A NEW IDENTITY FOR PARSEVAL FRAMES

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ABSTRACT. In this paper we establish a surprising new identity for Parseval frames in a Hilbert space. Several variations of this result are given, including an extension to general frames. Finally, we discuss the derived results.

1. INTRODUCTION

Frames are an essential tool for many emerging applications such as data transmission. Their main advantage is the fact that frames can be designed to be redundant while still providing reconstruction formulas. This makes them robust against noise and losses while allowing freedom in design (see, for example, [6, 11]). Due to their numerical stability, *tight frames* and *Parseval frames* are of increasing interest in applications (See Section 2.1 for definitions.). Particularly in image processing, tight frames have emerged as essential tool (compare [8]). In abstract frame theory, systems constituting tight frames and, in particular, Parseval frames have already been extensively explored [3, 6, 7, 10, 11, 12], yet many questions are still open.

For many years engineers believed that, in applications such as speech recognition, a signal can be reconstructed without information about the phase. In [1] this longstanding conjecture was verified by constructing new classes of Parseval frames for which a signal vector can reconstructed without noisy phase or its estimation. While working on efficient algorithms for signal reconstruction, the authors of [1] discovered a surprising identity for Parseval frames (see [2] for a detailed discussion of the origins of the identity).

Our Parseval frame identity can be stated as follows (Theorem 3.2): For any Parseval frame $\{f_i\}_{i \in I}$ in a Hilbert space $\mathbb{H}$, and for every subset $J \subset I$ and every $f \in \mathbb{H}$

$$(1.1) \qquad \sum_{i \in J} |\langle f, f_i \rangle|^2 - \|\sum_{i \in J} \langle f, f_i \rangle f_i \|^2 = \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \|\sum_{i \in J^c} \langle f, f_i \rangle f_i \|^2.$$

The proof given here, based on operator theory, admits an elegant extension to arbitrary frames (Theorem 3.1). However, our main focus will be on Parseval frames because of their importance in applications, particularly to signal processing. Several interesting variants of our result are presented; for example, we show that

overlapping divisions can be also used. Then the Parseval frame identity is discussed in detail; in particular, we derive intriguing equivalent conditions for both sides of the identity to be equal to zero.

2. NOTATION AND PRELIMINARY RESULTS

2.1. **Frames and Bessel sequences.** Throughout this paper $\mathbb{H}$ will always denote a Hilbert space and I an indexing set. The finite linear span of a sequence of elements $\{f_i\}_{i \in I}$ of $\mathbb{H}$ will be denoted by $\text{span}(\{f_i\}_{i \in I})$. The closure in $\mathbb{H}$ of this set will be denoted by $\overline{\text{span}}(\{f_i\}_{i \in I})$.

A system $\{f_i\}_{i \in I}$ in $\mathbb{H}$ is called a *frame* for $\mathbb{H}$, if there exist $0 < A \le B < \infty$ (*lower and upper frame bounds*) such that

$$A \|f\|^2 \le \sum_{i \in I} |\langle f, f_i \rangle|^2 \le B \|f\|^2 \quad \text{for all } f \in \mathbb{H}.$$

If A, B can be chosen such that $A = B$, then $\{f_i\}_{i \in I}$ is an *A-tight frame*, and if we can take $A = B = 1$, it is called a *Parseval frame*. A *Bessel sequence* $\{f_i\}_{i \in I}$ is only required to fulfill the upper frame bound estimate but not necessarily the lower estimate. And a sequence $\{f_i\}_{i \in I}$ is called a *frame sequence*, if it is a frame only for $\overline{\text{span}}(\{f_i\}_{i \in I})$.

The *frame operator* $Sf = \sum_{i \in I} \langle f, f_i \rangle f_i$ associated with $\{f_i\}_{i \in I}$ is a bounded, invertible, and positive mapping of $\mathbb{H}$ onto itself. This provides the frame decomposition

$$f = S^{-1} S f = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i,$$

where $\tilde{f}_i = S^{-1} f_i$. The family $\{\tilde{f}_i\}_{i \in I}$ is also a frame for $\mathbb{H}$, called the *canonical dual frame* of $\{f_i\}_{i \in I}$. If $\{f_i\}_{i \in I}$ is a Bessel sequence in $\mathbb{H}$, for every $J \subset I$ we define the operator S_J by

$$S_J f = \sum_{i \in J} \langle f, f_i \rangle f_i.$$

Finally, we state a known result (see, for example, [9]), since it will be employed several times.

Proposition 2.1. *Let $\{f_i\}_{i \in I}$ be a frame for $\mathbb{H}$ with frame operator S. For every $f \in \mathbb{H}$, we have*

(1) $\|\sum_{i \in I} \langle f, f_i \rangle f_i\|^2 \le \|S\| \sum_{i \in I} |\langle f, f_i \rangle|^2$.

(2) $\sum_{i \in I} |\langle f, f_i \rangle|^2 \le \|S^{-1}\| \|\sum_{i \in I} \langle f, f_i \rangle f_i\|^2$.

Moreover, both these inequalities are best possible.

We note that for Bessel sequences $\{f_i\}_{i \in I}$, the operator S above is well-defined, we still call it "the frame operator", and part (1) of Proposition 2.1 remains true. Also, without chance of confusion, we will use I to denote our index set and the identity operator on $\mathbb{H}$.

For more details on frame theory we refer to the survey article [4] and the book [9].

2.2. **Operator Theory.** We first state a basic result from Operator Theory, which is very useful for the proof of the fundamental identity.

Proposition 2.2. *If S, T are operators on $\mathbb{H}$ satisfying $S + T = I$, then $S - T = S^2 - T^2$.*

Proof. We compute

$$S - T = S - (I - S) = 2S - I = S^2 - (I - 2S + S^2) = S^2 - (I - S)^2 = S^2 - T^2.$$

$\square$

3. A NEW IDENTITY

3.1. **General frames.** We first study the situation of general frames in $\mathbb{H}$.

Theorem 3.1. *Let $\{f_i\}_{i \in I}$ be a frame for $\mathbb{H}$ with canonical dual frame $\{\tilde{f}_i\}_{i \in I}$. Then for all $J \subset I$ and all $f \in \mathbb{H}$ we have*

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 - \sum_{i \in I} |\langle S_J f, \tilde{f}_i \rangle|^2 = \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \sum_{i \in I} |\langle S_{J^c} f, \tilde{f}_i \rangle|^2.$$

Proof. Let S denote the frame operator for $\{f_i\}_{i \in I}$. Since $S = S_J + S_{J^c}$, it follows that $I = S^{-1} S_J + S^{-1} S_{J^c}$. Applying Proposition 2.2 to the two operators $S^{-1} S_J$ and $S^{-1} S_{J^c}$ yields

$$(3.1) \qquad S^{-1} S_J - S^{-1} S_J S^{-1} S_J = S^{-1} S_{J^c} - S^{-1} S_{J^c} S^{-1} S_{J^c}.$$

Further, for every $f, g \in \mathbb{H}$ we obtain

$$(3.2) \quad \langle S^{-1} S_J f, g \rangle - \langle S^{-1} S_J S^{-1} S_J f, g \rangle = \langle S_J f, S^{-1} g \rangle - \langle S^{-1} S_J f, S_J S^{-1} g \rangle.$$

Now we choose g to be $g = Sf$. Then we can continue the equality (3.2) in the following way:

$$= \langle S_J f, f \rangle - \langle S^{-1} S_J f, S_J f \rangle = \sum_{i \in J} |\langle f, f_i \rangle|^2 - \sum_{i \in I} |\langle S_J f, \tilde{f}_i \rangle|^2.$$

Setting equality (3.2) equal to the corresponding equality for J^c and using (3.1), we finally get

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 - \sum_{i \in I} |\langle S_J f, \tilde{f}_i \rangle|^2 = \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \sum_{i \in I} |\langle S_{J^c} f, \tilde{f}_i \rangle|^2.$$

$\square$

3.2. **Parseval Frames.** In the situation of Parseval frames the new identity is of a special form, which moreover enlightens the surprising nature of it.

Theorem 3.2 (Parseval Frame Identity). *Let $\{f_i\}_{i \in I}$ be a Parseval frame for $\mathbb{H}$. For every subset $J \subset I$ and every $f \in \mathbb{H}$, we have*

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 - \| \sum_{i \in J} \langle f, f_i \rangle f_i \|^2 = \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \| \sum_{i \in J^c} \langle f, f_i \rangle f_i \|^2.$$

Proof. We wish to apply Theorem 3.1. Let $\{\tilde{f}_i\}_{i \in I}$ denote the dual frame of $\{f_i\}_{i \in I}$. Since $\{f_i\}_{i \in I}$ is a Parseval frame, its frame operator equals the identity operator

and hence $\tilde{f}_i = f_i$ for all $i \in I$. Employing Theorem 3.1 and the fact that $\{f_i\}_{i \in I}$ is a Parseval frame yields

$$
\begin{aligned}
\sum_{i \in J} |\langle f, f_i \rangle|^2 - \|\sum_{i \in J} \langle f, f_i \rangle f_i\|^2 &= \sum_{i \in J} |\langle f, f_i \rangle|^2 - \|S_J f\|^2 \\
&= \sum_{i \in J} |\langle f, f_i \rangle|^2 - \sum_{i \in I} |\langle S_J f, f_i \rangle|^2 \\
&= \sum_{i \in J} |\langle f, f_i \rangle|^2 - \sum_{i \in I} |\langle S_J f, \tilde{f}_i \rangle|^2 \\
&= \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \sum_{i \in I} |\langle S_{J^c} f, \tilde{f}_i \rangle|^2 \\
&= \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \|S_{J^c} f\|^2 \\
&= \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \|\sum_{i \in J^c} \langle f, f_i \rangle f_i\|^2.
\end{aligned}
$$

$\square$

Note that each side of the Parseval Frame Identity is non-negative, because S_J has operator norm at most 1 (see Proposition 2.1 and the remarks that follow it).

A version of the Parseval Frame Identity for overlapping divisions is derived in the following result.

Proposition 3.3. *Let $\{f_i\}_{i \in I}$ be a Parseval frame for $\mathbb{H}$. For every $J \subset I$, every $E \subset J^c$, and every $f \in \mathbb{H}$, we have*

$$
\|\sum_{i \in J \cup E} \langle f, f_i \rangle f_i\|^2 - \|\sum_{i \in J^c \setminus E} \langle f, f_i \rangle f_i\|^2
$$

$$
= \|\sum_{i \in J} \langle f, f_i \rangle f_i\|^2 - \|\sum_{i \in J^c} \langle f, f_i \rangle f_i\|^2 + 2 \sum_{i \in E} |\langle f, f_i \rangle|^2.
$$

Proof. Applying Theorem 3.2 twice yields

$$
\begin{aligned}
\|\sum_{i \in J \cup E} \langle f, f_i \rangle f_i\|^2 &- \|\sum_{i \in J^c \setminus E} \langle f, f_i \rangle f_i\|^2 \\
&= \sum_{i \in J \cup E} |\langle f, f_i \rangle|^2 - \sum_{i \in J^c \setminus E} |\langle f, f_i \rangle|^2 \\
&= \sum_{i \in J} |\langle f, f_i \rangle|^2 - \sum_{i \in J^c} |\langle f, f_i \rangle|^2 + 2 \sum_{i \in E} |\langle f, f_i \rangle|^2 \\
&= \|\sum_{i \in J} \langle f, f_i \rangle f_i\|^2 - \|\sum_{i \in J^c} \langle f, f_i \rangle f_i\|^2 + 2 \sum_{i \in E} |\langle f, f_i \rangle|^2.
\end{aligned}
$$

$\square$

Since each λ-tight frame can be turned into a Parseval frame by a change of scale, we obtain the following corollary.

Corollary 3.4. *Let $\{f_i\}_{i \in I}$ be a λ-tight frame for $\mathbb{H}$. Then for every $J \subset I$ and every $f \in \mathbb{H}$ we have*

$$
\lambda \sum_{i \in J} |\langle f, f_i \rangle|^2 - \|\sum_{i \in J} \langle f, f_i \rangle f_i\|^2 = \lambda \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \|\sum_{i \in J^c} \langle f, f_i \rangle f_i\|^2.
$$

Proof. If $\{f_i\}_{i\in I}$ is a λ-tight frame for $\mathbb{H}$, then $\{\frac{1}{\sqrt{\lambda}}f_i\}_{i\in I}$ is a Parseval frame for $\mathbb{H}$. Applying Theorem 3.2 proves the result. $\qquad\square$

Furthermore, the identity in Theorem 3.2 remains true even for Parseval frame sequences.

Corollary 3.5. *Let* $\{f_i\}_{i\in I}$ *be a Parseval frame sequence in* $\mathbb{H}$. *Then for every* $J \subset I$ *and every* $f \in \mathbb{H}$ *we have*

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 - \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2 = \sum_{i\in J^c}|\langle f, f_i\rangle|^2 - \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2.$$

Proof. Let P denote the orthogonal projection of $\mathbb{H}$ onto $\overline{\text{span}}(\{f_i\}_{i\in I})$. By Theorem 3.2, for every $f \in \mathbb{H}$, we have

$$\sum_{i\in J}|\langle Pf, f_i\rangle|^2 - \|\sum_{i\in J}\langle Pf, f_i\rangle f_i\|^2 = \sum_{i\in J^c}|\langle Pf, f_i\rangle|^2 - \|\sum_{i\in J^c}\langle Pf, f_i\rangle f_i\|^2.$$

Since $\langle Pf, f_i\rangle = \langle f, Pf_i\rangle = \langle f, f_i\rangle$ for all $i \in I$, the result follows. $\qquad\square$

We finally remark that Theorem 3.2 is logically equivalent to Theorem 3.1. Indeed, let $\{f_i\}_{i\in I}$ be a frame for $\mathbb{H}$, with frame operator S. It is well-known (see, for example, [4, Theorem 4.2]) that $\{h_i := S^{-\frac{1}{2}}f_i\}_{i\in I}$ is a Parseval frame for $\mathbb{H}$. It is easy to check that Theorem 3.2 applied to $\{h_i\}_{i\in I}$ yields Theorem 3.1.

4. DISCUSSION OF THE PARSEVAL FRAME IDENTITY

The identity given in Theorem 3.2 is quite surprising in that the quantities on the two sides of the identity are not comparable to one another in general. For example, if J is the empty set, then the left-hand-side of this identity is zero because

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 = 0 = \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2.$$

The right-hand-side of this identity is also zero, but now because

$$\sum_{i\in J^c}|\langle f, f_i\rangle|^2 = \|f\|^2 = \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2.$$

Similarly, if $|J| = 1$, then both terms on the left-hand-side of this identity may be arbitrarily close to zero, while the two terms on the right-hand-side of the identity are nearly equal to $\|f\|^2$, and they are canceling precisely enough to produce the identity.

If $\{f_i\}_{i\in I}$ is a Parseval frame for $\mathbb{H}$, then for every $J \subset I$ and every $f \in \mathbb{H}$ we have

$$\|f\|^2 = \sum_{i\in J}|\langle f, f_i\rangle|^2 + \sum_{i\in J^c}|\langle f, f_i\rangle|^2.$$

Hence, one of the two terms on the right-hand-side of the above equality is greater than or equal to $\frac{1}{2}\|f\|^2$. It follows from Theorem 3.2 that for every $J \subset I$ and every $f \in \mathbb{H}$,

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 + \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2 = \sum_{i\in J^c}|\langle f, f_i\rangle|^2 + \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2 \geq \frac{1}{2}\|f\|^2.$$

We will now see that actually the right-hand-side of this inequality is in fact much larger.

Proposition 4.1. *If $\{f_i\}_{i \in I}$ is a Parseval frame for $\mathbb{H}$, then for every $J \subset I$ and every $f \in \mathbb{H}$ we have*

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 + \| \sum_{i \in J^c} \langle f, f_i \rangle f_i \|^2 \geq \tfrac{3}{4} \|f\|^2.$$

Further, we have equality for all $f \in \mathbb{H}$ if and only if $S_J = \frac{1}{2} I$.

Proof. By completing the square, we see that

$$S_J^2 + S_{J^c}^2 = 2 \left(S_J - \frac{1}{2} I \right)^2 + \frac{1}{2} I;$$

and so

$$S_J^2 + S_{J^c}^2 \geq \frac{1}{2} I,$$

with equality if and only if $S_J = \frac{1}{2} I$.

Since $S_J + S_{J^c} = I$, it follows that $S_J + S_{J^c}^2 + S_{J^c} + S_J^2 \geq \tfrac{3}{2} I$. Applying Proposition 2.2 to $S = S_J$ and $T = S_{J^c}$ yields $S_J + S_{J^c}^2 = S_{J^c} + S_J^2$. Thus

$$2(S_J + S_{J^c}^2) = S_J + S_{J^c}^2 + S_{J^c} + S_J^2 \geq \tfrac{3}{2} I.$$

Finally, for every $f \in \mathbb{H}$ we have

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 + \| \sum_{i \in J^c} \langle f, f_i \rangle f_i \|^2 = \langle S_J f, f \rangle + \langle S_{J^c} f, S_{J^c} f \rangle = \langle (S_J + S_{J^c}^2) f, f \rangle \geq \tfrac{3}{4} \|f\|^2.$$

From above, we clearly have equality for all $f \in \mathbb{H}$ if and only if $S_J = \frac{1}{2} I$. $\qquad \square$

We remark that a key part of the proof of Proposition 4.1 extends in a straightforward manner to prove the following fact. Let $\{f_i\}_{i \in I}$ be a Parseval frame for $\mathbb{H}$. Let $\{J_k\}_{k=1}^n$ be a finite partition of the index set I and let $S_k := S_{J_k}$ for each k. Then

$$S_1^2 + \cdots + S_n^2 \geq \frac{1}{n} I;$$

with equality if and only if $S_j = S_k$, for all $j, k \in \{1, \cdots, n\}$; if and only if $S_j = \frac{1}{n} I$, for all $j \in \{1, \cdots, n\}$.

Let $\{f_i\}_{i \in I}$ be a λ-tight frame for $\mathbb{H}$. Reformulating Corollary 3.4 yields that for every $J \subset I$ and every $f \in \mathbb{H}$ we have

$$\lambda \sum_{i \in J} |\langle f, f_i \rangle|^2 - \lambda \sum_{i \in J^c} |\langle f, f_i \rangle|^2 = \| \sum_{i \in J} \langle f, f_i \rangle f_i \|^2 - \| \sum_{i \in J^c} \langle f, f_i \rangle f_i \|^2.$$

We intend to study when both sides of this equality equal zero, which is closely related to questions concerning extending a frame to a tight frame. The proofs of these results use the next lemma as a main ingredient.

Lemma 4.2. *Let $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in K}$ be Bessel sequences in $\mathbb{H}$ with frame operators S and T, respectively. If $S = T$, then*

$$\overline{\operatorname{span}}(\{f_i\}_{i \in I}) = \overline{\operatorname{span}}(\{g_i\}_{i \in K}).$$

Proof. For any $f \in \mathbb{H}$, we have

$$\sum_{i \in I} |\langle f, f_i \rangle|^2 = \langle Sf, f \rangle = \langle Tf, f \rangle = \sum_{i \in K} |\langle f, g_i \rangle|^2.$$

It follows that $f \perp f_i$ for all $i \in I$ if and only if $f \perp g_i$ for all $i \in K$. $\qquad \square$

The following result is an immediate consequence of Lemma 4.2 and the above reformulation of Corollary 3.4.

Proposition 4.3. *Let $\{f_i\}_{i \in I}$ be a λ-tight frame for $\mathbb{H}$ and $J \subset I$. If $\|S_J f\| = \|S_{J^c} f\|$, for all $f \in \mathbb{H}$, then*

$$\overline{\mathrm{span}}(\{f_i\}_{i \in I}) = \overline{\mathrm{span}}(\{f_i\}_{i \in J^c}).$$

It is well known that given a frame $\{f_i\}_{i \in I}$ for a Hilbert space $\mathbb{H}$, there exists a sequence (and in fact there are many such sequences) $\{g_i\}_{i \in K}$ so that $\{f_i\}_{i \in I} \cup \{g_i\}_{i \in K}$ is a tight frame (see, for example, [5]). We will now see that, if we choose two different families to extend $\{f_i\}_{i \in I}$ to a tight frame, then these new families have several important properties in common.

Proposition 4.4. *Let $\{f_i\}_{i \in I}$ be a frame for $\mathbb{H}$. Assume that $\{f_i\}_{i \in I} \cup \{g_i\}_{i \in K}$ and $\{f_i\}_{i \in I} \cup \{h_i\}_{i \in L}$ are both λ-tight frames. Then the following conditions hold.*

(1) *For every $f \in \mathbb{H}$, $\sum_{i \in K} |\langle f, g_i \rangle|^2 = \sum_{i \in L} |\langle f, h_i \rangle|^2$.*

(2) *For every $f \in \mathbb{H}$, $\sum_{i \in K} \langle f, g_i \rangle g_i = \sum_{i \in L} \langle f, h_i \rangle h_i$.*

(3) *$\overline{\mathrm{span}}(\{g_i\}_{i \in K}) = \overline{\mathrm{span}}(\{h_i\}_{i \in L})$.*

Proof. For all $f \in \mathbb{H}$, we have

$$\sum_{i \in I} |\langle f, f_i \rangle|^2 + \sum_{i \in K} |\langle f, g_i \rangle|^2 = \lambda \|f\|^2 = \sum_{i \in I} |\langle f, f_i \rangle|^2 + \sum_{i \in L} |\langle f, h_i \rangle|^2.$$

This yields (1).

Similarly, for all $f \in H$,

$$\sum_{i \in I} \langle f, f_i \rangle f_i + \sum_{i \in K} \langle f, g_i \rangle g_i = \lambda f = \sum_{i \in I} \langle f, f_i \rangle f_i + \sum_{i \in L} \langle f, h_i \rangle h_i,$$

which proves (2).

Condition (3) follows immediately from (2) and Lemma 4.2. $\square$

In the next theorem we will derive many equivalent conditions for both sides of the Parseval Frame Identity (Theorem 3.2) to equal zero. For this, we first need a result concerning the operators S_J, S_{J^c}.

Proposition 4.5. *Let $\{f_i\}_{i \in I}$ be a Parseval frame for $\mathbb{H}$. For any $J \subset I$, $S_J S_{J^c}$ is a positive self-adjoint operator on $\mathbb{H}$ which satisfies*

$$S_J - S_J^2 = S_J S_{J^c} \geq 0.$$

Proof. Clearly,

$$S_J - S_J^2 = S_J(I - S_J) = S_J S_{J^c}.$$

Moreover, since S_J and S_{J^c} are commuting positive, self-adjoint operators, it follows that $S_J S_{J^c}$ is also positive and self-adjoint.

$\square$

Note that for any positive operator T on a Hilbert space $\mathbb{H}$ and any $f \in \mathbb{H}$, $Tf = 0$ implies $\langle Tf, f \rangle = 0$. The converse of this is also true. If $\langle Tf, f \rangle = 0$, then by a simple calculation

$$\langle Tf, f \rangle = \langle T^{1/2}f, T^{1/2}f \rangle = \|T^{1/2}f\|^2 = 0.$$

So $T^{1/2}f = 0$, and hence $Tf = 0$. Noting that one side of the Parseval Frame Identity is zero if and only if the other side is, we are led to the following result.

Theorem 4.6. *Let $\{f_i\}_{i \in I}$ be a Parseval frame for $\mathbb{H}$. For each $J \subset I$ and $f \in \mathbb{H}$, the following conditions are equivalent.*

(1) $\sum_{i \in J} |\langle f, f_i \rangle|^2 = \| \sum_{i \in J} \langle f, f_i \rangle f_i \|^2$.

(2) $\sum_{i \in J^c} |\langle f, f_i \rangle|^2 = \| \sum_{i \in J^c} \langle f, f_i \rangle f_i \|^2$.

(3) $\sum_{i \in J} \langle f, f_i \rangle f_i \perp \sum_{i \in J^c} \langle f, f_i \rangle f_i$.

(4) $f \perp S_J S_{J^c} f$.

(5) $S_J f = S_J^2 f$.

(6) $S_J S_{J^c} f = 0$.

Proof. $(1) \Leftrightarrow (2)$: This is follows immediately from Theorem 3.2.

$(3) \Leftrightarrow (4)$: This is proven by the following equality:

$$\langle \sum_{i \in J} \langle f, f_i \rangle f_i, \sum_{i \in J^c} \langle f, f_i \rangle f_i \rangle = \langle S_J f, S_{J^c} f \rangle = \langle f, S_J S_{J^c} f \rangle.$$

$(5) \Leftrightarrow (6)$: This follows from Proposition 4.5.

$(1) \Leftrightarrow (5)$: We have

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 - \| \sum_{i \in J} \langle f, f_i \rangle f_i \|^2 = \langle S_J f, f \rangle - \langle S_J f, S_J f \rangle = \langle (S_J - S_J^2) f, f \rangle.$$

By Proposition 4.5, $S_J - S_J^2 \geq 0$. Therefore the right-hand side of the above equality is zero if and only if $(S_J - S_J^2) f = 0$ by our discussion preceding the proposition.

$(1) \Rightarrow (4)$: By (1), $\langle S_J f, f \rangle = \langle S_J f, S_J f \rangle$. Hence $\langle S_J S_{J^c} f, f \rangle = \langle (S_J - S_J^2) f, f \rangle = 0$, which implies (4).

$(4) \Rightarrow (6)$: By Proposition 4.5, we have that $S_J S_{J^c} \geq 0$. Thus $\langle S_J S_{J^c} f, f \rangle = 0$ if and only if $S_J S_{J^c} f = 0$ by the discussion preceding this proposition. $\square$

Acknowledgments

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