Study Outline for Exam 2

Second Order Linear Differential Equations:

$$y'' + p(t)y' + q(t)y = r(t) \quad [*] \text{ (Inhomogeneous)}$$

$$y'' + p(t)y' + q(t)y = 0 \quad [**] \text{ (Homogeneous)}$$

where p, q and g are continuous functions on an interval.

1. Existence and Uniqueness of solutions to IVP:

$$y(t_0) = y_0,\ y'(t_0) = y'_0.\ [***]$$

For any t_0 in the interval on which the coeff fctns are continuous, there is exactly one solution of [*] satisfying the initial conditions [***]. The same is true for [**].

2. The set of solutions to [**] is a two-dimensional vector space, meaning that there are two linearly independent (neither is a multiple of the other) solutions so that EVERY solution is a linear combination of those two.

Two solutions y_1 and y_2 are linearly independent and so form a basis for the solution set exactly when the Wronskian, which is given by the following formula and enjoys the property that it is either identically zero or never zero:

$$W = y_1 y'_2 - y'_1 y_2$$

is NOT zero.

3. Constant Coefficient Homogeneous Equations

$$ay'' + by' + cy = 0.$$

(a) The characteristic polynomial is

$$ar^2 + br + c;$$

Its roots determine the solutions.

Distinct Real Roots: r_1, r_2 give

$$e^{(r_1 t)},\ e^{(r_2 t)}.$$

Complex Conjugate Roots: $\alpha \pm i\beta$ give

$$e^{(\alpha t)}\cos(\beta t),\ e^{(\alpha t)}\sin(\beta t).$$

Repeated Real Root: r_0 gives

$$e^{(r_0 t)},\ te^{(r_0 t)}.$$

4. Reduction of Order (or Order Reduction)

If y_1(t) is a solution of [**], then one gets a second linearly independent solution by substituting

y_2(t) = u(t)y_1(t)

into [**], noting that the result does not depend on u, then solving the resulting differential equation for u' and then integrating to get u.

5. Inhomogeneous Equations

If the inhomogeneous term r(t) is a sum of functions, then find a particular solution for each summand and then add them together to get a particular solution for the full equation. We have the following methods:

(a) Undetermined Coefficients (only for constant coeff eqns). When the inhomogeneous term is a product of a polynomial, exponential and a sinusoidal. Try the exact same kind of candidate for a solution using unknown coeffs; plug into the diff eqn to determine the coeffs. Don't forget to multiply the 'candidate' by t^s, where s is the smallest non-negative integer required to guarantee that no term in the candidate is a solution of the homogeneous eqn.

(b) Green Function (also for const coeff eqns). Selct g(t) to be the unique function that solves the homogeneous eqn and satisfies g(0)=0, g'(0)=1. Then a particular solution of the inhomog. eqn is given by

int_{t_0}^t g(t-s)r(s) ds

(c) Variation of Parameters (requires eqn to be normalized, but not const coeff). If y_1 and y_2 are lin ind sols of the homog eqn, then

-y_1(t) int^t y_2(s)g(s)/W(s) ds + y_2(t) int^t y_1(s)g(s)/W(s) ds

is a sol of the inhomog eqn.

6. Mass-Spring System

Unforced & Damped or Undamped; i.e.,

mu'' + ku = 0 or mu'' + gamma u' + ku = 0.

yielding harmonic motion or a damped oscillation accordingly.

Forced: Resonance and Beats; i.e.,

mu'' + ku = Rcos(omega t), omega_0 = sqrt(k/m).

Resonance occurs for omega = omega_0; Beats for omega very close to omega_0.

7. Finally, you may be expected to interpret a Matlab session involving some qualitative analysis of a linear, non-constant coeff 2^{nd} order ODE.