \delta \). Since \( f \) is increasing we see that \( \{f((a_k, b_k))\}_{k=1}^{n} \) are disjoint and by continuity \( m(f((a_k, b_k))) = f(b_k) - f(a_k) \). By the converse condition

\[ \sum_{k=1}^{n} |f(b_k) - f(a_k)| = m \left( \bigcup_{k=1}^{n}(a_k, b_k) \right) < \epsilon. \]

\[ \blacksquare \]

**Problem 43:** Define the functions \( f \) and \( g \) on \([-1, 1]\) by \( f(x) = x^{1/3} \) for \(-1 \leq x \leq 1\) and

\[ g(x) = \begin{cases} x^2 \cos(\pi/2x) & \text{if } x \neq 0, x \in [-1, 1] \\ 0 & \text{if } x = 0. \end{cases} \]

(i) Show that both \( f \) and \( g \) are absolutely continuous on \([-1, 1]\).

(ii) For the partition \( P_n = \{-1, 0, 1/2n, 1/[2n - 1], \ldots, 1/3, 1/2, 1\} \) of \([-1, 1]\), examine \( V(f \circ g, P_n) \).

(iii) Show that \( f \circ g \) fails to be of bounded variation, and hence also fails to be absolutely continuous, on \([-1, 1]\).

**Solution:**

(i) To show the absolute continuity of \( f \), we note that it is monotone on \([-1, 1]\), and so it will be absolutely continuous if

\[ \int_{-1}^{1} f' = f(1) - f(-1). \]

Since \( f'(x) = \frac{1}{3}x^{-2/3} \) is Riemann integrable on \([-1, -\epsilon] \cup [\epsilon, 1]\) for any \( \epsilon > 0 \), and \( f(1) = 1 \), and \( f(-1) = -1 \), then by monotone convergence

\[ \frac{1}{3} \int_{-1}^{1} x^{2/3} dx = \lim_{\epsilon \to 0} \left( (1)^{1/3} + \epsilon^{1/3} - (-\epsilon^{1/3}) - (-1)^{1/3} \right) = 2, \]

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and therefore \( f \) must be absolutely continuous.

To show that \( g \) is absolutely continuous, we simply show that it is the indefinite integral of some function over \([a, b]\). Define \( g(x) = 2x \cos(\pi/2x) + \frac{1}{2} \sin(\pi/2x) \), then we claim that

\[
\int_{-1}^{x} g(s) \, ds = x^2 \cos(\pi/2x) = f(x)
\]

Once again using the fact that \( g(x) \) is integrable on \([-1, -\epsilon] \cup [\epsilon, 1]\) for any \( \epsilon > 0 \), we can say that if \( x < 0 \), then we already have

\[
\int_{-1}^{x} g(s) \, ds = x^2 \cos(\pi/2x) = f(x).
\]

However if \( x \geq 0 \), then

\[
\int_{-1}^{x} g(s) \, ds = \lim_{\epsilon \to 0} \left[ x^2 \cos(\pi/2x) - \epsilon^2 \cos(-\pi/2\epsilon) + \epsilon^2 \cos(\pi/2\epsilon) - 0 \right]
\]

\[
= x^2 \cos(\pi/2x) = f(x).
\]

Therefore \( f(x) \) is absolutely continuous.

(ii) Let \( P_n = \{-1, 0, 1/2n, 1/[2n-1], \ldots, 1/3, 1/2, 1\} \) be a partition of \([-1, 1]\). We examine \( V(f \circ g, P_n) \),

\[
V(f \circ g, P_n) = V(x^{2/3} \cos^{1/3}(\pi/2x), P_n)
\]

\[
= \frac{2^{n-1}}{3} \sum_{k=1}^{2n-1} \left| \frac{\cos^{1/3}(n\pi - k\pi/2)}{(2n-k)^{2/3}} - \frac{\cos^{1/3}(n\pi - (k-1)\pi/2)}{(2n-k+1)^{2/3}} \right|
\]

Using the fact that

\[
\cos(n\pi - k\pi/2) = \cos(n\pi) \cos(k\pi/2) = \begin{cases} 0 & k \text{ odd} \\ (-1)^{n+k/2} & k \text{ even} \end{cases}
\]

We can consider the sum over only even indices in \( k \), each of which is counted twice in the sum

\[
V(f \circ g, P_n) = 2 \sum_{k=0}^{n-1} \left| \frac{(-1)^{(n+k)/3}}{(2(n-k))^{2/3}} \right|
\]

\[
= 2 \sum_{k=0}^{n-1} \frac{1}{(2(n-k))^{2/3}}
\]

\[
= 2^{1/3} \sum_{k=1}^{n} \frac{1}{k^{2/3}}
\]
(iii) The sum in the previous part for diverges as $n \to \infty$ Therefore, $f \circ g$ has unbounded variation, and cannot be absolutely continuous.

Problem 45: Let $f$ be absolutely continuous on $\mathbb{R}$ and $g$ be absolutely continuous and strictly monotone on $[a, b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a, b]$.

Solution: Let $\epsilon > 0$, $\{(a_k, b_k)\}$ be a collection of disjoint intervals of $[a, b]$. Since $g$ is strictly increasing, $\{(g(a_k), g(b_k))\}$ forms another disjoint collection of intervals, and by the absolute continuity of $f$, there exists a $\delta_1 > 0$ such that whenever

$$\sum_{k=1}^{n} |f(g(b_k)) - f(g(a_k))| < \epsilon \quad \text{whenever} \quad \sum_{k=1}^{n} |g(b_k) - g(a_k)| < \delta_1.$$  

However, by the absolute continuity of $g$, there exists a $\delta_2 > 0$ such that

$$\sum_{k=1}^{n} |g(b_k) - g(a_k)| < \delta_1 \quad \text{whenever} \quad \sum_{k=1}^{n} [b_k - a_k] < \delta_2.$$
