A vector space is a mathematical object whose elements may be added (vector addition) and multiplied by scalars (i.e., real or complex numbers) with the proviso that all the usual rules of arithmetic are valid—e.g., the commutative laws, associative laws, and distributive laws.

**Examples**

1. $\mathbb{R}^2 = \{ (x,y) : x, y \in \mathbb{R} \}$ where $(x,y) + (u,v) = (x+u, y+v)$ and $a(x,y) = (ax, ay)$. This is just the usual set of vectors emanating from the origin in the Cartesian plane with the usual vector operations.

2. $\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \}$ where $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$ and $a(x_1, x_2, \ldots, x_n) = (ax_1, ax_2, \ldots, ax_n)$.

3. All real polynomials of degree $\leq n = \{ q_0 + q_1 x + \cdots + q_n x^n : q_i \in \mathbb{R} \}$ with usual addition and scalar multiplication.

4. The set of all sets of the 1st order linear homogeneous ODE: \[ y' + p(t)y = 0 \]

5. $\mathbb{C}^2 = \{ (z, w) : z, w \in \mathbb{C} \}$ with the usual complex addition and multiplication.

Intuitively, you should see that the dimension of the vector space in each of these examples is: 2, $n$, $n+1$; 1, 2 respectively. To make that precise, we make a
Definition: A set of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) in a vector space \( V \) is called linearly independent if no non-trivial linear combination of them adds to \( \mathbf{0} \), i.e.,

\[
\sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{0} \iff c_i = 0, \forall i.
\]

The dimension of a vector space is the size of a maximal set of linearly independent vectors. Such a collection is called a basis of the vector space. In the five examples above, bases can be chosen as follows:

1. \( \{ \mathbf{e}_1, \mathbf{e}_2 \} \), where \( \mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1) \).

2. \( \{ \mathbf{e}_k \}_{k=1}^{n} \), where \( \mathbf{e}_k \) is the vector with all components \( 0 \) except the \( k \)-th, which is \( 1 \).

3. \( \{ 1, x, x^2, \ldots, x^n \} \).

4. \( \mu(b)^{-1} \), where \( \mu(b) = e^{\int b} \).

5. \( \{ (1,0), (0,1) \} \).

Note: Bases are not unique. See if you can write down another basis in each of the above examples.

Now let's turn to solutions of linear equations, which in a certain case, will lead to another crucial example of a vector space. We deal with real variables here; analogous results hold for complex variables.
A set of $n$ simultaneous linear equations in $n$ real variables is a family:

\[ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \]
\[ \vdots \]
\[ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n, \]

where the $a$'s and $b$'s are real constants, and the $x$'s are unknowns.

We use matrix terminology

\[ A = \begin{pmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{nn}
\end{pmatrix}, \quad X = \begin{pmatrix}
    x_1 \\
    \vdots \\
    x_n
\end{pmatrix}, \quad V = \begin{pmatrix}
    b_1 \\
    \vdots \\
    b_n
\end{pmatrix}. \]

Thus the family of equations can be expressed as a single matrix equation:

\[ AX = V. \]

(If you don't know or have forgotten how to multiply matrices, I am teaching you with the help of working it out.)

In analogy with differential equations, the equation $AX = V$ is called homogeneous if $V = 0$, inhomogeneous otherwise. Then by the linearity of matrix operation, it follows immediately that the set of solutions of a homogeneous equation, i.e.,

\[ \{X : AX = 0\}, \]

is a vector space. It will have dimension equal to some number between 0 and $n$. Moreover, if

\[ AX_i = 0, \quad i = 1, 2 \quad \text{and} \quad AX_p = V, \]
Thus for any scalars \( \alpha_1, \alpha_2 \)

\[
\alpha_1 X + \alpha_2 X_2 \text{ will be a soln of the homog eqn}
\]

and \( \alpha_1 X_1 + \alpha_2 X_2 + X_p \) will be a soln of the inhomog eqn.

This is a precise analog of the principle of superposition for solutions of

linear ODEs, namely, if \( y_1 \) and \( y_2 \) solve

\[
y^{(n)} + q_1(t)y^{(n-1)} + \cdots + q_n(t)y = 0 \quad (\ast)
\]

and \( y_p \) solves

\[
y^{(n)} + q_1(t)y^{(n-1)} + \cdots + q_n(t)y = b(t) \quad (\ast\ast)
\]

then for any scalars \( \alpha_1, \alpha_2 \) it is true that \( \alpha_1 y_1 + \alpha_2 y_2 \) solves (\ast) and

\[
\alpha_1 y_1 + \alpha_2 y_2 + y_p \text{ solves } (\ast\ast).
\]

Ex. Reverse the above with the facts that you have learned for 1st order linear ODEs.

Let's illustrate the assertions about algebraic equations in the cases

\( n = 1 \) and \( n = 2 \)

\( n = 1: \)

- \( ax + b = 0 \) has a unique solution if \( a \neq 0 \); 
- \( ax + b = 0 \) has no solution if \( a = 0 \) and \( b \neq 0 \); 
- \( ax + b = 0 \) has a one-parameter family of solutions if \( a \neq 0 \) and \( b = 0 \).

\( n = 2: \)

\[
ax + by = u \quad \text{or} \quad \mathbf{AX} = \mathbf{V} \text{ with } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{V} = \begin{pmatrix} u \\ v \end{pmatrix},
\]

\[
\text{ex and } dy = v
\]

These two linear equations represent two lines in the plane. In which case:
The equations have no solution if the lines are parallel;
- an infinite family of solutions if the lines are identical;
- exactly one solution if the lines are not parallel.

The two lines are not parallel exactly when their slopes $\frac{d}{b}$ and $\frac{c}{d}$ differ, or more accurately when $\text{det} A = ad - bc \neq 0$, which holds precisely when the matrix $A$ is invertible.

(Sec. Check that $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & b \\ c & a \end{pmatrix}$, meaning $AA^{-1} = A^{-1}A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Here are the facts in general.

Define $\text{det} A = \sum_{k=1}^{n} (-1)^{j+k} (\text{det} A_{jk}) a_{jk}$; $A_{jk}$ is the $k$th cofactor of $A$ obtained by deleting the $j$th row and $k$th column of $A$.

That is, it is a fact that you can evaluate a determinant by expanding along any row or any column. We shall never evaluate anything bigger than a $3 \times 3$ and in the vast majority of cases, no larger than $2 \times 2$.

Example: Compute $\text{det} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Then we have the following facts:
1. $\det(AB) = \det(A) \det(B)$ if $A$ and $B$ are square matrices of the same dimensions. (Recall that matrix multiplication is not commutative, i.e., $AB \neq BA$ in general.)

2. $\det A \neq 0 \Rightarrow A$ is invertible.

3. $AX = 0$ has a unique solution, namely $X = 0 \Rightarrow \det A \neq 0$.

4. Otherwise, the solution set of $AX = 0$ is a vector space.

5. If we set $r = \text{rank}(A) = \text{maximal number of linearly independent rows of } A$, which must = columns of $A$, then $r + (\dim \text{ of solution space of } AX = 0) = n$.

6. $AX = V$ has a unique solution, namely $X = A^{-1}V \Rightarrow \det A \neq 0$.

7. If $\det A = 0$, and $V \neq 0$, then $AX = V$ may have no solution, one solution, or infinitely many solutions.

Give examples of each when $n > 2$. 
Bonus Brief Introduction to Constant Coefficient Linear

Second Order ODEs (Notes, § 2.4)

This will be one of the very few instances of 2nd order ODEs for which we are able to develop a formula solution method. Consider

$$ay'' + by' + cy = 0 \quad (1)$$

where $a$, $b$, $c$ are given real constants and $y$ is unknown. We assume $a \neq 0$; otherwise we are dealing with a 1st order ODE.

Now, just to pick a specific example, we can easily verify that $e^{rt}$ and $e^{rt}$ solve the equation

$$y'' - 3y' + 2y = 0.$$  

(Check that!) So in the general case (1), let's try $e^{rt}$. Plug it in and let's see whether it yields a value of $r$ for which $e^{rt}$ is a solution of (1). We get

$$ar^2 e^{rt} + br e^{rt} + c e^{rt} = e^{rt} (ar^2 + br + c).$$

The latter will be zero precisely when $ar^2 + br + c = 0$, i.e., when $r$ is a root of the quadratic polynomial $ar^2 + br + c$. We call the latter the characteristic polynomial of the ODE (1). If $r_1$ and $r_2$ are two distinct real roots of the characteristic polynomial, then $e^{r_1 t}$ and $e^{r_2 t}$ will be sols. Moreover, they must be linearly independent because their Wronskian

$$W(e^{r_1 t}, e^{r_2 t}) = e^{r_1 t} (e^{r_2 t})' - (e^{r_1 t})' e^{r_2 t} = e^{(r_1 + r_2) t} (r_2 - r_1) \neq 0 \quad \text{if } r_1 \neq r_2,$$

we will explain in class why, as a consequence of the existence and
unique. Therefore, there cannot be a third linearly independent
solution, i.e. the solution set of (1) is a two-dimensional vector space.

So the study of (1) reduces to a knowledge of the roots of its
characteristic polynomial. But as we know, it may not be the case
that there are two distinct real roots. As usual, for quadratic
polynomials, there are three cases to consider:

(a) two distinct real roots;
(b) two complex conjugate roots;
(c) a single repeated real root.

We have seen above how to dispose with case (a). We will deal
with cases (b) and (c) next week in class.