Notes on Introduction to 2nd Order ODEs — in lieu of 3/3 classroom lecture

A general second order ODE is an equation $F(t, y, y', y'') = 0$, where y is an unknown function of t. The equation is said to be explicit if it can be written $y'' = f(t, y, y')$. As with 1st order eqns, there is an Existence & Uniqueness Theorem -- namely, if f and its partials with respect to its second and third variables are continuous on some open region S in $\mathbb{R}^3$, then for any $(t_0, y_0, y_0') \in S$, there exists a unique function $y = \varphi(t)$, defined on an interval I containing t_0, so that

$$\varphi''(t) = f(t, \varphi(t), \varphi'(t))$$
$$\varphi(t_0) = y_0$$
$$\varphi'(t_0) = y_0'.$$

Alas, for 2nd order equations, there are very few categories of eqns for which we can develop an algebraic/calculus method to construct formula solutions. In fact, we shall confine our attention almost exclusively to linear equations:

$$a(t)\, y'' + b(t)\, y' + c(t)\, y = d(t)$$

where the coeff fctns of t are continuous fctns. As we did with first order linear eqns, we normalize and adopt a slightly different syntax -- namely

$$y'' + p(t)\, y' + q(t)\, y = r(t),$$

where p, q, r are continuous functions on an interval I and $y(t)$ is to be determined. The existence and uniqueness theorem applies and in this case it turns out that domain of definition of the solution is the whole interval I.

Note. There is an analogous theory for nth order linear ODEs, $n > 2$ -- see the Notes for the details. We shall concentrate on $n = 2$.

Definition. A linear equation is called homogeneous if $r \equiv 0$ and inhomogeneous otherwise. Thus:

$$\text{homogeneous} \quad y'' + py' + qy = 0 \quad (*)$$
$$\text{inhomogeneous} \quad y'' + py' + qy = r \quad (**)$$

Principle of Superposition:

1. If y_1 and y_2 are sols of a homog eqn $(*)$ and c_1, c_2 any constants, then
$$y = c_1 y_1 + c_2 y_2$$
is also a soln of the homog eqn $(*)$.
2. If y_0 is a soln of a homog eqn $(*)$ and y_p is a soln to a corresponding inhomog eqn $(**)$ [same p & q], then $y_0 + y_p$ is also a soln to the inhomog eqn $(**)$.

In particular, the collection of all solns to a given homog eqn is a vector space - i.e. a mathematical object on which there is vector addition and scalar multiplication. We shall recall more about vector spaces in our lecture on II.3, but examples of vector spaces are:

- the plane $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ with coordinate-wise operations
- the solns of a set of linear algebraic eqns, $\{X : AX = 0,\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}\}$
- the polynomials of degree $\leq n$.

(More details on vector spaces next week.)

The set of solns to a homogeneous equation $(*)$ is a two-dimensional vector space. That means you can find two solutions φ_1, φ_2 that are linearly independent (i.e., neither is a multiple of the other) and every soln φ is a linear combination of φ_1 and φ_2 -- i.e. $\varphi = \alpha\varphi_1 + \beta\varphi_2$ for some scalars α, β.

Often, we encounter a homog eqn for which we can identify two solns φ_1, φ_2 and we want to be sure they are linearly independent (so that they span the set of all sols, ie any soln φ can be written $\varphi = \alpha\varphi_1 + \beta\varphi_2$ for some scalars α, β). Here is a test we shall employ.

Define the <u>Wronskian</u> $W(\varphi_1, \varphi_2)$ of φ_1 and φ_2 by

$$W(\varphi_1, \varphi_2) = \varphi_1\varphi_2' - \varphi_1'\varphi_2.$$

Now note the following computation:

$$\varphi_1'' + p\varphi_1' + q\varphi_1 = 0 \qquad \varphi_2'' + p\varphi_2' + q\varphi_2 = 0$$

Multiply first eqn by φ_2, second by φ_1, and subtract first from second to get

$$(\varphi_1\varphi_2'' - \varphi_1''\varphi_2) + p(\varphi_1\varphi_2' - \varphi_1'\varphi_2) = 0, \text{ ie.}$$

$$W' + pW = 0 \quad (*) \qquad (\text{check that } W' = \varphi_1\varphi_2'' - \varphi_1''\varphi_2.)$$

But $(*)$ is a simple linear homog 1st order eqn, whose sols are:

$$W = ce^{-\int p}.$$

Thus W is either identically zero, or never zero. It is precisely in the latter case that φ_1 and φ_2 are linearly independent. In that case, the set $\{\varphi_1, \varphi_2\}$ is called a <u>fundamental set of sols</u>, meaning they form a basis for the 2-dim vector space of sols, ie every soln of the eqn is of the form

$$\alpha\varphi_1 + \beta\varphi_2 \quad \text{for some scalars } \alpha, \beta.$$

We'll talk about inhomog eqns (ie where $r \not\equiv 0$) later, but for now we see that to find <u>all</u> sols of a 2nd order, linear, homog eqn, we need only find two linearly independent ones.

Sometimes there is a canonical choice. If a point $t_0 \in I$ is specified, then the <u>natural fund. solns</u> is a pair of sols $N_1(t), N_2(t)$ that satisfy

$$N_1(t_0)=1\ ,\ N_1'(t_0)=0$$
$$N_2(t_0)=0\ ,\ N_2'(t_0)=1.$$

Note that $W(N_1, N_2)(t_0)=1\neq 0$; so these must be linearly independent.

Finally there is a technique, called reduction of order, by which if you can find one solution, the method produces a second lin. indep. soln. It goes as follows:

If $\varphi_1''+p\varphi_1'+q\varphi_1=0$, then

try $\varphi_2(t)=u(t)\varphi_1(t)$, where $u(t)$ is an as yet undetermined fctn. If $u\equiv c$ is constant, then by linearity φ_2 is a soln. But now let u be variable. Plug into the equation:

$$(u\varphi_1)''+p(u\varphi_1)'+q(u\varphi_1)$$
$$=u\varphi_1''+2u'\varphi_1'+u''\varphi_1+p(u\varphi_1'+u'\varphi_1)+q(u\varphi_1)$$
$$=u''\varphi_1+u'(2\varphi_1'+p\varphi_1)+u(\varphi_1''+p\varphi_1'+q\varphi_1)$$
$$=u''\varphi_1+u'(2\varphi_1'+2p\varphi_1)=0 \quad \text{since } \varphi_1 \text{ is a soln}$$

But the latter is a first order eqn in u'. Set $v=u'$; solve the first order eqn for v; anti-differentiate to get u; multiply by φ_1 -- you get a second linearly indep. soln

Illustrative example of Reduction of Order

$y_1(t)=t$ solves $t^2y''+2ty'-2y=0$.

Set $y_2=uy_1=ut$. Then $y_2'=tu'+u$, $y_2''=tu''+2u'$. So

$$t^2(tu''+2u')+2t(tu'+u)-2tu=t^3u''+4t^2u'=0$$

Set $v=u'$: $t^2(tv'+4v)=0 \Rightarrow tv'+4v=0 \Rightarrow v=t^{-4}, u=t^{-3}, y_2=t^{-2}$ is a second ~~sol~~ lin indep soln

Exc: To illustrate this material & material in II.1, II.2, consider these problems:

II.1 – 4, 5, 6, 13 ; II.2 – 3, 8, 13, 18, 25