Notes on Vector Spaces and Linear Algebra

in lieu of canceled class on 3/5/15

A <u>vector space</u> is a mathematical object whose elements may be added (vector addition) and multiplied by scalars (i.e., real or complex numbers) with the proviso that all the usual rules of arithmetic are valid -- e.g., the commutative law, associative law and distributive law.

<u>Examples</u>

1. $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ where $(x,y) + (u,v) = (x+u, y+v)$ and $\alpha(x,y) = (\alpha x, \alpha y)$. This is just the usual set of vectors emanating from the origin in the Cartesian plane with the usual vector operations.

2. $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$ where $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1+y_1, \ldots, x_n+y_n)$ and $\alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$.

3. All real polynomials of degree $\leq n$ $= \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{R}\}$ with usual addition and scalar multiplication

4. The set of all sols of the 1st order linear homogeneous ODE: $y' + p(t)y = 0$

5. $\mathbb{C}^2 = \{(z, w) : z, w \in \mathbb{C}\}$ with the usual complex addition and multiplication.

Intuitively you sense that the dimension of the vector space in each of these examples is: 2, n, $n+1$, 1, 2 respectively. To make that precise, we make a

<u>Definition</u>: A set of vectors $v_1, \dots, v_k$ in a vector space V is called <u>linearly independent</u> if no non-trivial linear combination of them adds to 0, i.e.

$$\sum_{i=1}^{k} a_i v_i = 0 \Longrightarrow a_i = 0, \forall i = 1, \dots, k.$$

The dimension of a vector space is the size of a maximal set of linearly independent vectors. Such a collection is called a <u>basis</u> of the vector space. In the five examples above, bases can be chosen as follows:

1. $\{\vec{i}, \vec{j}\}$, where $\vec{i} = (1,0)$, $\vec{j} = (0,1)$.

2. $\{\vec{i}_k\}_{k=1}^{n}$, where $\vec{i}_k$ is the vector with all components $=0$ except the kth, which is 1.

3. $\{1, x, x^2, \dots, x^n\}$.

4. $\mu(t)^{-1}$, where $\mu(t) = e^{\int^t p(s)\,ds}$.

5. $\{(1,0), (0,i)\}$

Note: Bases are not unique. See if you can write down another basis in each of the above examples.

Now let's turn to solutions of linear equations, which in a certain case, will lead to another crucial example of a vector space. We deal with real variables here; analogous results hold for complex variables.

A set of n simultaneous linear equations in n real variables is a family:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where the a's and b's are real constants and the x's are unknown.

We use matrix terminology

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad V = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Then the family of equations can be expressed as a single matrix equation:

$$AX = V.$$

(If you don't know or have forgotten how to multiply matrices, I am tasking you with the chore of looking it up.)

In analogy with differential equations, the equation $AX = V$ is called <u>homogeneous</u> if $V = 0$, <u>inhomogeneous</u> otherwise. Then by the linearity of matrix operations, it follows immediately that the set of solutions of a homogeneous equation, i.e.

$$\{X : AX = 0\}$$

is a vector space. It will have dimension equal to some number between 0 and n. Moreover, if

$$AX_i = 0, \quad i = 1,2 \quad \text{and} \quad AX_p = V,$$

Then for any scalars α_1, α_2

$\alpha_1 X_1 + \alpha_2 X_2$ will be a soln of the homog eqn

and $\alpha_1 X_1 + \alpha_2 X_2 + X_p$ will be a soln of the inhomog eqn.

This is a precise analog of the Principle of Superposition for solutions of ~~1st~~ linear ODEs; namely, if y_1 and y_2 solve

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0 \quad (*)$$

and y_p solves

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = b(t) \quad (**),$$

then for any scalars α_1, α_2 it is true that $\alpha_1 y_1 + \alpha_2 y_2$ solves $(*)$ and $\alpha_1 y_1 + \alpha_2 y_2 + y_p$ solves $(**)$.

Exc Reconcile the above with the facts that you have learned for 1st order linear ODEs.

Let's illustrate the assertions about algebraic equations in the cases $n=1$ and $n=2$.

$n=1$:
- $ax=b$ has a unique solution if $a \neq 0$;
- $ax=b$ has no solution if $a=0$ and $b \neq 0$;
- $ax=b$ has a one-parameter family of sols if $a = 0$ and $b=0$.

$n=2$: $\begin{matrix} ax+by=u \\ cx+dy=v \end{matrix}$ or $AX=V$ with $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \end{pmatrix}$, $V=\begin{pmatrix} u \\ v \end{pmatrix}$.

These two linear equations represent two lines in the plane. In which case:

- The equations have no solution if the lines are (distinct but) parallel;
- ——— an infinite family of solns if the lines are identical;
- Exactly one solution if the lines are not parallel.

The two lines are not parallel exactly when their slopes $\frac{-a}{b}$ and $\frac{-c}{d}$ differ, or more accurately when $\det A = ad - bc \neq 0$; which holds precisely when the matrix A is invertible.

(Exc: Check that $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, meaning $AA^{-1} = A^{-1}A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.)

Here are the facts in general.

__Defn__ $\det A = \sum_{j=1}^{n} (-1)^{j+k} (\det A_{jk})\, a_{jk}$; A_{jk} = the jk co-matrix of A obtained by deleting the kth row and jth column of A

$$= \sum_{k=1}^{n} (-1)^{j+k} (\det A_{jk})\, a_{jk}.$$

That is, it is a fact that you can evaluate a determinant by expanding along any row or any column. We shall never evaluate anything bigger than a 3×3 and in the vast majority of cases, no larger than 2×2.

__Exc__ Compute $\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Then we have the following facts:

1. $\det(AB) = \det(A)\det(B)$ if A and B are square matrices of the same dimensions. (Recall that matrix multiplication is not commutative, i.e. $AB \neq BA$ in general.)

2. $\det A \neq 0 \iff A^{-1}$ exists

3. $AX = 0$ has a unique soln, namely $X = 0 \iff \det A \neq 0$

4. Otherwise, the solution set of $AX = 0$ is a vector space

5. If we set $r = \operatorname{rank}(A)$ = maximal number of linearly independent rows of A, which must = ---------- columns of A, then
r + (dim of soln space of $AX = 0$) $= n$.

6. $AX = V$ has a unique soln, namely $X = A^{-1}V \iff \det A \neq 0$

7. If $\det A = 0$, and $V \neq 0$, then $AX = V$ may have no soln, one soln, or infinitely many solns.
Exc Give examples of each when $n = 2$.

Bonus Brief Introduction to Constant Coefficient Linear Second Order ODEs (Notes, § II.4)

This will be one of the very few instances of 2nd order eqns for which we are able to develop a formula solution method. Consider

$$ay'' + by' + cy = 0 \quad (1)$$

where a, b, c are given real constants and y is unknown. We assume $a \neq 0$; otherwise we are dealing with a 1st order eqn.

Now, just to pick a specific example, we can easily verify that e^t and e^{2t} solve the equation

$$y'' - 3y' + 2y = 0.$$

(Check that!) So in the general case (1), let's try e^{rt}. Plug it in and let's see whether it yields a value of r for which e^{rt} is a soln of (1). We get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c).$$

The latter will be zero precisely when $ar^2 + br + c = 0$, ie when r is a root of the quadratic polynomial $ar^2 + br + c$. We call the latter the <u>characteristic polynomial</u> of the eqn (1). If r_1 and r_2 are two distinct real roots of the charac. poly., then

$$e^{r_1 t} \quad \& \quad e^{r_2 t}$$

will be solns. Moreover, they must be linearly independent because their Wronskian $W(e^{r_1 t}, e^{r_2 t}) = e^{r_1 t}(e^{r_2 t})' - (e^{r_1 t})' e^{r_2 t}$

$$= e^{(r_1+r_2)t}(r_2 - r_1) \neq 0 \quad \text{if } r_1 \neq r_2.$$

We will explain in class why, as a consequence of the existence and uniqueness theorem, there cannot be a third linearly independent soln; i.e. the solution set of (1) is a two-dimensional vector space

So the study of (1) reduces to a knowledge of the roots of its characteristic polynomial. But as we know, it may not be the case that there are two distinct real roots. As usual, for quadratic polynomials, there are three cases to consider:

(a) two distinct real roots

(b) two " complex conjugate roots

(c) a single repeated real root.

We have seen above how to dispense with case (a). We will deal with cases (b) and (c) next week in class.