Mellin Transforms and Mellin Barnes Integral Representations

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Abstract

The Mellin transform and its inverse are important tools in mathematics. It is closely related to the Fourier and bi-lateral Laplace transform, and is used in many diverse areas of mathematics, including analytic number theory, the study of difference equations, asymptotic expansions, and the study of special functions. A special type of Mellin inversion integral called the Mellin Barnes integral is often used to analyze the behavior of special functions. In this brief exposition we will explore properties of the Mellin transform and the Mellin-Barnes integral representation. We will discuss the relation of the Mellin-Barnes integral to the Gauss hypergeometric and Confluent hypergeometric functions, and explore how this representation can be used to obtain asymptotic expansions by simply analyzing the structure of the poles.

1 The Mellin Transform

1.1 Definition and Inversion

The Mellin transform, of a function \( f(x) \) defined on the interval \([0, \infty)\) is given by

\[
\mathcal{M}\{f(x)\} \equiv \int_0^\infty f(x)x^{s-1}dx.
\]

(1)

Many of the properties of the Mellin transform follow from the properties of the Bi-lateral (two-sided) Laplace transform and equivalently the Fourier Transform. We can realate the Mellin transform to the Fourier transform in the following way. Suppose that \( f \) is an \( L_1(\mathbb{R}) \) function such that

\[
\begin{cases}
A^+ e^{-\beta^+ x} & x \to \infty \\
A^- e^{-\beta^- x} & x \to -\infty
\end{cases}
\]

(2)

where \( \beta^+ > \beta^- \), then we know that the Fourier transform

\[
\mathcal{F}\{f\}(\lambda) \equiv \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx
\]
exists and is analytic in the region $\beta^- < \text{Im}(\lambda) < \beta^+$. This region of analyticity of the Fourier transform is known as the *strip of regularity*. The Fourier transform can be related to the Mellin transform by the substitution $u = e^{-x}$,

$$\mathcal{F}\{f\}(\lambda) = \int_0^\infty f(-\log u)u^{\lambda-1}du = \mathcal{M}\{f \circ (-\log)\}(i\lambda). \quad (3)$$

We denote the function $g \equiv f \circ (-\log x)$ and note that (2) requires that

$$g(x) \leq \begin{cases} A^-x^{-b} & x \to \infty \\ A^+x^{-a} & x \to 0 \end{cases}$$

where $b = -\beta^- < a = -\beta^+$. We also see that for $s = i\lambda$, $\mathcal{M}\{g\}(s)$ exists and is analytic in the region

$$a < \text{Re}(s) < b \quad (4)$$

This region is the equivalent *strip of regularity* for the Mellin transform. The strip of regularity here corresponds to a *sufficient* condition for existence and analyticity, however it is not necessary. There are many examples (two of which will be shown later) of functions whose Mellin transform exists and is meromorphic outside the strip of regularity. The Fourier inversion integral gives us that

$$g(x) = f(-\log(x)) = \frac{1}{2\pi i} \int_{-\gamma-i\infty}^{-\gamma+i\infty} \mathcal{F}\{f\}(\lambda)e^{-i\lambda\log x}d\lambda \quad (5)$$

where the integration line is such that $\beta^- < -\gamma < \beta^+$ lies in the strip of regularity. Using the relation to the Mellin transform (3), we can write (5) as

$$g(x) = \frac{1}{2\pi i} \int_{-\gamma-i\infty}^{-\gamma+i\infty} \mathcal{M}\{g\}(i\lambda)x^{-i\lambda}d\lambda,$$

where upon changing the variable to $s = i\lambda$ we obtain

$$g(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}\{g\}(s)x^{-s}\,ds \quad (6)$$

where $a < \gamma < b$. The formula (6) is the well-known result of the *Mellin Inversion Theorem*.

To illustrate the existence and analyticity of the Mellin transform in the strip of regularity, as well as glimpse its relationship to special functions we consider we consider a couple well-known examples.

**Example 1.1.1** We consider the function $f(x) = e^{-x}$. We note first that $f(x) \leq x^0$ as $x \to 0$ and $f(x) \leq x^{-\alpha}$ as $x \to \infty$ for any $\alpha > 0$. Therefore the therefore we expect to Mellin transform to exist and be analytic in the strip of regularity $0 < \text{Re}(s) < \infty$. In this case the Mellin transform is just the definition of the $\Gamma$ function

$$\mathcal{M}\{f\}(s) = \int_0^\infty e^{-x}x^{s-1}dx = \Gamma(s).$$
In this example, we can clearly see that \( \Gamma(s) \) is defined and analytic in the strip of regularity. However, we note that the \( \Gamma(z) \) is meromorphic in the entire complex plane. We note that without the strip of regularity, the Mellin transform, and hence the inversion is not unique. Instead of \( f(x) = e^{-x} \) in this example, we could consider \( f(x) = e^{-x} - 1 \), which has a the strip of regularity \(-1 < \text{Re}(s) < 0\), but the transform is
\[
\mathcal{M}\{f\}(s) = \int_0^\infty (e^{-x} - 1)x^{s-1}\,dx = \frac{(e^{-x} - 1)}{s}x^s \bigg|_0^\infty + \frac{1}{s} \int_0^\infty e^{-x}x^s\,dx = \frac{\Gamma(s + 1)}{s} = \Gamma(s).
\]
We also note that the inversion integrals for \( e^{-x} \) and \( e^{-x} - 1 \) are
\[
\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\,ds
\]
where the inversion line is to right of all poles in the Gamma function for \( e^{-x} \) and in between the poles at \(-1\) and \(0\) for \( e^{-x} - 1 \). We can catch here a glimpse of the asymptotic behavior of the Mellin inversion integral and that contribution of the rightmost pole gives the largest contribution to the integral as \( x \to \infty \).

**Example 1.1.2** We now consider the Mellin transform of the function \( f(x) = (1 + x)^{-a} \), where \( \text{Re}(a) > 0 \). To determine the strip of regularity, we see that \( f(x) \leq x^0 \) as \( x \to 0 \) and \( f(x) \leq x^{-\text{Re}(a)} \) as \( x \to \infty \), hence the strip of regularity is \( 0 < \text{Re}(s) < a \).

The Mellin transform is given by the integral
\[
\mathcal{M}\{(1 + x)^{-a}\}(s) = \int_0^\infty (1 + x)^{-a}x^{s-1}\,dx.
\]
This can be transformed in the well-known integral of the Beta function by the substitution \( u = x/(1 + x) \),
\[
\int_0^\infty (1 + x)^{-a}x^{s-1}\,dx = \int_0^1 u^{s-1}(1 - u)^{a-s-1}\,du = B(s, a - s).
\]
To simplify this further we note that the Beta function has the following representation in terms of the Gamma function
\[
B(s, a - s) = \frac{\Gamma(s)\Gamma(a - s)}{\Gamma(a)}.
\]
Therefore
\[
\mathcal{M}\{(1 + x)^{-a}\}(s) = \frac{\Gamma(s)\Gamma(a - s)}{\Gamma(a)}.
\]
The transformed function (7) is once again analytic in the strip of regularity, and meromorphic in the complex plane. It is also interesting to note that in the special case that \( a = 1 \), we may use Euler’s reflection formula \( \Gamma(z)\Gamma(1 - z) = \pi / \sin \pi z \) to simplify the solution
\[
\mathcal{M}\{(1 + x)^{-1}\}(s) = \frac{\pi}{\sin \pi s}.
\]
We note that in both of these examples, the transform can be written in terms of the Gamma function and trigonometric functions. This is often the case for the Mellin transform and suggests a deep connection between the Gamma function and the Mellin transform.
1.2 Fundamental Properties

Just like the Laplace and Fourier transforms, the Mellin transform has some fundamental properties when applied certain integrands that make it useful. Some of the most useful and interesting of these properties are included below without proof. We denote in the usual way, the Mellin transform of a function $f(x)$ as $F(s)$.

(a1) $\mathcal{M}\{f(ax)\} (s) = a^{-s}F(s)$

(b1) $\mathcal{M}\{x^a f(x)\} (s) = F(s + a)$

(c1) $\mathcal{M}\{f(x^a)\} = |a|^{-1}F(s/a), \quad a \neq 0$

(d1) $\mathcal{M}\{|\log x|^n f(x)\} (s) = F^{(n)}(s)$

where the first 3 results can be found by substitution, and the last can be found simply by differentiating $F(s)$ with respect to $s$. These properties can be used to find the transforms of a large number of functions. For instance it is now trivial to take the Mellin transform of the Gaussian

$$\mathcal{M}\{e^{-x^2}\} = \frac{1}{2} \Gamma(s/2).$$

Another interesting result of this is the application of (a1) to series of the following form

$$g(x) = \sum_k a_k \phi(b_k x)$$

called an Harmonic Sum by Flajolet et al. Under suitable conditions in the convergence, it is enough to assume uniform convergence, and the assumption that the transform $\Phi(s) \equiv \mathcal{M}\{\phi\} (s)$ exists and is analytic in some strip of regularity $a < \text{Re}(s) < b$, we have

$$\mathcal{M}\{g(x)\} (s) = \left[ \sum_k a_k b_k^{-s} \right] \Phi(s) \quad (9)$$

Assuming that the series

$$\beta(s) = \sum_k a_k b_k^{-s}$$

converges and is analytic in some sub-strip of the srip of regularity of $\Phi(s)$. We note that in the case that $b_k = k$, when $\beta(s)$ becomes a Dirichlet series.

**Example 1.2.1** A well known example of this is property is used to find an integral representation of the Riemann $\zeta$ function, given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Consider the following expansion as an harmonic sum

$$f(x) = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}$$
The strip of regularity is $1 < \text{Re}(s) < \infty$, and so by (9)

$$M \{ f \} (s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \sum_{n=1}^{\infty} \frac{1}{n^s} \Gamma(s) = \zeta(s) \Gamma(s).$$  \hspace{1cm} (10)$$

Solving (10) for $\zeta(s)$, we obtain the well known integral representation of the $\zeta$ function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}$$

for $\text{Re}(s) > 1$. This result can be used to show the analyticity of $\zeta(s)$ for $\text{Re}(s) > 0$.

Another representation of the zeta function in the critical strip $0 < \text{Re}(s) < 1$ that can be found by a Mellin transform that is very useful to analytical number theorists is

$$\zeta(s) = -\frac{1}{s} \int_0^\infty \frac{x^{s-1}}{x - 1} dx.$$

where $\text{frac}(x)$ is the fractional part of $x$. □

The mellin transform also has a few useful differential and integral properties,

\hspace{1cm} (a2)

$$M \left\{ f^{(n)}(x) \right\} (s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} F(s-n),$$

provided

$$\lim_{x \to 0, \infty} x^{s-r-1} f^{(n-r-1)}(x) = 0 \quad \text{for} \quad r = 0, 1, \ldots (n-1).$$

\hspace{1cm} (b2)

$$M \left\{ \left[ x \frac{d}{dx} \right]^n f(x) \right\} = (-s)^n F(s),$$

provided

$$\lim_{x \to 0, \infty} x^s f^{(n-r-1)}(x) = 0 \quad \text{for} \quad r = 0, 1, \ldots (n-1).$$

\hspace{1cm} (c2)

$$M \left\{ \int_0^x f(t) dt \right\} = -\frac{1}{s} F(s+1).$$

These results can be found via integration by parts under the restriction that the left over terms vanish at the limits of integration. A useful result of the integration formula (c) is the Mellin transform of the function $f(x) = \log(1+x)$. Since $\log(1+x) = \int_0^x \frac{1}{1+t} dt$, we see by using result (8)

$$M \{ \log(1+x) \} (s) = -\frac{1}{s} \frac{\pi}{\sin \pi(s+1)} = \frac{\pi}{s \sin \pi s}$$
1.3 The Parseval Formula

One useful property of the Fourier and Laplace transforms are the convolution
theorems. They allow one to find inverse transforms of more complicated functions
comprised of products of functions with known inversions. The following result is
the fundamental convolution result for the Mellin transform. Suppose that we have
two functions $f(x)$ and $g(x)$ with transforms $F(s)$ and $G(s)$, such that $F(s)$ and $G(1-s)$
have overlapping strips of regularity. We consider the integral along the vertical line
$\text{Re}(s) = c$ in this strip. Then Parseval’s formula is

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)G(1-s)ds \quad (11)$$

**Proof.** Consider the RHS of (11), we have

$$I = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}\{f\}(s)\mathcal{M}\{g\}(1-s)ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[ \int_0^\infty f(x)x^{s-1}dx \right] \mathcal{M}\{g\}(1-s)ds$$

$$= \frac{1}{2\pi i} \int_0^\infty f(x) \int_{c-i\infty}^{c+i\infty} \mathcal{M}\{g\}(1-s)x^{s-1}dsdx$$

Under the change of variables $u = (1-s)$ the inner integral becomes

$$\int_{c-1-i\infty}^{c-1+i\infty} \mathcal{M}\{g\}(u)x^{-u}du = 2\pi ig(x)$$

and therefore

$$I = \int_0^\infty f(x)g(x)dx.$$ 

Hence we have Parseval’s formula. \[\square\]

This equation can be changed into a slightly more general form for the right hand side
by use of property (b1), consider $g(x) = h(x)x^{a-1}$, then

$$\int_0^\infty f(x)h(x)x^{a-1}dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)H(a-s)ds \quad (12)$$

The form of the integrand in the right hand side of (12) is common for many Mellin-
inversion (Mellin-Barnes) integrals. This form is useful when the integrand can be
separated into two functions $F(s)$ with and $H(a-s)$ with an over-lapping strip of
regularity.

**Example 1.3.1** To see how formula (12) works, consider the following integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(a+s)\Gamma(b-s)ds$$
where $-a < c < b$ and $\text{Re}(a) > 0$. Using (12), where
\[
\mathcal{M} \{f\} (s) = \Gamma(a + s), \quad \mathcal{M} \{g\} (s) = \Gamma(s)
\]
and using property (b1), we obtain
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(a + s)\Gamma(b - s) ds = \int_{0}^{\infty} x^{a+b-1} e^{-2x} dx = 2^{-(a+b)} \Gamma(a + b).
\]
□

Another useful form of Parseval’s formula can be related directly to the Fourier convolution theorem. For functions $u$ and $v$ the Fourier convolution theorem is given by
\[
\mathcal{F} \left\{ \int_{-\infty}^{\infty} u(t)v(x-t) dt \right\} = \mathcal{F} \{u\} \mathcal{F} \{v\}.
\]
Using (3) and defining new functions $f = u \circ (-\log)$ and $g = v \circ (-\log)$, we can write (13) in terms of the Mellin transform
\[
\mathcal{M} \left\{ \int_{-\infty}^{\infty} u(t)v(- \log x-t) dt \right\} = \mathcal{M} \{f\} \mathcal{M} \{g\}.
\]
Under the substitution $\log s = \log x + t, dt = ds/s$, the equation becomes
\[
\mathcal{M} \left\{ \int_{0}^{\infty} f(x/s)g(s) \frac{ds}{s} \right\} = F(s)G(s).
\]
which can be written in the form
\[
\int_{0}^{\infty} f(x/t)g(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)G(s)x^{-s} ds.
\]
This is the alternate form of Parseval’s formula. The conditions on the overlapping strips of regularity of the functions $F(s)$, and $G(s)$ can be relaxed somewhat by considering meromorphic extensions of $F(s)$ and $G(s)$, and deforming the contours in the strips of regularity of both $F$ and $G$ to meet such that the separation of the poles of both $F$ and $G$ is preserved with respect to themselves. Clearly, the conditions on meromorphic extensions of $F$ and $G$ must be such that they preserve the separation of the poles with respect to themselves. Therefore, in most applications of Parseval’s formula, we won’t worry about imposing overlapping strip restrictions, and simply look to make sure that the poles can be separated by a deformed contour. Each of the formulas (11), (12) and (14) are equivalent, and equally useful in evaluating certain integrals. Parsevals formula is one of the central results in the study of Mellin transforms, and is used frequently in the evaluation of Mellin-Barnes integral (Sec 2.1).
2 The Mellin-Barnes Integral Representation

2.1 Mellin Barnes integrals

A Melling Barnes integral is an inverse mellin integral involving Gamma functions in the integrand. Most generally it is of the form

$$\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\alpha_1 + A_1 s) \cdots \Gamma(\alpha_n + A_n s)}{\Gamma(\gamma_1 + C_1 s) \cdots \Gamma(\gamma_n + C_n s)} \times \frac{\Gamma(\beta_1 - B_1 s) \cdots \Gamma(\beta_n - B_n s)}{\Gamma(\delta_1 - D_1 s) \cdots \Gamma(\delta_n - D_n s)} z^{-s} ds$$

where the poles of $\Gamma(\alpha_i + A_is)$ can be separated from the poles of $\Gamma(\beta_j - B_js)$ and the integration contour is taken to the right of the $\Gamma(\alpha_i + A_is)$ poles and the left of the $\Gamma(\beta_j - B_js)$ poles in the common strip of analytcity. Integrals of this form lead to a surprising variety of special functions. Often times the conditions of separation of the poles can be relaxed to a more general case in which the poles on the left overlap the poles on the right. In this case the integration contour is take along the imaginary axis, and appropriately indented to separated the left hand and right hand poles.

Often times these integrals can be evaluated by straightforward applications of Parsevals formula and the thoery of residues to pick up the poles in one half of the plane. Before we look at a couple of examples of Mellin Barnes integrals, we find it useful to have the following analytical results.

**Beta function**

One function which can be useful in evaluating certain Mellin-Barnes integrals is the
Beta function, given by the integral

\[ B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \]  

(16)

Under the substitution \( t = 1/(1+u) \), the integral can be written in an alternate form

\[ B(x, y) = \int_0^\infty u^{y-1}(1+u)^{-x-y}du. \]  

(17)

The Beta function also has a very useful representation in terms of the \( \Gamma \) function,

\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \]  

(18)

Proof: To see how (18) is obtained, we consider the product of two \( \Gamma \) functions

\[ \Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty x^{a-1}y^{b-1}e^{-x-y}dxdy \]

under the change of variables \( x = uv, y = u(1-v) \), where \( 0 < u < \infty, 0 < v < 1 \), the Jacobian is

\[ \left| \frac{\partial(x,y)}{\partial(v,u)} \right| = \det \begin{bmatrix} u & v \\ -u & (1-v) \end{bmatrix} = u \]

we obtain

\[ \Gamma(a)\Gamma(b) = \int_0^\infty \int_0^1 u^{a-1}v^{a-1}u^{b-1}(1-v)^{b-1}e^{-u}uvdu \]

\[ = \int_0^\infty ua + b - 1e^{-u}du \int_0^1 v^{a-1}(1-v)^{b-1}dv \]

\[ = \Gamma(a+b)B(a,b). \]

Therefore we may write the two following useful Melling transforms using the Beta function, consider the function \( x^a(1-x)^b u(1-x) \), where \( u(x) \) is the Heaviside function, then

\[ \mathcal{M} \left\{ x^a(1-x)^{b-a-1}u(1-x) \right\} = \int_0^1 x^{a+s-1}(1-x)^{b-a-1}dx = \frac{\Gamma(a+s)\Gamma(b-a)}{\Gamma(b+s)} . \]  

(19)

and similarly consider the function \( x^a(1+x)^{-b-a} \),

\[ \mathcal{M} \left\{ x^a(1+x)^{-a-b} \right\} = \int_0^\infty x^{a+s-1}(1+x)^{-(a+s-1)-(b+1-s)}dx = \frac{\Gamma(a+s-1)\Gamma(b+1-s)}{\Gamma(b+a)} . \]  

(20)

Equations (19) and (20) are very useful when used in conjunction with Parseval’s formula.

Example 2.1.1 We consider the following Melling Barnes integral

\[ I = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s+a)\Gamma(c-s)}{\Gamma(s+b)\Gamma(d-s)}ds \]  

(21)
If we choose
\[ F(s) = M \{ f \} (s) = \frac{\Gamma(s + a)}{\Gamma(s + b)}, \quad G(s) = M \{ g \} (s) = \frac{\Gamma(c - 1 + s)}{\Gamma(d - 1 + s)} \]
then by (19)
\[ f(x) = \frac{1}{\Gamma(b - a)} x^a (1 - x)^{b-a-1} u(1 - x), \quad g(x) = \frac{1}{\Gamma(c - d)} x^{c-1} (1 - x)^{c-d-1} u(1 - x). \]

Therefore, by Parseval’s formula (11) we have
\[ I = \int_0^1 x^{a+c-1} (1 - x)^{b-a+c-d-2} dx = \frac{\Gamma(a + c) \Gamma(b - a + c - d - 1)}{\Gamma(b - a) \Gamma(c - d) \Gamma(b - d - 1)}. \]

\[ \square \]

2.2 Relation to Hypergeometric functions

Gauss hypergeometric

The real power of the Mellin-Barnes integral comes from its connection with the Hypergeometric function. Suppose that we have the Mellin-Barnes integral
\[ I = \frac{1}{2\pi i} \int_C \frac{\Gamma(a + s) \Gamma(b + s) \Gamma(-s)}{\Gamma(c + s)} (-z)^s ds \]
where the integration contour \( C \) lies along the imaginary axis, and is suitably indented to separate poles of \( \Gamma(-s) \) from poles of \( \Gamma(a + s) \Gamma(b + s) \) as show in figure 2.2. There are sequences of poles at
\[ s = -a - n \quad s = -b - n \quad \text{and} \quad s = n, \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots. \]

We then close the contour \( C \) to the right to a rectangular contour that contains the first \( N \) poles of \( \Gamma(-s) \) as shown in figure 2.2. This contour is comprised of a vertical contour \( C_N \) and a two horizontal contours \( C_1 \) and \( C_2 \) at heights \( \pm iN \) above the real axis. Therefore we may write
\[ \frac{1}{2\pi i} \left[ \int_C + \int_{C_1} + \int_{C_2} + \int_{C_N} \right] \frac{\Gamma(a + s) \Gamma(b + s) \Gamma(-s)}{\Gamma(c + s)} (-z)^s ds = I_N \]
where
\[ I_N = \sum_{n=0}^N \text{Res}_{s=n} \frac{\Gamma(a + s) \Gamma(b + s) \Gamma(-s)}{\Gamma(c + s)} (-z)^s \]

To compute the residue of \( \Gamma(-s) \) at \( s = n \) we note that
\[ \lim_{s \to n} (s - n) \Gamma(-s) = \lim_{s \to n} \frac{1}{\Gamma(1 + s)} \frac{\pi(s - n)}{\sin \pi s} = \lim_{s \to n} \frac{(-1)^n}{\Gamma(1 + s)} \frac{\pi(s - n)}{\sin (\pi(s - n))} = \frac{(-1)^n}{n!} \]
and therefore

\[ I_N = \sum_{n=0}^{N} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)} \frac{z^n}{n!} \]

\[ = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \sum_{n=0}^{N} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \]

where \((a)_n\) is the Pochhammer symbol defined by

\[ (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1) \ldots (a + n - 1). \]

It can be shown the integrals along the contours \(C_1, C_2,\) and \(C_N\) vanish as \(N \to \infty,\) though the details are too technical to fit in here. For a detailed analysis of this see Slater p22.

In the limit as \(N \to \infty\) we see then that

\[ \lim_{N \to \infty} I_N = I = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \]

which is just the well known series representation of Gauss’s Hypergeometric function \(2F_1(a, b, c; z),\) therefore we now have the well-known Mellin-Barnes integral representation of the Hypergeometric function,

\[ 2F_1(a, b, c; z) = \frac{\Gamma(c)}{2\pi i\Gamma(a)\Gamma(b)} \int_C \frac{\Gamma(a + s)\Gamma(b + s)\Gamma(-s)}{\Gamma(c + s)} (-z)^s ds \quad (22) \]

**Confluent hypergeometric:**
In a very similarly fashion to the Gauss hypergeometric function $2F_1(a, b, c; z)$, we can also find a Mellin-Barnes integral representation of the Confluent hypergeometric function $1F_1(a, b; z)$ defined by the series

$$1F_1(a, b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \quad (23)$$

By closing the contour in the left half-plane, of the integral

$$I_C = \frac{1}{2\pi i} \int_C \frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(b+s)} (-z)^s ds$$

where the contour $C$ is taken parallel to the imaginary axis and indented to separate the poles of $\Gamma(a+s)$ from the poles of $\Gamma(-s)$. Once again using the fact that

$$\text{Res}_{s=n} \frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(b+s)} (-z)^s = \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{z^n}{n!}$$

It can be shown that

$$1F_1(a, b; z) = \frac{\Gamma(b)}{2\pi i \Gamma(a)} \int_C \frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(b+s)} (-z)^s ds. \quad (24)$$

The Confluent hypergeometric function $1F_1(a, b; z)$ is a very important and universal special function in analysis. In particular it is well-known for its connection to numerous other special functions, for example the incomplete gamma function can be given by

$$\gamma(a, z) = \frac{z^a e^{-z}}{a} 1F_1(1, a+1, z) = z^a e^{-z} \frac{\Gamma(a)}{2\pi i} \int_C \frac{\Gamma(1+s)\Gamma(-s)}{\Gamma(a+s+1)} (-z)^s ds. \quad (25)$$

Also the modified Bessel function $I_\nu(z)$ can be given by

$$I_\nu(z) = e^{-\frac{z}{2}} \left(\frac{2}{z}\right)^{\nu/2} 1F_1(\nu+\frac{1}{2}, 2\nu+1, 2z)$$

$$= e^{-z} \left(\frac{z}{2}\right)^{\nu} \frac{\Gamma(2\nu+1)}{2\pi i \Gamma(\nu+\frac{1}{2}) \Gamma(1+\nu)} \int_C \frac{\Gamma(\nu+s)\Gamma(-s)}{\Gamma(2\nu+s+1)} (-2z)^s ds. \quad (26)$$

For a full list of relations of the confluent hypergeometric function consult the section on confluent hypergeometric functions in the DLMF (Digital Library of Mathematical functions) at http://dlmf.nist.gov/13.6.

### 2.3 Asymptotics

We have now seen that the Mellin-Barnes integral representation can be useful in defining a convergent series representation by closing the contour in the right half plane. However, it is possible to displace the contour in the left half plane past each of the poles, in an similar fashion to the Laplace transform to obtain and asymptotic series for large $|z| \to \infty$. However, instead of obtainin an exponential asymptotic series as
one gets with the Laplace transform we obtain a formal inverse power series. We show this method to obtain asymptotic expansions from the Mellin-Barnes integral representation on the confluent hypergeometric function representation (24).

\[ _1 F_1(a, b; z) = \frac{\Gamma(b)}{2\pi i \Gamma(a)} \int_C \frac{\Gamma(a + s) \Gamma(-s)}{\Gamma(b + s)} (-z)^s ds \]

Suppose that we push the contour \( C \) to the left, past one of the poles of \( \Gamma(a + s) \), then we will obtain an integral along a new contour \( C_1 \) along with the residue of the integrand at that pole. The residue of the integrand

\[ \frac{\Gamma(a + s) \Gamma(-s)}{\Gamma(b + s)} (-z)^s \]

at the poles \( s = -a - n, \ n = 0 \pm 1 \pm 2 \ldots \) is given by

\[
\lim_{s \to -a-n} \frac{(s+a+n)\Gamma(a+s)\Gamma(-s)}{\Gamma(b+s)} (-z)^s = \lim_{s \to -a-n} \frac{\pi(s+a+n)\Gamma(-s)}{\pi(s+a+n)\Gamma(1-a-s)\Gamma(b+s)} (-z)^s
\]

\[ = \frac{\Gamma(a+n)}{n!\Gamma(b-a-n)} (-1)^n (-z)^{-a-n} \]

\[ = (-z)^{-a} \frac{\Gamma(a+n)}{\Gamma(b-a-n)} \frac{z^{-n}}{n!} \]

\[ = (-z)^{-a} \frac{\Gamma(a)}{\Gamma(b-a)} (a)_n (b-a)_n \frac{z^{-n}}{n!} \]  \hspace{1cm} (27)

\[ = (-z)^{-a} \frac{\Gamma(a)}{\Gamma(b-a)} (a)_n (b-a)_n \frac{z^{-n}}{n!} \]  \hspace{1cm} (28)

therefore if we shift the contour to the left past \( N \) poles, we may write the confluent hypergeometric function as a formal asymptotic series by summing up the res

\[ _1 F_1(a, b; z) = (-z)^{-a} \frac{\Gamma(b)}{\Gamma(b-a)} \sum_{n=0}^{N} (a)_n (b-a)_n \frac{z^{-n}}{n!} + R_N \]  \hspace{1cm} (29)

where the remainder term is

\[ R_N = \frac{\Gamma(b)}{2\pi i \Gamma(a)} \int_{C_N} \frac{\Gamma(a + s) \Gamma(-s)}{\Gamma(b + s)} (-z)^s ds \]

where \( C_N \) is the orginal contour \( C \) shifted to the left by \( N \), and satisfies \( |R_N| = \mathcal{O}(z^{-N}) \) as \( |z| \to \infty \) in the sector \( \arg z < \pi/2 \).

References


