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Measure functions for frames

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Abstract

This paper addresses the natural question: "How should frames be compared?" We answer this question by quantifying the overcompleteness of all frames with the same index set. We introduce the concept of a *frame measure function*: a function which maps each frame to a continuous function. The comparison of these functions induces an equivalence and partial order that allows for a meaningful comparison of frames indexed by the same set. We define the *ultrafilter measure function*, an explicit frame measure function that we show is contained both algebraically and topologically inside all frame measure functions. We explore additional properties of frame measure functions, showing that they are additive on a large class of supersets—those that come from so called *non-expansive* frames. We apply our results to the Gabor setting, computing the frame measure function of Gabor frames and establishing a new result about supersets of Gabor frames.
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1. Introduction

Let H be a separable Hilbert space and I a countable index set. A sequence $\mathcal{F} = \{f_i\}_{i \in I}$ of elements of H is a *frame* for H if there exist constants A, $B > 0$ such that

$$\forall h \in H, \quad A\|h\|^2 \leqslant \sum_{i \in I} |\langle h, f_i \rangle|^2 \leqslant B\|h\|^2. \tag{1}$$

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The numbers A, B are called *lower* and *upper frame bounds*, respectively. Frames were first introduced by Duffin and Schaeffer [18] in the context of nonharmonic Fourier series, and today frames play important roles in many applications in mathematics, science, and engineering. We refer to the monograph of Daubechies [15] or the research-tutorial [11] for basic properties of frames.

Central, both theoretically and practically, to the interest in frames has been their overcomplete nature; the strength of this overcompleteness is the ability of a frame to express arbitrary vectors as a linear combination in a "redundant" way. Until recently, for infinite-dimensional frames, the overcompleteness or redundancy has only referred to a qualitative feature of frames. A notable exception is in the case of Gabor frames where many works have connected essential features of the frames to quantities related to the density of the associated lattice of time and frequency shifts ([24] and references therein). Recently, the work in [2,3,8,9] examined and explored the notion of *excess* of a frame, i.e. the maximal number of frame elements that could be removed while keeping the remaining elements a frame for the same span. A quantitative approach to certain frames with infinite excess was given in [2,3] which introduced a general notion of a localized frame and, among other results, provided nice quantitative measures associated to this class of frames.

This paper addresses the natural question: "How should frames be compared?" We answer this question by quantifying the overcompleteness of all frames with the same index set. We describe a new equivalence relation and partial order on these frames. We introduce the central tool for working with this partial order: the *frame measure function* which maps each frame to a continuous function. The frame measure functions are compatible with our equivalence relation, namely two frames are equivalent if and only if their frame measure functions are equal (pointwise as continuous functions) and one frame dominates another if their frame measure functions have the corresponding dominance (pointwise). This results in a quantification of frames that reflects the partial order and leads to a meaningful quantitative definition of the overcompleteness of a frame.

Though equivalence of frames with an infinite number of elements has been considered previously (see [6,14]) and a standard notion of equivalence for frames exist, the size of each equivalence class is too small; it is fundamentally unsatisfying as it distinguishes frames, that from a signal processing point of view, are equivalent.

In contrast, the equivalence relation, partial order and frame measure function introduced here have the following desirable properties (that are not present in the standard equivalence relation):

- The equivalence relation groups together all Riesz bases.
- The equivalence relation groups together all frames that differ by a finite permutation of their elements or by arbitrary phase change of their elements.
- From an information theory point of view, the equivalence relation groups together frames that transmit signals with similar variances due to noise.
- The values of the measure function are linked to the amount of excess of the frame.
- For a large class of frames (those that are called *non-expansive*) any frame measure function is additive on supersets, namely, the frame measure function applied to the frame $\{f_i \oplus g_i\}_{i \in I}$ acting on $H_1 \oplus H_2$ is equal to the sum of the frame measure function applied to the two frames $\{f_i\}_{i \in I}$ acting on H_1 and $\{g_i\}_{i \in I}$ acting on H_2.
- The values of the frame measure function for Gabor frames are shown to correspond to the density in the time–frequency plane of the shifts associated to the frame.

The focus of this work is to explore the properties of the equivalence relation, partial order, and frame measure functions. In addition to showing the above listed facts, we describe a specific frame measure function, the *ultrafilter frame measure function*—a function from the set of all frames indexed by a set I (denoted by $\mathcal{F}[I]$) to the set of continuous functions on the compact space consisting of the free ultrafilters. We show that every frame measure function contains a copy of the ultrafilter frame measure function. In addition, as with representation theory, we define separable, reducible, and minimal frame measure functions and show that all minimal frame measure functions are topologically equivalent to the ultrafilter frame measure function.

We apply this theory to the Gabor setting. In addition to computing the measure of Gabor frames, we apply our results to Gabor supersets, showing new necessary conditions on the densities of the time–frequency shifts of the individual Gabor frames.

Finally we propose that the reciprocal of the measure function be defined to be the redundancy for an infinite frame. Redundancy, an often referred to qualitative feature of frames, has eluded a meaningful quantitative definition for infinite frames. Using the results of this work, we justify our definition of redundancy by both showing it to be quantitatively meaningful and a natural generalization of redundancy for finite frames.

A striking feature of these ideas is the variety of mathematical areas that are involved. The fundamental objects, frames, are objects of considerable interest to the signal processing community. The motivation for our definitions of frame equivalence and comparison come from both information theoretic and operator theoretic considerations. The ideas and tools that drive the results are mainly operator theoretic and topological.

The equivalence relation, partial order, and frame measure functions introduced here are a function of certain averages of the terms $\langle f_i, \tilde{f}_i \rangle$ of a given frame $\{f_i\}_{i \in I}$ (where $\{\tilde{f}_i\}_{i \in I}$ is the canonical dual frame to $\{f_I\}_{I \in I}$). These are the same averages that play a central role in the two papers [2,3] which introduce the notion of localized frames. In this work, our goal is to compare all frames that are indexed by the same fixed index set but which possibly lie in different Hilbert spaces; we require no special localized structure for the frames. In contrast, in [2,3] the situation considered is that of frames which all lie in the same Hilbert space that are indexed by different sets. An index set map is introduced and when this index map is chosen so that the frame is localized, powerful results are obtained relating a feature of the index map (density), to certain averages of $\langle f_i, \tilde{f}_i \rangle$ (relative measure). Despite the differences in approach between [2,3] and this work, there is significant intersection and interelation of ideas. Specifically, where the settings are compatible, the notion of a *non-expansive frame* introduced here is the same as the notion of a l^2 localized frame of [2,3]. In addition, we use specific results of [3] to compute the ultrafilter frame measure function of Gabor frames.

The work is organized as follows. The equivalence relation and partial order is introduced and initially explored in Section 3. Section 4 defines and proves essential properties of the *ultrafilter frame measure function*. The general notion of a frame measure function is defined and core properties are proven in Section 5. Of particular note is Corollary 5.24 which shows that every frame measure function contains an algebraic copy of the ultrafilter frame measure function. Section 6 examines the topological properties of the frame measure function, showing, among other things, that the ultrafilter frame measure function is, in a certain sense, the unique minimal frame measure function. We extend the frame measure functions ideas to the space of operators in Section 7 and introduce the core concept of a non-expansive operator. Section 7.3 applies these ideas to supersets to prove Theorem 7.14 which establishes that frame measure functions are additive on superframes comprised of non-expansive frames. Section 8 examines the connection between the measure function and the index set. Section 9 applies the results to the Gabor

setting, computing the frame measure function of Gabor frames and establishing a new result about supersets of Gabor frames. Finally, Section 10 defines and explores the properties of the redundancy function for infinite frames. Appendices A and B cover some background material on supersets and ultrafilters.

2. Notation and preliminaries

2.1. Basic notation

For any set S, $|S|$ will denote the number of elements in S. Throughout this paper I will be a fixed countable index set accompanied by a decomposition into a nested union (indexed by the positive integers $1, 2, \ldots$) of finite subsets. That is,

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset I_{n+1} \subset \cdots \subset I, \tag{2}$$

$$|I_n| < \infty, \tag{3}$$

$$\bigcup_{n \geqslant 1} I_n = I. \tag{4}$$

Though not explicit in the notation, the index set I will always have the above decomposition associated with it. The variable $\mathbf{i}$ shall denote the sequence $\mathbf{i} = (|I_1|, |I_2|, \ldots)$. We denote by $l^2(I)$ the Hilbert space of square summable sequences indexed by I with inner product defined as $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i \in I} x_i \bar{y}_i$. We denote by δ_i the sequence whose ith entry is one and is zero otherwise; thus $\{\delta_i\}_{i \in I}$ is the canonical orthonormal basis for $l^2(I)$.

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

Equality of two functions $f = g$ that have the same domain shall mean that the two functions agree for every point in the domain.

Given two sequences $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$ and a scalar c, $\mathbf{x} + \mathbf{y}$ shall denote the sequence $(x_1 + y_1, x_2 + y_2, \ldots)$, $c\mathbf{x}$ shall denote the sequence $(cx_1, cx_2, cx_3, \ldots)$, $\frac{\mathbf{x}}{\mathbf{y}}$ shall denote the sequence $(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots)$, and $\lfloor \mathbf{x} \rfloor$ shall denote the sequence $(\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots)$.

H shall denote a Hilbert space. For a subset $S \subset H$, $\overline{\text{span}\{S\}}$ shall denote the closure of the linear subspace of H spanned by the elements of S. Given $h \in H$, $\|h\| = (\langle h, h \rangle)^{\frac{1}{2}}$ shall denote the Hilbert space norm of h. Given $A : H \to H$, a bounded linear operator, $\|A\| = \sup_{h, g \in H, \|h\|=1, \|g\|=1} |\langle Ah, g \rangle|$ shall be the operator norm of A.

Appendix B contains a summary of some basic notation and properties of ultrafilters.

Finally, we remark that occasionally, when a result is straightforward to verify, we will state it without providing a proof.

2.2. Frames

We use standard notations for frames as found in the texts of Gröchenig [21], or Daubechies [15]; see also the research-tutorials [23] or [11], and the introductory book [13] for background on frames and Riesz bases.

We shall use the following particular notation.

The definition of a *frame* is given in (1). A sequence $\mathcal{F} = \{f_i\}_{i \in I}$ that is a frame for $\overline{\text{span}\{\mathcal{F}\}}$ which might not be all of H shall be called a *frame sequence*.

A frame is *finite* if the size of the index set I is finite and *infinite* if the size of the index set I is infinite. A frame is said to be *tight* if we can choose equal frame bounds $A = B$. When $A = B = 1$, the frame is called a *Parseval frame*. We denote by $\mathcal{F}[I]$ the set of all frame sequences indexed by I.

In the case of a frame or a frame sequence $\mathcal{F}$, the *frame operator* S, defined by $Sf = \sum_{i \in I} \langle f, f_i \rangle f_i$ is a bounded, positive, and invertible mapping of $\overline{\mathrm{span}\{\mathcal{F}\}}$ onto itself. The *Gram operator* G in $l^2(I)$ is defined to be:

$$G : l^2(I) \to l^2(I), \quad \{G(\{c_j\}_{j \in I})\}_i = \sum_{j \in I} \langle f_i, f_j \rangle c_j. \tag{5}$$

The *analysis operator* $T : \overline{\mathrm{span}\{\mathcal{F}\}} \to l^2(I)$, and *synthesis operator* $T^* : l^2(I) \to \overline{\mathrm{span}\{\mathcal{F}\}}$, are defined by $Tf = \{\langle f, f_i \rangle\}_{i \in I}$, respectively by $T^*c = \sum_{i \in I} c_i f_i$.

The following terminology is standardly applied to frames, however it applies equally well to frame sequences; rather than introduce additional notation, we shall associate to a frame or frame sequence $\mathcal{F}$:

- the *canonical* (or *standard*) *dual frame* $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i \in I}$ where $\tilde{f}_i = S^{-1} f_i$.
- the *associated Parseval frame* $\{S^{-\frac{1}{2}} f_i\}_{i \in I}$ which has the property that it is equal to its canonical dual frame and has upper and lower frame bounds equal to 1.

The *associated Gram projection* to a frame or frame sequence $\mathcal{F}$ will be the orthogonal projection in $l^2(I)$ onto the range of the Gram operator G. Equivalently, this is the Gram operator of the associated Parseval frame.

A frame is a basis if and only if it is a Riesz basis, i.e., it is the image of an orthonormal basis for H under a continuous, invertible mapping of H onto itself. A Riesz sequence shall refer to a sequence that is a Riesz basis for its closed linear span.

For a frame $\mathcal{F} = \{f_i\}_{i \in I}$ with canonical dual $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i \in I}$, $\langle f_i, \tilde{f}_j \rangle$ is the (j, i) entry of the matrix representation of the associated Gram projection in the canonical basis of $l^2(I)$. Consequently

$$0 \leqslant \langle f_i, \tilde{f}_i \rangle = \langle S^{-1/2} f_i, S^{-1/2} f_i \rangle = \| S^{-1/2} f_i \|^2 \leqslant 1 \tag{6}$$

and $\langle f_i, \tilde{f}_i \rangle = 1$ if and only if $f_i \perp \overline{\mathrm{span}\{\{f_j\}_{j \neq i}\}}$.

For two frames $\mathcal{F}$ and $\mathcal{G}$, the *superset* $\mathcal{F} \oplus \mathcal{G}$ shall denote the set $\{f_i \oplus g_i\}_{i \in I}$. Appendix A contains some basic notation and results pertaining to supersets.

Note the upper bound inequality in (1) is equivalent to $\| \sum_i c_i f_i \|^2 \leqslant B \sum_i |c_i|^2$ for any $(c_i)_i \in \ell^2(I)$.

2.3. The sequences $a(\mathcal{F})$ and $b(\mathcal{F})$ associated to a frame

In this paper, frames will be compared using the data $\{f_i, \tilde{f}_i\}_{i \in I}$. Specifically, for each frame $\mathcal{F} \in \mathcal{F}[I]$, the sequence

$$a(\mathcal{F}) = \{a_n(\mathcal{F})\}_{n \in \mathbf{N}}, \quad a_n(\mathcal{F}) = \frac{1}{|I_n|} \sum_{i \in I_n} \langle f_i, \tilde{f}_i \rangle, \tag{7}$$

shall play a central role. The related "unnormalized" sequence

$$b(\mathcal{F}) = \big\{b_n(\mathcal{F})\big\}_{n \in \mathbf{N}}, \quad b_n(\mathcal{F}) = \sum_{i \in I_n} \langle f_i, \tilde{f}_i \rangle = |I_n| a_n(\mathcal{F}), \tag{8}$$

shall be used frequently.

3. A new notion of frame equivalence

In this section we define the equivalence and partial ordering of frames. These concepts will only depend on the sequences $b(\mathcal{F})$ (or equivalently $a(\mathcal{F})$). The ideas and proofs about this equivalence are more naturally viewed as properties of sequences. Consequently we begin by defining a class of sequences, called *frame compatible sequences* and showing that all sequences $b(\mathcal{F})$ arising from frames are frame compatible and that all frame compatible sequences are "close" to $b(\mathcal{F})$ for some frame $\mathcal{F}$ (Theorem 3.4). We then define an equivalence and partial order on frame compatible sequences (Definition 3.5) which naturally pulls back to an equivalence and partial order of frames (Definition 3.7). We compare this equivalence to the well studied standard equivalence. Section 3.3 shows the advantages of the new equivalence. Finally, in Section 3.4 we establish the frame-sequence correspondence which relates the addition of sequences to the superset operation $\oplus$ of certain frames (Theorem 3.17). This correspondence will repeatedly be used later in proofs about frame measure functions.

3.1. Frame compatible sequences, equivalence and partial order

Definition 3.1. A sequence of nonnegative real numbers $\mathbf{x} = (x_1, x_2, \ldots)$ will be called *frame compatible* if

(1) $0 \leqslant x_1 \leqslant |I_1|$,
(2) $0 \leqslant x_i - x_{i-1} \leqslant |I_i \setminus I_{i-1}|$ for all $i \geqslant 2$.

We shall denote by X the set of all frame compatible sequences.

Remark 3.2. Note that if $\mathbf{x}$ is frame compatible then so is $\lfloor \mathbf{x} \rfloor$.

Definition 3.3. A frame will be called *normal-orthogonal* if all nonzero elements of it are distinct elements of an orthonormal set.

Theorem 3.4.

(1) *Given a frame $\mathcal{F}$, the sequence $b(\mathcal{F})$ is frame compatible.*
(2) *For any frame compatible sequence $\mathbf{x}$, there exists a normal-orthogonal frame denoted by $\mathcal{G}^{\mathbf{x}}$ with $b(\mathcal{G}^{\mathbf{x}}) = \lfloor \mathbf{x} \rfloor$.*

Proof. Statement (1) follows simply from $0 \leqslant \langle f_i, \tilde{f}_i \rangle \leqslant 1$ (6).

To prove (2) choose $S_1 \subset I_1$ such that $|S_1| = \lfloor x_1 \rfloor$. Choose $S_n \subset I_n \setminus I_{n-1}$, $n \geqslant 2$, such that $|S_n| = \lfloor x_n \rfloor - \lfloor x_{n-1} \rfloor$ (this can be done precisely because $\mathbf{x}$ is frame compatible). Set $S = \bigcup_n S_n \subset I$ and let B be an arbitrary countable orthonormal set. Define a frame $\mathcal{G}^{\mathbf{x}} = \{g_i^{\mathbf{x}}\}_{i \in I}$ such that $\{g_i^{\mathbf{x}}\}_{i \in S}$ are distinct elements of B and $g_i^{\mathbf{x}} = 0$ for $i \in I \setminus S$. The frame $\mathcal{G}^{\mathbf{x}}$ is Parseval since it is the union of distinct orthonormal elements and zeroes. It follows therefore that $\langle g_i^{\mathbf{x}}, \tilde{g}_i^{\mathbf{x}} \rangle = \|g_i^{\mathbf{x}}\|^2$ which is 1 for $i \in S$ and 0, otherwise. Thus $b_n(\mathcal{G}^{\mathbf{x}}) = \sum_{j=1}^n |S_j| = \lfloor x_n \rfloor$. $\quad \square$

The following defines an important equivalence relation and partial order on the set of frame compatible sequences. We combine these definitions with the map b to produce the central object of this paper: an equivalence and partial order on the set of frames $\mathcal{F}[I]$.

Definition 3.5 *(Sequence equivalence and partial ordering).*

(1) Given two sequences $\mathbf{x}$, $\mathbf{y}$ with complex entries we say

$$\mathbf{x} \approx \mathbf{y} \quad \text{if } \lim_{n \to \infty} \frac{1}{|I_n|}(x_n - y_n) = 0.$$

(2) Given two sequences $\mathbf{x}$, $\mathbf{y}$ with non-negative real entries we say

$$\mathbf{y} \lhd \mathbf{x} \quad \text{if } \liminf_{n \to \infty} \frac{1}{|I_n|}(x_n - y_n) \geqslant 0.$$

Remark 3.6. For the moment, the equivalence relation and partial order will be applied to frame compatible sequences. However, later we shall be considering this relation on a larger collection of sequences.

Definition 3.7 *(Ultrafilter frame equivalence and partial ordering).*

(1) We shall say two frames $\mathcal{F}, \mathcal{G} \in \mathcal{F}[I]$ are ultrafilter equivalent, denoted $\mathcal{F} \approx \mathcal{G}$, if $b(\mathcal{F}) \approx b(\mathcal{G})$.
(2) For two frames $\mathcal{F}, \mathcal{G} \in \mathcal{F}[I]$ we say $\mathcal{F} \lhd \mathcal{G}$ if $b(\mathcal{F}) \lhd b(\mathcal{G})$.

The next two subsections provide some motivation for this definition. We begin by reviewing the standard notion of equivalence and then discuss some advantages of the ultrafilter equivalence.

3.2. The standard equivalence of frames.

A different notion of equivalence of frames that has been studied quite extensively is as follows (see [1,6,22]).

Definition 3.8. Given two frames $\mathcal{F} = \{f_i\}_{i \in I} \subset H_1$, $\mathcal{G} = \{g_i\}_{i \in I} \subset H_2$, we say $\mathcal{F} \sim \mathcal{G}$ if there is a bounded invertible operator $S : H_1 \to H_2$ such that $Sf_i = g_i$ for every $i \in I$.

It is easy to verify that $\sim$ is an equivalence relation (namely it is reflexive, symmetric and transitive). Moreover, it admits the following geometric interpretation that says that two frames are $\sim$ equivalent if and only if the ranges of their Gram operators are the same.

Theorem 3.9. *(See [6,14].) Consider $\mathcal{F}, \mathcal{G}$ in $\mathcal{F}[I]$ and let P, Q be their associated Gram projections. Then $\mathcal{F} \sim \mathcal{G}$ if and only if $P = Q$.*

Using this theorem and the remark before (6), it is simple to verify that the equivalence in the $\sim$ relation implies equivalence in the $\approx$ relation:

Proposition 3.10. *Given two frames $\mathcal{F}, \mathcal{G} \in \mathcal{F}[I]$, $\mathcal{F} \sim \mathcal{G}$ implies $\mathcal{F} \approx \mathcal{G}$.*

The $\sim$ equivalence relation is a very strong notion of equivalence. For instance, in the following examples, the closely related frames $\mathcal{F}$ and $\mathcal{G}$ are *not* $\sim$ equivalent.

Example 3.11. Let the elements of $\mathcal{G}$ differ from those in $\mathcal{F}$ by scalars of modulus one, i.e. $\mathcal{G} = \{g_j = e^{i\phi_j} f_j \colon j \in I\}$. In most cases, these frames are not $\sim$ equivalent (unless $\mathcal{F}$ was a Riesz basis for its span). In fact, this is true even when we require that $e^{i\phi_j} \in \{-1, 1\}$.

Example 3.12. Let the elements of $\mathcal{G}$ be a finite permutation of those in $\mathcal{F}$, i.e. let $\pi \colon I \to I$ be a bijection so that there exists a finite subset $J \subset I$ so that $\pi(i) = i$ for all $i \in I \setminus J$, and set $\mathcal{G} = \{g_i = f_{\pi(i)} \colon i \in I\}$. In almost all cases $\mathcal{F}$ and $\mathcal{G}$ are not $\sim$ equivalent.

Example 3.13. Let $\mathcal{F} = \{f_i\}$, and $\mathcal{G} = \{g_i\}$ be two Parseval frames so that $\|f_i\| = \|g_i\|$, for all $i \in I$. In most cases these two frames are not $\sim$ equivalent.

3.3. The advantages of the ultrafilter equivalence $\approx$

The following proposition is straightforward and shows that unlike the $\sim$ equivalence, the $\approx$ equivalence identifies the frames in Examples 3.11–3.13 as equivalent.

Proposition 3.14.

(1) *If $\mathcal{G} = \{g_j = e^{i\phi_j} f_j \colon j \in I\}$, then $\mathcal{G} \approx \mathcal{F}$.*
(2) *If $\mathcal{G} = \{g_i = f_{\pi(i)} \colon i \in I\}$ for a finite permutation $\pi \colon I \to I$ then $\mathcal{G} \approx \mathcal{F}$.*
(3) *Let $\mathcal{F}$ and $\mathcal{G}$ be Parseval frames so that $\|f_i\| = \|g_i\|$. Then $\mathcal{G} \approx \mathcal{F}$.*

Remark 3.15. It is instructive to note that claim (1) of the previous proposition follows from $\tilde{g}_j = e^{i\theta} \tilde{f}_j$.

The $\approx$ equivalence of frames holds for a much larger class of permutations:

Proposition 3.16. *Let π be a permutation (not necessarily finite) with the property that*

$$\lim_{n \to \infty} \frac{|I_n \cap \pi(I_n)|}{|I_n|} = 1.$$

If $\mathcal{G} = \{g_i = f_{\pi(i)} \colon i \in I\}$, then $\mathcal{G} \approx \mathcal{F}$.

Proof. Let $J_n = I_n \cap \pi(I_n)$, thus the sets $\{f_j \colon j \in J_n\}$ and $\{g_j \colon j \in J_n\}$ are identical. The result follows from the fact that:

$$\left| a_n(\mathcal{F}) - a_n(\mathcal{G}) \right| \leqslant \left| a_n(\mathcal{F}) - \frac{1}{|I_n|} \sum_{j \in J_n} \langle f_j, \tilde{f}_j \rangle \right| + \left| \frac{1}{|I_n|} \sum_{j \in J_n} \langle g_j, \tilde{g}_j \rangle - a_n(\mathcal{G}) \right|$$

$$\leqslant \frac{1}{|I_n|} \sum_{i \in I_n \setminus J_n} \langle f_j, \tilde{f}_j \rangle + \frac{1}{|I_n|} \sum_{j \colon \pi(j) \in I_n \setminus J_n} \langle f_j, \tilde{f}_j \rangle \leqslant 2 \frac{|I_n \setminus J_n|}{|I_n|} \leqslant 2 \left(1 - \frac{|J_n|}{|I_n|} \right),$$

the last inequality following from the fact that $\langle f_j, \tilde{f}_j \rangle, \langle g_j, \tilde{g}_j \rangle \leqslant 1$ (cf. (6)). $\quad \square$
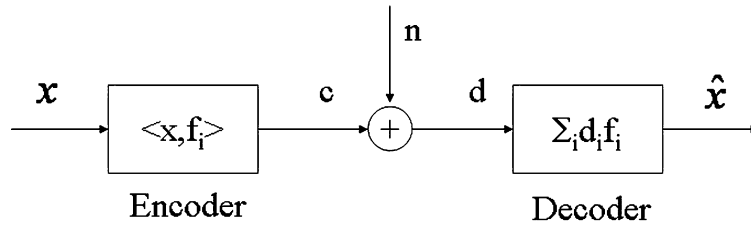
Fig. 1. The transmission encoding–decoding scheme used to suggest the importance of averages (14).

At the heart of the ultrafilter equivalence is the sequence $a(\mathcal{F})$ (or equivalently $b(\mathcal{F})$). Here we give an interpretation of $a(\mathcal{F})$ from a stochastic signal analysis perspective. This interpretation further justifies the ultrafilter equivalence $\approx$.

We shall consider a Parseval frame $\mathcal{F} \in \mathcal{F}[I]$. Since every frame is $\sim$ equivalent (and thus $\approx$ equivalent by Proposition 3.10), to its associated Parseval frame, the behavior of both equivalence relations is captured on the set of Parseval frames. Suppose the span H of $\mathcal{F}$ models a class of signals we are interested in transmitting using an encoding and decoding scheme based on $\mathcal{F}$ as in Fig. 1.

More specifically, a "signal," that is a vector $x \in H$, is "encoded" through the sequence of coefficients $\mathbf{c} = \{\langle x, f_i \rangle\}_{i \in I}$ given by the analysis operator $T : H \to l^2(I)$. These coefficients are sent through a communication channel to a receiver and there they are "decoded" using a linear reconstruction scheme $\hat{x} = \sum_{i \in I} d_i f_i$ furnished by the reconstruction operator T^*. It is common to consider what happens if the transmitted coefficients $\mathbf{c} = (c_i)_{i \in I}$ are perturbed by some (channel) noise. In this case, the received coefficients $\mathbf{d} = (d_i)_{i \in I}$ are not the same as the transmitted coefficients $\mathbf{c}$. We shall assume the system behaves as an *additive white noise channel* model, meaning the transmitted coefficients are perturbed additively by unit variance white noise. Thus we can write

$$d_i = c_i + n_i, \tag{9}$$

$$\mathbf{E}[n_i] = 0, \tag{10}$$

$$\mathbf{E}[n_i \overline{n_j}] = \delta_{i,j}, \tag{11}$$

where $\mathbf{E}$ is the expectation operator and n_i represents the independent noise component at the ith coefficient. The reconstructed signal $\hat{x}$ has two components, one due to the transmitted coefficients $\sum_i c_i f_i = x$ and the other due to the noise $\varepsilon = \sum_i n_i f_i$. We analyse the noise component. Since its variance is infinite in general, we consider the case that only finitely many coefficients are transmitted, say a finite subset $I_n \subset I$. Then the *average variance per coefficient of the noise-due-error* is defined by

$$a_n' = \frac{\mathbf{E}[|\varepsilon_n|^2]}{|I_n|}, \tag{12}$$

where

$$\varepsilon_n = \sum_{i \in I_n} n_i f_i. \tag{13}$$

Using the assumptions (10), (11) we obtain

$$a'_n = \frac{1}{|I_n|} \sum_{i \in I_n} \| f_i \|^2 \tag{14}$$

which is exactly the quantity $a_n(\mathcal{F})$ used to define the ultrafilter frame equivalence. Since $\| f_i \| \leqslant 1$ it follows $a'_n \leqslant 1$. For an orthonormal basis the average noise-due-error variance per coefficient would have been 1 for all n (since $\| f_i \|^2 = 1$ for all i). Hence $a'_n = a_n(\mathcal{F})$ gives a measure of how much the channel noise variance is reduced when a frame is used instead of an orthonormal basis. In channel encoding theory, the noise reduction phenomenon described earlier is attributed to the redundancy a frame has compared to an orthonormal basis (see for instance [17]). Hence, any measure of redundancy has to be connected to the averages $a'_n = a_n(\mathcal{F})$ from (14).

It follows that two frames that are ultrafilter frame equivalent have the same noise-due-error limiting behavior and if $\mathcal{F} \lhd \mathcal{G}$ then $\mathcal{F}$ has better noise-due-error limiting behavior. The ultrafilter frame measure function, which we introduce in Section 4.1, is defined using the limiting behavior of $a(\mathcal{F})$ to give an important quantitative measure of frames.

3.4. The frame sequence correspondence

The following theorem describes the correspondence between frames and frame sequences and shows that addition of frame sequences can be realized by the superset operation ($\oplus$) of certain frames.

Theorem 3.17 *(Frame–sequence correspondence).*

(1) *For every frame $\mathcal{F}$ there exists a normal-orthogonal frame $\mathcal{G}$ with $\lfloor b(\mathcal{F}) \rfloor = b(\mathcal{G})$ and thus $\mathcal{F} \approx \mathcal{G}$.*
(2) *Given frame compatible sequences $\mathbf{x}^1, \ldots, \mathbf{x}^k$, and $\mathbf{z} = \sum_{i=1}^k \mathbf{x}^i$, there exist frames $\mathcal{F}^{x^1}, \ldots, \mathcal{F}^{x^k}, \mathcal{F}^z$ such that*
    (a) *$\mathcal{F}^z = \bigoplus_{i=1}^k \mathcal{F}^{x^i}$,*
    (b) *$b(\mathcal{F}^{x^i}) \approx \mathbf{x}^i$ for all $1 \leqslant i \leqslant k$, and $b(\mathcal{F}^z) \approx \mathbf{z}$.*

Proof. (1) Given $\mathcal{F}$, the existence of $\mathcal{G}$ is given by Theorem 3.4. It remains to show that the sequences $b(\mathcal{F})$ and $\lfloor b(\mathcal{F}) \rfloor$ are $\approx$ compatible which follows from

$$\frac{b_n(\mathcal{F}) - \lfloor b(\mathcal{F}) \rfloor_n}{|I_n|} \leqslant \frac{1}{|I_n|}.$$

(2) We present the proof only for the case $k = 2$; the general case follows along same lines. We simplify the notation to $\mathbf{x}^1 = \mathbf{x}$ and $\mathbf{x}^2 = \mathbf{y}$. Let $\mathbf{b}$ be the sequence defined by: $\mathbf{b} = \lfloor \mathbf{z} \rfloor - \lfloor \mathbf{x} \rfloor - \lfloor \mathbf{y} \rfloor$; notice that $b_i \in \{0, 1\}$. Define $\tilde{x}_i = (\tilde{x}_i)_i$ recursively as follows:

$$\tilde{x}_1 = \lfloor x_1 \rfloor, \qquad \tilde{x}_i = \min\left(\tilde{x}_{i-1} + \lfloor z_i \rfloor - \lfloor z_{i-1} \rfloor, \lfloor x_i \rfloor\right).$$

Using the fact that $\mathbf{z}$ is frame compatible, it is straightforward to verify that $\tilde{\mathbf{x}}$ is frame compatible. By definition, $\tilde{x}_i \leqslant \lfloor x_i \rfloor$, we now show that $\tilde{x}_i \geqslant \lfloor x_i \rfloor - 1 + b_i$. Suppose this is not the case then let j be the smallest index for which $\tilde{x}_j < \lfloor x_j \rfloor - 1 + b_j$. Thus

$$\tilde{x}_j = \tilde{x}_j + \lfloor z_j \rfloor - \lfloor z_{j-1} \rfloor \tag{15}$$

$$= \tilde{x}_{j-1} + \lfloor x_j \rfloor - \lfloor x_{j-1} \rfloor + \lfloor y_j \rfloor - \lfloor y_{j-1} \rfloor + b_j - b_{j-1} \tag{16}$$

$$\geqslant \lfloor x_{j-1} \rfloor - 1 + b_{j-1} + \lfloor x_j \rfloor - \lfloor x_{j-1} \rfloor + \lfloor y_j \rfloor - \lfloor y_{j-1} \rfloor + b_j - b_{j-1} \tag{17}$$

$$= \lfloor x_j \rfloor - 1 + \lfloor y_j \rfloor - \lfloor y_{j-1} \rfloor + b_j \geqslant \lfloor x_j \rfloor - 1 + b_j \tag{18}$$

which contradicts the assumption on j. Thus $\lfloor x_i \rfloor - 1 \leqslant \tilde{x}_i \leqslant \lfloor x_i \rfloor$ for all i, and hence $\tilde{\mathbf{x}} \approx \lfloor \mathbf{x} \rfloor \approx \mathbf{x}$. Define $\tilde{\mathbf{y}} = \lfloor \mathbf{z} \rfloor - \tilde{\mathbf{x}}$. It is straightforward to verify from the definition of $\tilde{\mathbf{x}}$ that $\tilde{\mathbf{y}}$ is frame compatible and since $\tilde{\mathbf{x}} \approx \mathbf{x}$, we can conclude $\tilde{\mathbf{y}} \approx \mathbf{y}$.

By Theorem 3.4, since $\lfloor \mathbf{z} \rfloor$ is frame compatible, we can find a normal-orthogonal frame $\mathcal{F}^z = \{f_i^z\}_{i \in I}$ with $b(\mathcal{F}^z) = \lfloor \mathbf{z} \rfloor$. Define $T \subset I$ to be the subset of I for which $f_i^z \neq 0$, i.e. $T = \{i \in I: f_i^z \neq 0\}$. Write $T = T_1 \cup T_2$ such that T_1 and T_2 are disjoint and $|T_1 \cap I_i| = \tilde{x}_i$, $|T_2 \cap I_i| = \tilde{y}_i$; this can be done since $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$ and $\lfloor \mathbf{z} \rfloor = \tilde{\mathbf{x}} + \tilde{\mathbf{y}}$ are frame compatible. Define $\mathcal{F}^x = \{f_i^x\}_{i \in I}$, $\mathcal{F}^y = \{f_i^z\}_{i \in I}$ as follows: $f_i^x = f_i^z$ for $i \in T_1$, $f_i^x = 0$, otherwise, $f_i^y = f_i^z$ for $i \in T_2$, $f_i^y = 0$, otherwise. We have $\mathcal{F}^x \oplus \mathcal{F}^y = \mathcal{F}^z$ (since the elements of $\mathcal{F}^z$ are orthogonal), and by construction $b(\mathcal{F}^x) = \tilde{\mathbf{x}} \approx \mathbf{x}$, $b(\mathcal{F}^y) = \tilde{\mathbf{y}} \approx \mathbf{y}$ and $b(\mathcal{F}^z) = \lfloor \mathbf{z} \rfloor \approx \mathbf{z}$.

The proof of (3) follows along the same lines. $\quad\square$

4. A measure of frames

In this section we introduce our main tool for a quantitative comparison of frames: the ultrafilter frame measure function. We give its definition in Section 4.1, and then we examine its connection with the notion of excess in 4.2. Appendix B gives a brief description of ultrafilters. Here we shall denote by N^* the set of free ultrafilters and for $p \in N^*$ and $\mathbf{x} = (x_1, x_2, \ldots)$ a sequence, the limit of $\mathbf{x}$ along p shall be denoted by $p\text{-lim}\,\mathbf{x}$. Finally $\mathcal{C}^*(\mathbf{N}^*)$ shall denote the set of continuous functions on N^*.

4.1. The ultrafilter frame measure function

We shall now use ultrafilters to give a new measure for frames.

Definition 4.1. Fix $(I_n)_{n \geqslant 0}$ as in Section 2.1. The *ultrafilter frame measure function* will be the map

$$\mu : \mathcal{F}[I] \to \mathcal{C}^*(\mathbf{N}^*); \quad \mu(\mathcal{F})(p) = p\text{-lim}\,a(\mathcal{F}) = p\text{-lim}\,\frac{1}{|I_n|} \sum_{i \in I_n} \langle f_i, \tilde{f}_i \rangle, \quad \forall p \in \mathbf{N}^*. \tag{19}$$

Theorem 4.2. *The ultrafilter frame measure function has the following properties*:

(1) $\mu(\mathcal{F}_1) = \mu(\mathcal{F}_2)$ *if and only if* $\mathcal{F}_1 \approx \mathcal{F}_2$.
(2) $\mu(\mathcal{F}_1)(p) \leqslant \mu(\mathcal{F}_2)(p)$ *for all* $p \in \mathbf{N}^*$ *if and only if* $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$.
(3) *If* $\mathcal{F}$ *is a Riesz basis for its span then* $\mu(\mathcal{F}) = 1$.
(4) *If* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ *are such that* $(\mathcal{F}_1, \mathcal{F}_2)$ *are orthogonal in the sense of supersets then* $\mu(\mathcal{F}_1 \oplus \mathcal{F}_2) = \mu(\mathcal{F}_1) + \mu(\mathcal{F}_2)$. (*See Appendix* A *for definitions involving supersets*.)

Proof. (1) We first note that the following statements are equivalent:

(a) $\mu(\mathcal{F}_1) = \mu(\mathcal{F}_2)$,
(b) $p\text{-lim}(a(\mathcal{F}_1)) = p\text{-lim}(a(\mathcal{F}_2))$ for all free ultrafilters p,
(c) $p\text{-lim}(a(\mathcal{F}_1) - a(\mathcal{F}_2)) = 0$ for all free ultrafilters p,
(d) the sequence $a(\mathcal{F}_1) - a(\mathcal{F}_2)$ has a single accumulation point at 0,
(e) $\lim_{n\to\infty}(a_n(\mathcal{F}_1) - a_n(\mathcal{F}_2)) = 0$,
(f) $\mathcal{F}_1 \approx \mathcal{F}_2$.

Indeed: (a) $\Leftrightarrow$ (b) and (e) $\Leftrightarrow$ (f) follow from the definitions of μ and $\approx$, (b) $\Leftrightarrow$ (c) follows from statement (2) of Proposition B.3, (c) $\Leftrightarrow$ (d) is due to statement (3) of Proposition B.3, (d) $\Leftrightarrow$ (e) follows from the fact that $0 \leqslant a_n(\mathcal{F}_1), a_n(\mathcal{F}_2) \leqslant 1$.

(2) The proof is very similar to (1); we omit the details.

(3) If $\mathcal{F}$ is a Riesz basis for its span, $\langle f_i, \tilde{f}_i \rangle = 1$ for all $i \in I$. Thus $a_n(\mathcal{F}) = 1$ for all $n \in \mathbf{N}$ and so since $\lim_{n\to\infty} a_n(\mathcal{F}) = 1$, statement (3) of Proposition B.3 implies $\mu(\mathcal{F}) = 1$.

(4) Since $\mathcal{F}^1$ and $\mathcal{F}^2$ are orthogonal in the sense of supersets, the canonical dual frame of $\mathcal{F}^1 \oplus \mathcal{F}^2$ is $\{\tilde{f}_i^1 \oplus \tilde{f}_i^2\}_{i \in I}$, the direct sum of the canonical duals for $\mathcal{F}^1$ and $\mathcal{F}^2$. Since $\langle f_i^1 \oplus f_i^2, \tilde{f}_i^1 \oplus \tilde{f}_i^2 \rangle = \langle f_i^1, \tilde{f}_i^1 \rangle + \langle f_i^2, \tilde{f}_i^2 \rangle$, we have $a_n(\mathcal{F}^1 \oplus \mathcal{F}^2) = a_n(\mathcal{F}^1) + a_n(\mathcal{F}^2)$ and the result follows. $\square$

4.2. The ultrafilter frame measure function and the excess of frames

The ultrafilter frame measure function gives information about the excess of a frame—a notion defined in [8]. We begin by summarizing the relevant ideas and results of [8].

The *excess* of a frame $\mathcal{F} \in \mathcal{F}[I]$ with span H is the supremum over the cardinalities of all subsets $J \subset I$ so that $\{f_i : i \in I \setminus J\}$ is complete in H. Since we consider only countable sets I, the excess is either a finite number or ∞. This supremum is always achieved [8]. Furthermore, for finite excess, J can be always chosen so that $\{f_i : i \in I \setminus J\}$ is also frame for H. However this property no longer holds true in general for infinite excess. A characterization of when this remains true was also given in [8]:

Theorem 4.3. *(See [8].) Let $\mathcal{F} \in \mathcal{F}[I]$ be a frame for H and $\tilde{\mathcal{F}}$ its canonical dual. Then the following are equivalent*:

(a) *There is an infinite subset $J \subset I$ such that $\{f_i; i \in I \setminus J\}$ is frame for H.*
(b) *There is an infinite subset $J' \subset I$ and $a < 1$ so that $\langle f_i, \tilde{f}_i \rangle \leqslant a$ for all $i \in J'$.*

We now show that condition (b) is implied when the ultrafilter frame measure function is not identically 1.

Theorem 4.4. *Let $\mathcal{F} \in \mathcal{F}[I]$ be a frame for H. If the ultrafilter frame measure function $\mu(\mathcal{F})$ is not identically one, then there is an infinite subset $J \subset I$ so that $\{f_i; i \in I \setminus J\}$ is frame for H.*

Proof. Since $\mu(\mathcal{F})$ takes on values in the interval $[0, 1]$, the hypothesis assumes that there exists some ultrafilter p such that $\mu(\mathcal{F})(p) < 1$. Thus we can find an infinite set $J \in p$ and a constant $\epsilon > 0$ such that $a_j(\mathcal{F}) < 1 - 2\epsilon$ for all $j \in J$. $a_j(\mathcal{F})$ is an average of terms between 0 and 1 and thus it follows that at least $\frac{\epsilon}{1-\epsilon}|I_j|$ of the terms $\langle f_i, \tilde{f}_i \rangle$, $i \in I_j$, are smaller than or equal to $1 - \epsilon$. Since J is an infinite set, it follows that an infinite number of the terms

$\langle f_i, \tilde{f_i} \rangle$ are bounded above by $1 - \epsilon$. This establishes criterion (b) of 4.3 and our result then follows. $\square$

Subsequent papers [3,9] analyzed the excess problem for Gabor frames. In particular if the upper Beurling density is strictly larger than one then there always exists an infinite subset that can be removed and leave the remaining set frame. Furthermore, if the generating window belongs to the modulation space M^1 (see [21] for definition of modulation spaces) and the lower Beurling density is strictly larger than one, then one can find an infinite subset of positive uniform Beurling density that can be removed and leave the remaining set a frame for L^2. These results came as applications of the general theory developed in [2]. That paper analyzed the excess and overcompleteness for a larger class of frames, namely those called localized frames. It gave a completely new relation connecting the density of the index set to averages of the sequence $\{\langle f_i, \tilde{f_i} \rangle\}$. We shall return to this connection in Section 9 in the context of Gabor frames.

Here we state one result from [2] in our context. To simplify the notation, assume the index set I is embedded in $\mathbf{Z}^d$.

Definition 4.5. A frame $\mathcal{F} \in \mathcal{F}[I]$ is called l^1-*localized* (*with respect to its canonical dual frame*) if there is a sequence $r \in l^1(\mathbf{Z}^d)$ so that $|\langle f_i, \tilde{f_j} \rangle| \leqslant r(i - j)$.

For a subset $J \subset I \subset \mathbf{Z}^d$, we define its upper and lower *densities* as the following numbers:

$$D^+(J) = \lim_{n \to \infty} \sup_{c \in \mathbf{Z}^d} \frac{|J \cap B_n(c)|}{|B_n(c)|}, \qquad D^-(J) = \lim_{n \to \infty} \inf_{c \in \mathbf{Z}^d} \frac{|J \cap B_n(c)|}{|B_n(c)|}$$

where $B_n(c)$ denotes the ball of radius n centered at c in $\mathbf{Z}^d$. The set J is said to have *uniform density* D if $D^-(J) = D^+(J) = D$. Now we restate Theorem 8 from [2] using the ultrafilter frame measure function.

Theorem 4.6. *Assume $I \subset \mathbf{Z}^d$ for some integer d. Let $\mathcal{F} \in \mathcal{F}[I]$ be a l^1-localized frame for H. If $\mu(\mathcal{F}) < 1$ then there is an infinite subset $J \subset I$ of positive uniform density so that $\{f_i; \ i \in I \setminus J\}$ is frame for H.*

Moreover, if $\mu(\mathcal{F}) < \alpha < 1$ then for each $0 < \varepsilon < 1 - \alpha$ the set J can be chosen as a subset of $\{i \in I; \ \langle f_i, \tilde{f_i} \rangle \leqslant \alpha\}$ and the frame $\{f_i; \ i \in I \setminus J\}$ has a lower frame bound $A(1 - \varepsilon - \alpha)$, where A is a lower frame bound of $\mathcal{F}$.

5. Sequence and frame measure functions

The ultrafilter frame measure function provides a quantitative measure for all frames indexed by the same set I. In this section we introduce the general notion of a frame measure function: a quantitative measure of frames defined by some general properties (Proposition 5.7). We prove some general facts about frame measure functions (Section 5.1) and prove that the ultrafilter frame measure function has a lattice structure (Section 5.2). The natural way to view frame measure functions is as linear maps on the sequences $a(\mathcal{F})$ via the frame sequence correspondence (Theorem 3.17). For this reason we present the frame measure function via related maps on sequences—*sequence measure functions* (Definition 5.5). The technique of proving results about sequence measure functions and "pulling the results back" to frame measure functions will be used repeatedly through the rest of this work.

We begin by extending frame compatible sequences to a larger space of sequences.

Definition 5.1. For the set X of frame compatible sequences, we define:

$$X^+ = \{c\mathbf{x}: \mathbf{x} \in X, \ c \geqslant 0\}, \tag{20}$$

$$X^{\mathbf{R}} = \{\mathbf{x}^1 - \mathbf{x}^2: \mathbf{x}^j \in X^+ \text{ for } j = 1, 2\}. \tag{21}$$

Proposition 5.2.

(1) *The set X of frame compatible sequences is convex.*
(2) *If $0 \leqslant c \leqslant 1$ and $\mathbf{x} \in X$, then $c\mathbf{x} \in X$.*
(3) *X^+ is a positive cone, that is, for $c_1, c_2 \geqslant 0$, and $\mathbf{x}_1, \mathbf{x}_2 \in X^+$, we have $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \in X^+$.*
(4) *$X^{\mathbf{R}}$ is the real vector space spanned by X, that is for any $c_1, c_2 \in \mathbf{R}$, $\mathbf{x}_1, \mathbf{x}_2 \in X^{\mathbf{R}}$, we have $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \in X^{\mathbf{R}}$.*

Proof. Property (1) is a consequence of the fact that the constraints of the definition of frame compatibility (Definition 3.1) are convex. Property (2) follows from convexity of X, since both 0 and $\mathbf{x}$ belong to X. Property (3) follows from (1) and (2) Finally property (4) follows from definition of $X^{\mathbf{R}}$ and (3). □

Theorem 5.3. *Given a linear function m on the frame compatible sequences, there exists a unique linear extension $\tilde{m}$ of m to $X^{\mathbf{R}}$.*

Proof. Since m is linear on X, it is clear that defining $\tilde{m}(c\mathbf{x}) \equiv cm(\mathbf{x})$ for $\mathbf{x}$ frame compatible and $c \geqslant 0$ uniquely extends m to X^+. Linearity of m on X implies linearity of $\tilde{m}$ on X^+ as follows: for $\mathbf{x}, \mathbf{y} \in X$, $c, d > 0$ we have

$$\tilde{m}(c\mathbf{x} + d\mathbf{y}) = (c+d)m\left(\frac{c}{c+d}\mathbf{x} + \frac{d}{c+d}\mathbf{y}\right) = (c+d)\left(\frac{c}{c+d}m(\mathbf{x}) + \frac{d}{c+d}m(\mathbf{y})\right)$$
$$= cm(\mathbf{x}) + dm(\mathbf{y}),$$

since $\frac{c}{c+d}\mathbf{x}$, $\frac{d}{c+d}\mathbf{y}$, $\frac{c}{c+d}\mathbf{x} + \frac{d}{c+d}\mathbf{y} \in X$. If a linear extension to $X^{\mathbf{R}}$ existed, it would have to be unique since $\mathbf{x} \in X^{\mathbf{R}}$ implies $\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^2$ for some $\mathbf{x}^j \in X^+$, $1 \leqslant j \leqslant 2$. Hence by linearity we need to have

$$\tilde{m}(\mathbf{x}) = \tilde{m}(\mathbf{x}^1) - \tilde{m}(\mathbf{x}^2). \tag{22}$$

It remains to show that (22) is well defined. Suppose $\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^2 = \mathbf{y}^1 - \mathbf{y}^2$ for $\mathbf{x}^j, \mathbf{y}^j \in X^+$, $1 \leqslant j \leqslant 2$. Then $\mathbf{x}^1 + \mathbf{y}^2 = \mathbf{y}^1 + \mathbf{x}^2$. By the linearity of $\tilde{m}$ on X^+ we have $\tilde{m}(\mathbf{x}^1) + \tilde{m}(\mathbf{y}^2) = \tilde{m}(\mathbf{y}^1) + \tilde{m}(\mathbf{x}^2)$. Rearranging terms yields

$$\tilde{m}(\mathbf{x}^1) - \tilde{m}(\mathbf{x}^2) = \tilde{m}(\mathbf{y}^1) - \tilde{m}(\mathbf{y}^2),$$

and thus (22) is well defined. □

Definition 5.4. Let W be a compact Hausdorff space; denote by $C^*(W)$ the set of real-valued continuous functions over W.

We now define the notions of a sequence and frame measure function.

Definition 5.5. A sequence measure function $m : X^{\mathbf{R}} \to C^*(W)$ will be a function which satisfies:

(1) For $x, y \in X^{\mathbf{R}}$, $m(\mathbf{x}) = m(\mathbf{y})$ if and only if $\mathbf{x} \approx \mathbf{y}$,
(2) For $x, y \in X^+$, $m(\mathbf{x}) \leqslant m(\mathbf{y})$ if and only if $\mathbf{x} \lhd \mathbf{y}$,
(3) For $\mathbf{i} = (|I_1|, |I_2|, |I_3|, \ldots)$, $m(\mathbf{i}) = 1$,
(4) m is linear.

Definition 5.6. A *frame measure function* will be a function $m_f : \mathcal{F}[I] \to C^*(W)$ which is the composition of the map $b : \mathcal{F}[I] \to X$ and a sequence measure function m, i.e. $m_f(\mathcal{F}) = m(b(\mathcal{F}))$, for all $\mathcal{F} \in \mathcal{F}[I]$.

The ultrafilter frame measure function is a frame measure function as we prove in Corollary 5.10.

An equivalent description of a frame measure function is contained in the following proposition.

Proposition 5.7. *A map $m_f : \mathcal{F}[I] \to C^*(W)$ is a frame measure function if and only if it satisfies the following properties*:

(A) $m_f(\mathcal{F}_1) = m_f(\mathcal{F}_2)$ *if and only if $\mathcal{F}_1 \approx \mathcal{F}_2$.*
(B) $m_f(\mathcal{F}_1)(x) \leqslant m_f(\mathcal{F}_2)(x)$ *for all $x \in M$ if and only if $\mathcal{F}_1 \lhd \mathcal{F}_2$.*
(C) *If $\mathcal{F}$ is a Riesz basis for its span then $m_f(\mathcal{F}) = 1$.*
(D) *If $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ are such that $(\mathcal{F}_1, \mathcal{F}_2)$ are orthogonal in the sense of supersets then $m_f(\mathcal{F}_1 \oplus \mathcal{F}_2) = m_f(\mathcal{F}_1) + m(\mathcal{F}_2)$.*

Proof. Given a frame measure function $m_f = m \circ b$, properties (A) and (B) follow immediately from properties (1) and (2) of Definition 5.5. Given a Riesz basis for its span $\mathcal{F}$, we have $\langle f_i, \tilde{f}_i \rangle = 1$ for all $i \in I$ and hence $b(\mathcal{F}) = \{|I_1|, |I_2|, \ldots\}$. Thus property (C) above follows from property (3) of Definition 5.5. Finally, if $\mathcal{F}_1, \mathcal{F}_2$ are orthogonal in the sense of supersets we have $b(\mathcal{F}_1 \oplus \mathcal{F}_2) = b(\mathcal{F}_1) + b(\mathcal{F}_2)$ and the linearity (property (4)) of m implies property (D) above.

We are left to show that a map m_f satisfying the above 4 properties implies that the existence of a sequence measure function m with $m_f = m \circ b$. We first define m on the frame compatible sequences from m_f as follows. Given $\mathbf{x} \in X$, by Theorem 3.4 there is a frame $\mathcal{G}^{\mathbf{x}}$ with $b(\mathcal{G}^{\mathbf{x}}) = \lfloor \mathbf{x} \rfloor$, we define $m(\mathbf{x}) = m_f(\mathcal{G}^{\mathbf{x}})$. Now for any frame $\mathcal{F}$, if we let $\mathbf{x} = b(\mathcal{F})$, we have $\mathcal{F} \approx \mathcal{G}^{\mathbf{x}}$ since $b(\mathcal{F}) \approx \lfloor \mathbf{x} \rfloor = b(\mathcal{G}^{\mathbf{x}})$. Thus by condition (A), $m_f(\mathcal{F}) = m_f(\mathcal{G}^{\mathbf{x}}) = m(\mathbf{x}) = m(b(\mathcal{F}))$ and thus $m_f = m \circ b$.

We now show this map m is linear on the set of frame compatible sequences, i.e.

(1) if $\mathbf{x}$ and $c\mathbf{x}$ are frame compatible then $cm(\mathbf{x}) = m(c\mathbf{x})$,
(2) if $\mathbf{x}, \mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ are frame compatible then $m(\mathbf{x}) + m(\mathbf{y}) = m(\mathbf{x} + \mathbf{y})$.

For any $\frac{a}{b} < c$, $a, b \in \mathbf{N}$, set $\mathbf{y} = \frac{1}{b}\mathbf{x} \in X$. Applying part (3) of Theorem 3.17 to the case $k = b$, $\mathbf{x}^i = \mathbf{y}$, $1 \leqslant i \leqslant k$, yields $bm(\mathbf{y}) = m(\mathbf{x})$. Similarly $am(\mathbf{y}) = m(\frac{a}{b}\mathbf{x})$; combining these conditions yields $\frac{a}{b}m(\mathbf{x}) = m(\frac{a}{b}\mathbf{x})$. Since $\frac{a}{b}\mathbf{x} \lhd c\mathbf{x}$, properties (A) and (B) imply $m(\frac{a}{b}\mathbf{x}) \leqslant m(c\mathbf{x})$. Coupling this with the above two relations yields $\frac{a}{b}m(\mathbf{x}) \leqslant m(c\mathbf{x})$. Applying this to a sequence of rational $\frac{a}{b}$ that approach c from below yield $cm(\mathbf{x}) \leqslant m(c\mathbf{x})$. A similar argument can be made for any rational fraction greater than or equal to c and we conclude $cm(\mathbf{x}) \leqslant m(c\mathbf{x}) \leqslant cm(\mathbf{x})$ and thus $cm(\mathbf{x}) = m(c\mathbf{x})$.

Statement (2) above follows directly from property (D) and part (2) of Theorem 3.17.

Thus m is linear on the set of frame compatible sequences and by Theorem 5.3 we can uniquely extend m to a linear map on $X^{\mathbf{R}}$; we will call this extended map m as well. It remains to show that m satisfies properties (1)–(3) of Definition 5.5. Property (3) follows from the fact that for an orthonormal basis $\mathcal{F}$, $m_f(\mathcal{F}) = 1$ and $b(\mathcal{F}) = \{|I_1|, |I_2|, |I_3|, \ldots\}$. We now establish property (1) of Definition 5.5. Given $\mathbf{x}, \mathbf{y} \in X^{\mathbf{R}}$, write $\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^2$, $\mathbf{y} = \mathbf{y}^1 - \mathbf{y}^2$, with $\mathbf{x}^j, \mathbf{y}^j \in X^+$ and $\frac{1}{c}\mathbf{x}^j, \frac{1}{c}\mathbf{y}^j$ frame compatible sequences. It is straightforward to verify that

$$
\begin{aligned}
m(\mathbf{x}) = m(\mathbf{y}) \quad &\Leftrightarrow \quad m(\mathbf{x}^1 + \mathbf{y}^2) = m(\mathbf{y}^1 + \mathbf{x}^2) \\
&\Leftrightarrow \quad m\left(\frac{1}{2c}(\mathbf{x}^1 + \mathbf{y}^2)\right) = m\left(\frac{1}{2c}(\mathbf{x}^2 + \mathbf{y}^1)\right) \\
&\Leftrightarrow \quad \frac{1}{2c}(\mathbf{x}^1 + \mathbf{y}^2) \approx \frac{1}{2c}(\mathbf{x}^2 + \mathbf{y}^1) \\
&\Leftrightarrow \quad \mathbf{x}^1 + \mathbf{y}^2 \approx \mathbf{x}^2 + \mathbf{y}^1 \\
&\Leftrightarrow \quad \mathbf{x} \approx \mathbf{y}
\end{aligned}
$$

where the third double implication comes from property (1) of a frame measure function and all other implications follow from the linearity of m. Finally we show property (2) of Definition 5.5. Given $\mathbf{x}, \mathbf{y} \in X^+$, there exists a constant c such that $\frac{1}{c}\mathbf{x}$, $\frac{1}{c}\mathbf{y}$ are both frame compatible. It is then straightforward that

$$
m(\mathbf{x}) \leqslant m(\mathbf{y}) \quad \Leftrightarrow \quad m\left(\frac{1}{c}\mathbf{x}\right) \leqslant m\left(\frac{1}{c}\mathbf{y}\right) \quad \Leftrightarrow \quad \left(\frac{1}{c}\mathbf{x}\right) \lhd \left(\frac{1}{c}\mathbf{y}\right) \quad \Leftrightarrow \quad \mathbf{x} \lhd \mathbf{y}. \qquad \square
$$

Remark 5.8. Condition (D) in Proposition 5.7 can be viewed as a linearity condition on supersets of certain pairs of frames. One might hope for more, namely that one could find a map with conditions A, B and C with the added property that the map was linear on supersets of all pairs of frames. This turns out to be too much to hope for as the following example shows:

Example 5.9. Let H be a Hilbert space with orthonormal basis $\{e_i\}_{i \in \mathbf{N}}$, let $I_n = \{1, \ldots, n\}$. Define $\mathcal{F} = \{f_i\}_{i \in \mathbf{N}}$ and $\mathcal{G} = \{g_i\}_{i \in \mathbf{N}}$ as follows:

$$
f_i = \begin{cases} e_i, & i \text{ even,} \\ 0, & i \text{ odd,} \end{cases}
$$

$$
g_i = \begin{cases} \frac{1}{2}e_i + \frac{1}{2}e_{i^2+1}, & i \text{ even,} \\ \frac{1}{2}e_{\sqrt{i-1}} + \frac{1}{2}e_i, & \sqrt{i-1} \text{ even,} \\ 0, & \text{otherwise.} \end{cases}
$$

Let $H_1 = \overline{\text{span}\{\mathcal{F}\}}$, $H_2 = \overline{\text{span}\{\mathcal{G}\}}$. The following facts about $\mathcal{F}$ and $\mathcal{G}$ can be verified:

- $\overline{\text{span}\{\{f_i \oplus g_i\}\}_{i \in \mathbf{N}}} = H_1 \oplus H_2$;
- $\mathcal{F} \oplus \mathcal{G} = \{f_i \oplus g_i\}_{i \in \mathbf{N}}$ is a frame for $H_1 \oplus H_2$ (this is verified using Theorem A.2 or checking that $\tilde{h}_i$ below are the dual frame elements);
- the canonical dual frame $\{\tilde{h}_i\}$ is given by

$$\tilde{h}_i = \begin{cases} e_i \oplus 0, & i \text{ even,} \\ -e_{\sqrt{i-1}} \oplus e_{\sqrt{i-1}} + e_i, & \sqrt{i-1} \text{ even,} \\ 0, & \text{otherwise;} \end{cases}$$

- $\mu(\mathcal{F}) = \frac{1}{2}$, $\mu(\mathcal{G}) = \frac{1}{4}$, $\mu(\mathcal{F} \oplus \mathcal{G}) = \frac{1}{2}$.
  Thus μ is not additive in the sense of supersets in this case.
  Though this shows no map of the above form can be linear on supersets of all pairs of frames, a main result of Section 7 shows that for index sets I with a little added structure, frame measure functions are linear on supersets of pairs of frames coming from a large subset of all frames that includes Gabor frames.

It is straightforward to verify using Proposition 5.7 that:

Corollary 5.10. *The ultrafilter frame measure function is a frame measure function.*

We define the corresponding sequence measure function:

Definition 5.11. The *ultrafilter sequence measure function* shall be the sequence measure function corresponding to the ultrafilter frame measure function, i.e. the map

$$\mu: X^{\mathbf{R}} \to C^*(\mathbf{N}^*) \quad \text{given by } \mu(\mathbf{x})(p) = p\text{-}\lim \frac{x_n}{|I_n|}.$$

We will use the same μ to denote both the ultrafilter sequence measure function and the ultrafilter frame measure function.

5.1. General properties of sequence and frame measure functions.

Proposition 5.12. *Suppose $\mathbf{x} \in X^{\mathbf{R}}$ and m is a sequence measure function, then*

(1) *If $c = \lim_{i \to \infty} \frac{x_i}{|I_i|}$ exists then $m(\mathbf{x})$ is the constant function of value c.*

(2) $\liminf \frac{x_i}{|I_i|} \leqslant m(\mathbf{x})(w) \leqslant \limsup \frac{x_i}{|I_i|}$ *for all $w \in W$.*

(3) *There exist $v, w \in W$ (different for different $\mathbf{x}$) such that $m(\mathbf{x})(v) = \liminf \frac{x_i}{|I_i|}$, $m(\mathbf{x})(w) = \limsup \frac{x_i}{|I_i|}$.*

Proof. (1) Recall $\mathbf{i} = (|I_i|)_i$. Set $\mathbf{y} = c\mathbf{i}$. It follows from Definition 3.5 that $\mathbf{x} \approx \mathbf{y}$ and so $m(\mathbf{x}) = m(\mathbf{y}) = cm(\mathbf{i}) = c \cdot 1$, the last two equalities following from the linearity of m and condition (3) of Definition 5.5.

(2) and (3). Let l be the greatest number for which $l\mathbf{i} \lhd \mathbf{x}$ and let L be the smallest number for which $\mathbf{x} \lhd L\mathbf{i}$. From the definition of $\lhd$, it follows that $l = \liminf \frac{\mathbf{x}}{\mathbf{i}}$ and $L = \limsup \frac{\mathbf{x}}{\mathbf{i}}$;

result (2) then follows since $m(c\mathbf{i})(w) = c$ for all $w \in W$. Furthermore, property (2) of the definition of a sequence measure function (Definition 5.5) ensures that $l = \liminf_{w \in W} m(\mathbf{x})(w)$ and $L = \limsup_{w \in W} m(\mathbf{x})(w)$. The continuity of m and the compactness of W ensures that there exist points $v, w \in W$ for which $m(\mathbf{x})(w)$ achieves the lower and upper bounds, i.e. $m(x)(v) = l$, $m(x)(w) = L$. $\quad\square$

Proposition 5.12 "pulls back" via the map b and the frame sequence correspondence (Theorem 3.17) to the following statement about frame measure functions.

Proposition 5.13. *Suppose $\mathcal{F} \in \mathcal{F}[I]$ and m is a frame measure function, then*:

(1) *If $c = \lim_{i \to \infty} a_i(\mathcal{F})$ exists then $m(\mathcal{F})$ is the constant function of value c.*
(2) $\liminf a(\mathcal{F}) \leqslant m(\mathcal{F})(w) \leqslant \limsup a(\mathcal{F})$ *for all $w \in W$.*
(3) *There exist $v, w \in W$ (different for different $\mathcal{F}$) such that $m(\mathcal{F})(v) = \liminf a(\mathcal{F})$, $m(\mathcal{F})(w) = \limsup a(\mathcal{F})$.*

5.2. Sequences and lattices

Proposition 5.14. *A real-valued sequence $\mathbf{x}$ is in $X^{\mathbf{R}}$ if and only if there exists a constant c such that $|x_1| \leqslant c|I_1|$ and $|x_i - x_{i-1}| \leqslant c(|I_i| - |I_{i-1}|)$ for all $i \geqslant 2$.*

Proof. If $\mathbf{x} \in X^{\mathbf{R}}$ then $\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^2$ with $\mathbf{x}^1, \mathbf{x}^2 \in X^+$. Thus there exists a constant c such that $\frac{2}{c}\mathbf{x}^1$, $\frac{2}{c}\mathbf{x}^2 \in X$ and therefore, $|x_1^k| \leqslant \frac{c}{2}|I_1|$ and $|x_i^k - x_{i-1}^k| \leqslant \frac{c}{2}(|I_i| - |I_{i-1}|)$ for $k = 1, 2$. It follows that $|x_1| \leqslant c|I_1|$ and $|x_i - x_{i-1}| \leqslant c(|I_i| - |I_{i-1}|)$.

Given a sequence $\mathbf{x}$ such that there is a constant c for which $|x_1| \leqslant c|I_1|$, $|x_i - x_{i-1}| \leqslant c(|I_i| - |I_{i-1}|)$, $i \geqslant 2$, set $d_1 = x_1$, $d_i = x_i - x_{i-1}$ for $i \geqslant 2$. Inductively define $\mathbf{x}^1, \mathbf{x}^2$ as follows: $x_1^1 = \max(d_1, 0)$, $x_1^2 = \max(-d_1, 0)$, $x_i^1 = x_{i-1}^1 + \max(d_i, 0)$, $x_i^2 = x_i^2 + \max(-d_i, 0)$. By construction $\mathbf{x}^1 - \mathbf{x}^2 = \mathbf{x}$. In addition, $x_i^k - x_{i-1}^k \in \{0, |d_i|\}$ and thus $\mathbf{x}^1, \mathbf{x}^2 \in X^+$. $\quad\square$

Definition 5.15. For real-valued sequences $\mathbf{x}, \mathbf{y}$, define the sequences $\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \vee \mathbf{y}$ as follows:

$$(\mathbf{x} \wedge \mathbf{y})_i = \min(x_i, y_i), \qquad (\mathbf{x} \vee \mathbf{y})_i = \max(x_i, y_i) \quad \text{for all } i \geqslant 1.$$

Remark 5.16. It follows from the definitions that $(\mathbf{x} \wedge \mathbf{y}) \vartriangleleft \mathbf{x} \vartriangleleft (\mathbf{x} \vee \mathbf{y})$ and $(\mathbf{x} \wedge \mathbf{y}) \vartriangleleft \mathbf{y} \vartriangleleft (\mathbf{x} \vee \mathbf{y})$.

Proposition 5.17. *The sets X, X^+, $X^{\mathbf{R}}$ are all closed under the binary operations $\wedge$ and $\vee$. Consequently each set forms a lattice.*

Proof. Given $\mathbf{x}, \mathbf{y} \in X$ and $i \geqslant 1$, without loss of generality we can assume $(\mathbf{x} \wedge \mathbf{y})_{i-1} = x_{i-1}$. So

$$(\mathbf{x} \wedge \mathbf{y})_i - (\mathbf{x} \wedge \mathbf{y})_{i-1} = \min(x_i, y_i) - x_{i-1} \leqslant x_i - x_{i-1} \leqslant |I_i| - |I_{i-1}|,$$

and so $\mathbf{x} \wedge \mathbf{y} \in X$. The result for X^+ follows from the result for X by noting that $c(\mathbf{x} \wedge \mathbf{y}) = c\mathbf{x} \wedge c\mathbf{y}$.

For $\mathbf{x}, \mathbf{y} \in X^{\mathbf{R}}$ let c be as in Proposition 5.14 so that $|x_1|, |y_1| \leqslant c|I_1|$ and $|x_i - x_{i-1}|, |y_i - y_{i-1}| \leqslant c(|I_i| - |I_{i-1}|)$ for $i \geqslant 2$. We now consider the two cases (a) $(\mathbf{x} \wedge \mathbf{y})_i \geqslant (\mathbf{x} \wedge \mathbf{y})_{i-1}$, (b) $(\mathbf{x} \wedge \mathbf{y})_i < (\mathbf{x} \wedge \mathbf{y})_{i-1}$.

In case (a) we can assume $(\mathbf{x} \wedge \mathbf{y})_{i-1} = x_{i-1}$ and again

$$0 \leqslant (\mathbf{x} \wedge \mathbf{y})_i - (\mathbf{x} \wedge \mathbf{y})_{i-1} = \min(x_i, y_i) - x_{i-1} \leqslant x_i - x_{i-1} \leqslant c(|I_i| - |I_{i-1}|).$$

In case (b) we can assume $(\mathbf{x} \wedge \mathbf{y})_i = x_i$ and thus

$$0 \geqslant (\mathbf{x} \wedge \mathbf{y})_i - (\mathbf{x} \wedge \mathbf{y})_{i-1} = x_i - \min(x_{i-1}, y_{i-1}) \geqslant x_i - x_{i-1} \geqslant -c(|I_i| - |I_{i-1}|).$$

These two cases establish that $\mathbf{x} \wedge \mathbf{y}$ satisfy the conditions of Proposition 5.14 and thus $\mathbf{x} \wedge \mathbf{y} \in X^{\mathbf{R}}$.

The corresponding result for $\mathbf{x} \vee \mathbf{y}$ can be proven in a similar fashion. □

Proposition 5.18. *The ultrafilter sequence measure function has the properties*:

(1) $\mu(\mathbf{x} \wedge \mathbf{y})(p) = \min(\mu(x)(p), \mu(y)(p))$.
(2) $\mu(\mathbf{x} \vee \mathbf{y})(p) = \max(\mu(x)(p), \mu(y)(p))$.

The lattice structure on sequences induces a lattice structure on frames:

Definition 5.19. Given two frame $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F} \vee \mathcal{G}$ will denote any frame that has the property that $b(\mathcal{F} \vee \mathcal{G}) \approx b(\mathcal{F}) \vee b(\mathcal{G})$. Similarly, denote by $\mathcal{F} \wedge \mathcal{G}$ any frame that has the property that $b(\mathcal{F} \wedge \mathcal{G}) \approx b(\mathcal{F}) \wedge b(\mathcal{G})$.

Remark 5.20. Theorem 3.17 and Proposition 5.17 guarantee the existence of the frames $\mathcal{F} \wedge \mathcal{G}$ and $\mathcal{F} \vee \mathcal{G}$.

With this notation Proposition 5.18 implies:

Proposition 5.21. *The ultrafilter frame measure function has the properties*:

(1) $\mu(\mathcal{F} \wedge \mathcal{G})(p) = \min(\mu(\mathcal{F})(p), \mu(\mathcal{G})(p))$.
(2) $\mu(\mathcal{F} \vee \mathcal{G})(p) = \max(\mu(\mathcal{F})(p), \mu(\mathcal{G})(p))$.

5.3. Universality of the ultrafilter sequence and frame measure function

We now show that a copy of the ultrafilter sequence measure function is embedded in any sequence measure function and consequently a copy of the ultrafilter frame measure function is embedded in any frame measure function.

Theorem 5.22. *Given a sequence measure function m, and an ultrafilter p, there exists an element $w_p \in W$ such that $\mu(\mathbf{x})(p) = m(\mathbf{x})(w_p)$ for all $\mathbf{x} \in X^{\mathbf{R}}$.*

Proof. Given an ultrafilter p, denote by Y_p all sequences for which the ultrafilter limit along p is the $\limsup$ of the sequence, i.e.

$$Y_p = \left\{ \mathbf{y} \in X^{\mathbf{R}} \colon \mu(\mathbf{y})(p) = \limsup \frac{\mathbf{y}}{\mathbf{i}} \right\}.$$

Set $W_p = \{ w \in W \colon m(\mathbf{y})(w) = \limsup \frac{\mathbf{y}}{\mathbf{i}}$ for all $\mathbf{y} \in Y_p \}$. We will eventually show that every point $w \in W_p$ satisfies $m(\mathbf{x})(w) = \mu(\mathbf{x})(p)$ for all $\mathbf{x} \in X^{\mathbf{R}}$. We begin by showing that W_p is nonempty.

Lemma 5.23. *For all free ultrafilters p, W_p is nonempty.*

Proof. Suppose $W_p = \emptyset$ for some p. Thus for every point $w \in W$ there exists a sequence $\mathbf{y}^w \in Y_p$ such that $m(\mathbf{y}^w)(w) < \mu(\mathbf{y}^w)(p) = \limsup \frac{\mathbf{y}^w}{\mathbf{i}}$. Since m is continuous we can find an open set V_w around w such that $m(\mathbf{y}^w)(v) \leqslant c_w < \mu(\mathbf{y}^w)(p)$ for all $v \in V_w$. Thus $\bigcup_{w \in W} V_w$ is an open cover of W. Since W is compact we can find $w_1, \ldots, w_n$ such that $\bigcup_{i=1}^{n} V_{w_i} = W$ and therefore for all $w \in W$ there exists an $i(w) \in \{1, 2, \ldots, n\}$ such that $m(\mathbf{y}^{w_{i(w)}})(w) \leqslant c_{w_{i(w)}} < \mu(\mathbf{y}^{w_{i(w)}})(p)$. Setting $\mathbf{z} = \sum_{i=1}^{n} \frac{1}{n} \mathbf{y}^{w_i}$ we have

$$m(\mathbf{z})(w) = \sum_{i=1}^{n} \frac{1}{n} m(\mathbf{y}^{w_i})(w) \leqslant \sum_{i \neq i(w)} \frac{1}{n} m(\mathbf{y}^{w_i})(w) + \frac{1}{n} c_{w_{i(w)}} < \frac{1}{n} \sum_{i=1}^{n} \mu(\mathbf{y}^{w_i})(p) = \mu(\mathbf{z})(p)$$

for all w. This however contradicts Proposition 5.12 since it shows that $m(\mathbf{z})$ cannot achieve $\limsup \frac{z_n}{|I_n|}$ since it is strictly less than $\mu(\mathbf{z})(p)$. $\quad\square$

The lemma established that W_p is nonempty; we now show that each $w \in W_p$ has the property that $m(\mathbf{x})(w) = \mu(\mathbf{x})(p)$ for all $\mathbf{x} \in X^{\mathbf{R}}$. Suppose this is not the case, i.e. there is an $\mathbf{x}$ such that $m(\mathbf{x})(w) \neq \mu(\mathbf{x})(p)$. Assume first that $m(\mathbf{x})(w) < r < \mu(\mathbf{x})(p)$. Set $\mathbf{y} = \mathbf{x} \wedge r\mathbf{i}$ (see Definition 5.15). Remark 5.16 then implies that $m(\mathbf{y})(w) \leqslant m(\mathbf{x})(w) < r$. In addition $\mu(\mathbf{y})(p) = r$ by Proposition 5.21. However, since

$$r = p\text{-}\lim \frac{\mathbf{y}}{\mathbf{i}} \leqslant \limsup \left(\frac{\mathbf{y}}{\mathbf{i}} \right) = \limsup \left(\min \left(\frac{x_n}{|I_n|}, r \right) \right) \leqslant r$$

we have $\mathbf{y} \in Y_p$ and thus by the definition of W_p we must have $m(\mathbf{y})(w) = r$, a contradiction.

The case $m(\mathbf{x})(w) > \mu(\mathbf{x})(p)$ reduces to the previous case by noting that for $\mathbf{x}' = \mathbf{i} - \mathbf{x}$ we have $m(\mathbf{x}')(w) = 1 - m(\mathbf{x})(w) < 1 - \mu(\mathbf{x})(p) = \mu(\mathbf{x}')(p)$. $\quad\square$

The following corollary follows from the frame–sequence correspondence (Theorem 3.17).

Corollary 5.24. *Given a frame measure function m, and an ultrafilter p, there exists an element $w_p \in W$ such that $\mu(\mathcal{F})(p) = m(\mathcal{F})(w_p)$ for all $\mathcal{F} \in \mathcal{F}[I]$.*

6. Topological results

We now examine sequence and frame measure functions from a topological point of view. Corollary 5.24 says that a copy of the ultrafilter frame measure function μ can be found inside any frame measure function. However, this is only an algebraic copy and nothing has been shown about the topological compatibilities between the two measure functions. We partially address these issues in this section. In 6.1 we introduce some natural additional properties (separable, irreducible, minimal) that a sequence or frame measure function could have and we define a canonical minimal measure function μ^0 related to μ. We also give a canonical construction for turning an arbitrary sequence or frame measure function into a separable one. In 6.2 we prove two important results:

- Corollary 6.18 which says that μ^0 is the unique (up to a homeomorphism) minimal measure function.
- Corollary 6.15 which gives a partial characterization of which continuous functions are realized as $\mu(\mathcal{F})$ for some $\mathcal{F} \in \mathcal{F}[I]$.

As has often been the case, the technique for proving these results is to prove the corresponding result for sequences and sequence measure functions and then apply the frame–sequence correspondence.

6.1. Separable, irreducible and minimal sequence and frame measure functions

We begin by defining some natural classes of sequence and frame measure functions.

Definition 6.1. A sequence measure function $m : X^{\mathbf{R}} \to \mathcal{C}^*(W)$ is

- *separable* if for every $v, w \in W$, $v \neq w$ there is $\mathbf{x} \in X^{\mathbf{R}}$ such that $m(\mathbf{x})(v) \neq m(\mathbf{x})(w)$,
- *reducible* if there is a compact $V \subsetneq W$ such that $m' : X^{\mathbf{R}} \to \mathcal{C}^*(V)$ is a sequence measure function, where $m'(\mathbf{x}) = m(\mathbf{x})|_V$,
- *irreducible* if it is not reducible,
- *minimal* if it is separable and irreducible.

Definition 6.2. A frame measure function $m_f = m \circ b$ is (*separable, reducible, irreducible, minimal*) if the corresponding sequence measure function m is (separable, reducible, irreducible, minimal).

The ultrafilter sequence and frame measure functions are not always *separable* as the following example shows.

Example 6.3. Suppose $I = \mathbf{N}$ and $I_n = \{1, 2, \ldots, n\}$; therefore $|I_n| = n$. Consider $p_1 \in \mathbf{N}^*$ a free ultrafilter on $\mathbf{N}$, and define

$$p_2 = \left\{ s + 1, \ (s + 1) \cup \{0\} \colon s \in p_1 \right\}$$

where $s + 1 = \{n + 1 \colon n \in s\}$.

Notice $p_1 \neq p_2$ since, for instance, $\{2k: k \in \mathbf{N}\}$ and $\{2k + 1: k \in \mathbf{N}\}$ would both be in p_1 which is impossible since their intersection is empty.

For any $\mathbf{x} \in X$ we have $x_n - x_{n-1} \leqslant |I_n| - |I_{n-1}| = 1$ and $\frac{x_n}{|I_n|} = \frac{x_n}{n} \leqslant 1$. Suppose p_1-lim $\frac{\mathbf{x}}{\mathbf{i}} = a$ thus for all $\epsilon > 0$, there is a set $s \in p_1$ for which $|\frac{x_n}{n} - a| < \epsilon$ for all $n \in s$. Let N be such that $\frac{1}{N} < \epsilon$. Note that $s' = \{n \in s: n \geqslant N\} \in p_1$ and set $t = s' + 1 \in p_2$. For $n \in t$,

$$\left| \frac{x_n}{n} - a \right| = \left| \frac{x_n}{n} - \frac{x_{n-1}}{n-1} + \frac{x_{n-1}}{n-1} - a \right| = \left| \frac{x_n - x_{n-1}}{n} - \frac{x_{n-1}}{n(n-1)} + \left(\frac{x_{n-1}}{n-1} - a \right) \right|$$

$$\leqslant \frac{1}{n} + \frac{1}{n} + \epsilon < 3\epsilon.$$

Thus p_2-lim $\frac{\mathbf{x}}{\mathbf{i}} = a$ as well, and so $\mu(X)(p_2) = \mu(X)(p_1)$. Therefore the set of continuous functions $\mu(X)$ in $C^*(W)$ does not separate p_1 from p_2 and thus μ is an example of a non-separable sequence measure function.

We would like to use μ to construct a separable measure function. Thus we are interested in grouping together all points in $\mathbf{N}^*$ that produce the same values for all sequences. To this end we introduce the following equivalence relation on $\mathbf{N}^*$:

Definition 6.4. For any $p_1, p_2 \in \mathbf{N}^*$, we say $p_1 \sim p_2$ if $\mu(\mathbf{x})(p_1) = \mu(\mathbf{x})(p_2)$ for all $\mathbf{x} \in X^{\mathbf{R}}$.

It is easy to check that $\sim$ is an equivalence relation. Let $\mathbf{N}^0 = \mathbf{N}^*/\sim$. We consider $\mathbf{N}^0$ endowed with the *quotient topology*: the finest topology such that the canonical projection $\pi: \mathbf{N}^* \to \mathbf{N}^0$, $\pi(p) = \hat{p} = \{p' \mid p' \in \mathbf{N}^*, \ p' \sim p\}$, is continuous. The open sets of $\mathbf{N}^0$ are therefore given by $\{U \subset \mathbf{N}^0: \pi^{-1}(U) \text{ open in } \mathbf{N}^*\}$.

Considering $\mathbf{N}^0$ with the quotient topology we have:

- $\mathbf{N}^0$ *is compact* since it is the continuous image of the compact space $\mathbf{N}^*$.
- The map $\mu^0(\mathbf{x}): \mathbf{N}^0 \to \mathbf{R}$ *defined by* $\mu^0(\mathbf{x})(\hat{p}) = \mu(\mathbf{x})(p)$ *is continuous for all* $\mathbf{x} \in X^{\mathbf{R}}$ since $\mu(\mathbf{x})$ is continuous on $\mathbf{N}^*$.
- $\mathbf{N}^0$ *is Hausdorff* as we now check. For $p_1 \neq p_2 \in \mathbf{N}^0$, there must be a sequence $\mathbf{x}$ for which $\mu^0(\mathbf{x})(p_1) \neq \mu^0(\mathbf{x})(p_2)$, and therefore there exist disjoint open sets $U_1, U_2 \subset \mathbf{R}$ such that $\mu^0(\mathbf{x})(p_1) \in U_1$, $\mu^0(\mathbf{x})(p_2) \in U_2$. It follows that the open sets $(\mu^0(\mathbf{x}))^{-1}(U_1)$, $(\mu^0(\mathbf{x}))^{-1}(U_2)$ separate p_1 and p_2.

The above allows us to define a new measure function:

Definition 6.5. Denote by $\mu^0: X^{\mathbf{R}} \to C^*(\mathbf{N}^0)$ the sequence measure function defined as

$$\mu^0(\mathbf{x})(\hat{p}) = \mu(\mathbf{x})(p). \tag{23}$$

Denote by μ^0 as well the corresponding frame measure function $\mu^0 \circ b$.

We now show that μ^0 is minimal; in Section 6.2 we will show that μ^0 is essentially the unique minimal sequence and frame measure function. We begin by stating a trivial consequence of Theorem 5.22.

Corollary 6.6. *For any sequence measure function $m : X^{\mathbf{R}} \to C^*(W)$ there exists an injection $\varphi : \mathbf{N}^0 \to W$ such that $m(\mathbf{x})(\varphi(p)) = \mu^0(\mathbf{x})(p)$ for all $\mathbf{x} \in X^{\mathbf{R}}$, $p \in \mathbf{N}^0$.*

Proof. The result follows trivially from Theorem 5.22 and the definition of μ^0 which just eliminates the indistinguishable points of $\mathbf{N}^*$.

Proposition 6.7. *The map $\mu^0 : X^{\mathbf{R}} \to C^*(\mathbf{N}^0)$ is a minimal sequence measure function.*

Proof. The definition of μ^0 assures that it is separable. Assume that μ^0 is not irreducible. Thus there is a compact $\mathbf{N}' \subsetneq \mathbf{N}^0$ so that $\mu' : X^{\mathbf{R}} \to C^*(\mathbf{N}')$ defined by $\mu'(\mathbf{x}) = \mu^0(\mathbf{x})|_{\mathbf{N}'}$ is again a sequence measure function. Now consider a point $p \in \mathbf{N}^0 \setminus \mathbf{N}'$. Denote by $\varphi : \mathbf{N}^0 \to \mathbf{N}'$ the map given in Corollary 6.6. Thus $\mu'(\mathbf{x})(\varphi(p)) = \mu(\mathbf{x})(p)$ for all $\mathbf{x} \in X^{\mathbf{R}}$. Since $\mu'(\mathbf{x})(\varphi(p)) = \mu^0(\mathbf{x})(\varphi(p))$ for all $\mathbf{x} \in X^{\mathbf{R}}$, the separability of μ^0 implies $\varphi(p) = p$, a contradiction since $\varphi(p) \in \mathbf{N}'$, $p \in \mathbf{N}^0 \setminus \mathbf{N}'$. Thus μ^0 must be irreducible and thus minimal. $\square$

As usual the above implies the corresponding result for frame measure functions.

Corollary 6.8. *The map $\mu^0 : \mathcal{F}[I] \to C^*(\mathbf{N}^0)$ is a minimal frame measure function.*

The construction above for getting $\mathbf{N}^0$ from $\mathbf{N}^*$ can be used for any sequence or frame measure function $m : X^{\mathbf{R}} \to C^*(W)$ to construct a separable sequence or frame measure function. Define on W the equivalence relation $v \sim w$ if $m(\mathbf{x})(v) = m(\mathbf{x})(w)$ for all $\mathbf{x} \in X^{\mathbf{R}}$. The quotient space $W^0 = W/\!\sim$ is then compact Hausdorff with respect to the quotient topology. We denote by π the continuous map $\pi : W \to W^0$ defined by $\pi(v) = \pi(w)$ if and only if $v \sim w$. The sequence measure function m induces a map $m^0 : X^{\mathbf{R}} \to C^*(W^0)$ with

$$m^0(\mathbf{x})(p) = m(\mathbf{x})(q), \quad \text{for } q \in \pi^{-1}(p). \tag{24}$$

The definition of m^0 yields:

Proposition 6.9. *The map $m^0 : X^{\mathbf{R}} \to C^*(W^0)$ is a separable sequence measure function. Consequently the map $m_f^0 = m^0 \circ b$ that can be constructed from a given frame measure function $m_f = m \circ b$ is a separable frame measure function.*

6.2. Uniqueness of the minimal sequence and frame measure function

Lemma 6.10. *If $m : X^{\mathbf{R}} \to C^*(W)$ is minimal, then $\varphi : \mathbf{N}^0 \to W$ described in Corollary 6.6 is injective with dense range.*

Proof. Injectivity is a result of Corollary 6.6. If the range $\varphi(\mathbf{N}^0)$ is not dense in W, then m restricted to the closure of $\varphi(\mathbf{N}^0)$ would also be a sequence measure function which would contradict the minimality of m. $\square$

Corollary 6.11. *For a minimal sequence measure function $m : X^{\mathbf{R}} \to C^*(W)$, $m(\mathbf{x} \wedge \mathbf{y}) = \min(m(\mathbf{x}), m(\mathbf{y}))$, $m(\mathbf{x} \vee \mathbf{y}) = \max(m(\mathbf{x}), m(\mathbf{y}))$ for any two sequences $\mathbf{x}, \mathbf{y} \in X^{\mathbf{R}}$.*

Proof. It follows from Proposition 5.18 that the result is true for the minimal sequence measure function μ^0. The result follows from Lemma 6.10 and the continuity of the maps $m(\mathbf{x})$, $m(\mathbf{x} \wedge \mathbf{y})$ and $m(\mathbf{x} \vee \mathbf{y})$. $\quad\square$

Lemma 6.12. *Let* $m : X^{\mathbf{R}} \to C^*(W)$ *be a minimal sequence measure function. For any* $a, b \in \mathbf{R}$ *and* $v, w \in W$, *there is an* $\mathbf{x} \in X^{\mathbf{R}}$ *such that* $m(\mathbf{x})(v) = a$ *and* $m(\mathbf{x})(w) = b$.

Proof. Recall $\mathbf{i} = (|I_1|, |I_2|, \ldots)$; $\mathbf{i}$ is sequence compatible and $m(\mathbf{i})(w) = 1$ for all $w \in W$.

The case $a = b$ is simple since $m(a\mathbf{i})(w) = a$ for all $w \in W$. For the case $a \neq b$, since m is separable, there exists $\mathbf{x}^0 \in X^{\mathbf{R}}$ such that $m(\mathbf{x}^0)(v) \neq m(\mathbf{x}^0)(w)$. Let $c_1, c_2 \in \mathbf{R}$ be determined by the linear system:

$$c_1 m(\mathbf{x}^0)(v) + c_2 = a, \qquad c_1 m(\mathbf{x}^0)(w) + c_2 = b.$$

Set $\mathbf{x} = c_1 \mathbf{x}^0 + c_2 \mathbf{i} \in X^{\mathbf{R}}$. It follows by linearity of the sequence measure function that $m(x)(v) = a$, $m(x)(w) = b$. $\quad\square$

Theorem 6.13 *(Density of range). Assume* $m : X^{\mathbf{R}} \to C^*(W)$ *is a minimal sequence measure function. Then for every bounded real-valued continuous function* $f \in C^*(W)$, *and every* $\varepsilon > 0$ *there exists* $\mathbf{x} \in X^{\mathbf{R}}$ *so that* $\|m(\mathbf{x}) - f\|_\infty < \varepsilon$.

Proof. Lemma 6.12 coupled with the fact that $X^{\mathbf{R}}$ is a lattice with respect to $\vee$, $\wedge$ (Proposition 5.17) allows for the application of the lattice version of Stone's theorem [27, Chapter I, Section 2, 10.II]; the result is then immediate. $\quad\square$

Corollary 6.14. *Given* $m : X^{\mathbf{R}} \to C^*(W)$ *a minimal sequence measure function, for every real-valued continuous function* $f \in C^*(W)$ *and every* $\varepsilon > 0$ *there exists a constant* c *and two frame compatible sequences* $\mathbf{y}^1$, $\mathbf{y}^2$, *such that* $\|c(m(\mathbf{y}^1) - m(\mathbf{y}^2)) - f\|_\infty < \varepsilon$.

Proof. Theorem 6.13 establishes the existence of $\mathbf{x} \in X^{\mathbf{R}}$ for which $\|m(\mathbf{x}) - f\|_\infty < \varepsilon$. The result follows from the fact that any $\mathbf{x} \in X^{\mathbf{R}}$ can be written as $\mathbf{x} = c(\mathbf{y}^1 - \mathbf{y}^2)$ with $\mathbf{y}^1$, $\mathbf{y}^2$ frame compatible. $\quad\square$

As usual the above yields the corresponding result for frame measure functions:

Corollary 6.15. *Given* $m : \mathcal{F}[I] \to C^*(W)$ *a minimal frame measure function, for every real-valued continuous function* $f \in C^*(W)$ *and every* $\varepsilon > 0$ *there exists a constant* c *and two frames* $\mathcal{F}^1$, $\mathcal{F}^2$, *such that* $\|c(m(\mathcal{F}^1) - m(\mathcal{F}^2)) - f\|_\infty < \varepsilon$.

Lemma 6.16. *If* $m : X^{\mathbf{R}} \to C^*(W)$ *is minimal, then* $\varphi : \mathbf{N}^0 \to W$ *described in Corollary 6.6 is continuous.*

Proof. To show continuity of φ we will show that for all open sets $V \subset W$ and all $p \in \varphi^{-1}(V)$ there exists an open set $U_p \in \mathbf{N}^0$ with $p \in U_p$ and $\varphi(U_p) \subset V$. By Urysohn's lemma, since $W \setminus V$ is closed, there is a continuous function $\tilde{f} \in C^*(W)$, so that $0 \leqslant \tilde{f} \leqslant 1$ on W, $\tilde{f}|_{W \setminus V} = 1$, and $\tilde{f}(\varphi(p)) = 0$. By Theorem 6.13 there exist $\mathbf{x} \in X^{\mathbf{R}}$ such that $\|m(\mathbf{x}) - \tilde{f}\|_\infty \leqslant \frac{1}{3}$. Thus $m(\mathbf{x})|_{W \setminus V} \geqslant \frac{2}{3}$ and $|m(\mathbf{x})(\varphi(p))| \leqslant \frac{1}{3}$. Set $U_p = \mu^0(\mathbf{x})^{-1}((-\frac{1}{2}, \frac{1}{2}))$; U_p is open (since $\mu^0(\mathbf{x})$ is

continuous) and $p \in U_p$ (since $|\mu^0(\mathbf{x})(p)| = |m(\mathbf{x})(\varphi(p))| = 0 \leqslant \frac{1}{3}$), and $\varphi(U_p) = \mu^0(\mathbf{x})(U_p) \subset (-\frac{1}{2}, \frac{1}{2}))$ whereas $m(x)(W \setminus V) \geqslant \frac{2}{3}$. $\quad\square$

Theorem 6.17. *All minimal sequence measure functions* $m: X^{\mathbf{R}} \to C^*(W)$ *are topologically equivalent to* μ^0, *i.e. there exists a continuous bijection with continuous inverse* $\varphi: \mathbf{N}^0 \to M$, *such that* $m(\mathbf{x})(\varphi(p)) = \mu^0(\mathbf{x})(p)$ *for all* $p \in \mathbf{N}^0$, $\mathbf{x} \in X^{\mathbf{R}}$.

Proof. We let $\varphi: \mathbf{N}^0 \to M$ be the map given in Corollary 6.6, Lemmas 6.10, and 6.16. From these results we have that φ is injective, has dense range, and is continuous. Since $\mathbf{N}^0$ is compact it follows from the continuity of φ that $\varphi(\mathbf{N}^0)$ is compact and thus it must be all of M (since it is dense in M). Thus φ is a bijection. Having established this bijection, we denote by $\varphi^{-1}: M \to \mathbf{N}^0$ the inverse map. The continuity of φ^{-1} is shown the same way as in Lemma 6.16. $\quad\square$

Corollary 6.18. *All minimal frame measure functions* $m: \mathcal{F}[I] \to C^*(W)$ *are topologically equivalent to* μ^0, *i.e. there exists a continuous, bijection with continuous inverse* $\varphi: \mathbf{N}^0 \to M$, *such that* $m(\mathcal{F})(\varphi(p)) = \mu^0(\mathcal{F})(p)$ *for all* $p \in \mathbf{N}^0$, $\mathcal{F} \in \mathcal{F}[I]$.

Remark 6.19. We provide an example of a sequence measure function that is separable but not minimal (that is, it is not irreducible). This implies the existence of a frame measure function that is separable but not minimal. Let $|I_n| = 2^n$ and consider the minimal measure function $\mu^0: X^{\mathbf{R}} \to C^*(\mathbf{N}^0)$. Let $W = \mathbf{N}^0 \cup \{w_0\}$ be the union of $\mathbf{N}^0$ with one extra point w_0. Pick two distinct $p_1, p_2 \in \mathbf{N}^*$ so that p_1 contains the set of odd integers, and p_2 contains the set of even integers. Define $m(\mathbf{x})(w_0) = \frac{1}{2}(\mu^0(x)(p_1) + \mu^0(x)(p_2))$ and define $m(\mathbf{x})(p) = \mu^0(x)(p)$. Since $\mathbf{N}^0$ is a proper subset of W, m is not minimal. Now consider the frame compatible sequence $\tilde{\mathbf{x}}$ defined by

$$\begin{cases} \tilde{x}_1 = 0, \\ \tilde{x}_{2n} = \tilde{x}_{2n-1} + |I_{2n} \setminus I_{2n-1}|, \\ \tilde{x}_{2n+1} = \tilde{x}_{2n}. \end{cases}$$

Explicitly, $\tilde{x}_{2n+1} = \tilde{x}_{2n} = \frac{2}{3}(4^n - 1)$. Notice that $\lim_{n \to \infty} \frac{\tilde{x}_{2n}}{|I_{2n}|} = \frac{2}{3}$ whereas $\lim_{n \to \infty} \frac{\tilde{x}_{2n+1}}{|I_{2n+1}|} = \frac{1}{3}$. Now take a $p \in \mathbf{N}^0$. Then $m(\tilde{\mathbf{x}})(p)$ equals either $\frac{1}{3}$ or $\frac{2}{3}$ depending on whether p contains the set of odd integers, or not. In either case $\tilde{\mathbf{x}}$ separates w_0 from p,

$$m(\tilde{\mathbf{x}})(w_0) = \frac{1}{2}\left(\frac{1}{3} + \frac{2}{3}\right) = \frac{1}{2} \neq m(\tilde{\mathbf{x}})(p).$$

Thus m is a separable but not minimal frame measure function.

7. The C^* algebra of non-expansive operators

Our approach to the classification of frames has been to examine the sequence $b(\mathcal{F})$ associated to a frame $\mathcal{F}$ via (8). The sequence $b(\mathcal{F})$ can be seen to be certain averages of the diagonal elements of the Gram matrix $\{\langle f_i, \tilde{f}_j \rangle\}_{i,j \in I}$. We now extend the definition of b to all $I \times I$ matrices and then compose this extended b map with a sequence measure function m to give a measure on $I \times I$ matrices. The result is an *operator measure function* that resembles a trace on

a large subalgebra of operators. In conjunction with some added structure on the index set I, this expanded viewpoint leads to Theorem 7.14 which states that $m(\mathcal{F}_1 \oplus \mathcal{F}_2) = m(\mathcal{F}_1) + m(\mathcal{F}_2)$ for a superframe $\mathcal{F}_1 \oplus \mathcal{F}_2$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ need not be orthogonal but merely *non-expansive* (see Definition 7.8). This in turn leads to a necessary density inequality for supersets of Gabor frames (Theorem 9.10 and Corollary 9.11).

We begin in Section 7.1 by extending the definitions of measure function and b to the set of bounded operators. We define the important notion of *non-expansive* operators and frames and show that the set of non-expansive operators is a large C^* subalgebra of the set of bounded linear operators acting on $l^2(I)$. We use this set up to prove the aforementioned result about supersets in Section 7.3.

7.1. Operator measure functions

We begin by defining $X^{\mathbf{C}} = \{x^1 + ix^2 : x^1, x^2 \in X^{\mathbf{R}}\}$. Recall the equivalence relation $\approx$ introduced in Definition 3.5 applies to sequences in $X^{\mathbf{C}}$ as well. Thus $\mathbf{x} \approx \mathbf{y}$, $\mathbf{x}, \mathbf{y} \in X^{\mathbf{C}}$, if $\lim_{n \to \infty} (x_n - y_n)/|I_n| = 0$.

The following extends the map b to operators.

Definition 7.1. Let b_{op} be the map from bounded linear operators on $l^2(I)$ to sequences defined by

$$b_{\mathrm{op}}(A) = \left\{ \sum_{i \in I_n} \langle A\delta_i, \delta_i \rangle \right\}_{n \in \mathbf{N}}$$

where $\{\delta_i\}_{i \in I}$ is the canonical basis of $l^2(I)$.

The range of b_{op} lies in $X^{\mathbf{C}}$:

Proposition 7.2. *For all $A \in B(l^2(I))$, $b(A) \in X^{\mathbf{C}}$.*

Proof. Define

$$a_j^+ = \max\big(\mathrm{Re}\big(\langle A\delta_j, \delta_j \rangle\big), 0\big), \qquad a_j^- = \min\big(\mathrm{Re}\big(\langle A\delta_j, \delta_j \rangle\big), 0\big),$$
$$a_j^i = \max\big(\mathrm{Im}\big(\langle A\delta_j, \delta_j \rangle\big), 0\big), \qquad a_j^{-i} = \min\big(\mathrm{Im}\big(\langle A\delta_j, \delta_j \rangle\big), 0\big),$$

thus $a_j^+ + a_j^- + i(a_j^i + a_j^{-i}) = \langle A\delta_j, \delta_j \rangle$ with $a_j^+, -a_j^-, a_j^i, -a_j^{-i} \leqslant \|A\|$. Define

$$x_n^+ = \sum_{j \in I_n} a_j^+, \qquad x_n^- = \sum_{j \in I_n} a_j^-, \qquad x_n^i = \sum_{j \in I_n} a_j^i, \qquad x_n^{-i} = \sum_{j \in I_n} a_j^{-i}.$$

It follows then that $b_n(A) = x_n^+ - (-x_n^-) + i x_n^i - i(-x_n^{-i})$. It is straightforward to verify that the sequences $\{x_n^+\}_{n \in \mathbf{N}}$, $\{-x_n^-\}_{n \in \mathbf{N}}$, $\{x_n^i\}_{n \in \mathbf{N}}$, $\{-x_n^{-i}\}_{n \in \mathbf{N}}$ are all in X^+ (the appropriate c being $\|A\|$) and thus $b(A) = \{b_n(A)\}_{n \in N} \in X^{\mathbf{C}}$. $\quad\square$

Remark 7.3. We note that given a frame $\mathcal{F}$ and its associated Gram projection $P \in B(l^2(I))$, we have $b(\mathcal{F}) = b_{\mathrm{op}}(P)$.

For the rest of this paper we will write b for b_{op}. Thus b is both a map from frames to sequences (previous notation) and the related map from linear operators to sequences.

Denote by $\mathcal{C}_{\mathbf{C}}^*(W)$ the set of complex valued continuous maps on W. We now show that any sequence measure function has a unique linear extension to $X^{\mathbf{C}}$.

Proposition 7.4. *Given a sequence function $m : X^{\mathbf{R}} \to \mathcal{C}^*(W)$, there exists a unique linear map $\tilde{m} : X^{\mathbf{C}} \to \mathcal{C}_{\mathbf{C}}^*(W)$ such that $\tilde{m}|X^{\mathbf{R}} = m$.*

Proof. For any $\mathbf{x} \in X^{\mathbf{C}}$, the decomposition of $\mathbf{x} = \mathbf{x}^1 + i\mathbf{x}^2$, $\mathbf{x}^1, \mathbf{x}^2 \in X^{\mathbf{R}}$, is unique with $\mathbf{x}_i^1 = \mathrm{Re}(\mathbf{x}_i)$, $\mathbf{x}_i^2 = \mathrm{Im}(\mathbf{x}_i)$. Define $\tilde{m} = m(\mathbf{x}^1) + im(\mathbf{x}^2)$. Thus $\tilde{m}$ is linear (since m was linear) and $\tilde{m}|_{X^{\mathbf{R}}} = m$. In addition $\tilde{m}$ is the unique linear extension since there is only one way to write $\mathbf{x} = \mathbf{x}^1 + i\mathbf{x}^2$. □

We now define an *operator measure* function.

Definition 7.5. An operator measure function, $\bar{m} : B(l^2) \to X^{\mathbf{C}}$ is a map of the form $\bar{m} = \tilde{m} \circ b$ where $\tilde{m}$ is the linear extension of a sequence measure function described in Proposition 7.4.

We note that an operator measure function $\overline{m}$ is linear since it is the composition of two linear maps. The next few sections examine the behaviour of $\overline{m}$. We show that with added structure on the index set I, there exists a large C^* algebra $\mathcal{C} \subset B(l^2(I))$ for which $\overline{m}$ is tracial, i.e. $\overline{m}(AB) = \overline{m}(BA)$ for $A, B \in \mathcal{C}$. This tracial property is then used to prove Theorem 7.14 which states that for a superframe $\mathcal{F}_1 \oplus \mathcal{F}_2$ of two non-expansive frames (see Definition 7.8) $\mathcal{F}_1, \mathcal{F}_2$, the equation $m(\mathcal{F}_1 \oplus \mathcal{F}_2) = m(\mathcal{F}_1) + m(\mathcal{F}_2)$ holds.

7.2. The C^* algebra of non-expansive operators

By a *quasi-distance d* on I we shall mean a map $d : I \times I \to \mathbf{R}^+$ that satisfies: (i) $d(i, i) = 0$, $d(i, j) \geqslant 0$; (ii) $d(i, j) = d(j, i)$; (iii) $d(i, j) \leqslant d(i, k) + d(k, j)$, for any $i, j, k \in I$.

For this section we shall consider an index set I equipped with a quasi-distance d. We call (I, d) a *quasi-metric index set*. We denote the ball of radius R from $i \in I$ by

$$B_R(i) = \{ j \in I : d(j, i) \leqslant R \}. \tag{25}$$

We shall say that I has finite upper density with respect to d if $\sup_{i \in I} |B_R(i)| < \infty$ for all $R > 0$.

Recall an algebra $S \subset B(l^2(I))$ that is invariant under the adjoint operation (i.e. $A^* \in S$ for any $A \in S$) is called a C^* *algebra* if it is closed in the operator norm topology.

Definition 7.6.

(1) An operator $A \in B(l^2(I))$ is row *non-expansive* if for any $\epsilon > 0$, there exists an $N(A, \varepsilon) > 0$ such that

$$\sum_{j \in I \setminus B_{N(A, \varepsilon)}(i)} \left| \langle A\delta_i, \delta_j \rangle \right|^2 < \varepsilon \tag{26}$$

for all $i \in I$.

(2) An operator is non-expansive if both A and A^* are row non-expansive. Denote by $\mathcal{C} \subset B(l^2(I))$ the set of non-expansive operators.

Theorem 7.7. *Suppose I has finite upper density with respect to d. Then $\mathcal{C}$ is*

(1) *closed under addition and scalar multiplication, i.e. if $A, B \in \mathcal{C}$ and $c \in \mathbf{C}$ then $A + B \in \mathcal{C}$ and $cA \in \mathcal{C}$,*
(2) *closed under multiplication, i.e. if $A, B \in \mathcal{C}$ then $AB \in \mathcal{C}$,*
(3) *closed in the operator norm topology, i.e. given a filter $\mathcal{J}$ on some set S with $A_j \in \mathcal{C}$ for all $j \in S$ and $\lim_{j \to \mathcal{J}} \|A - A_j\| = 0$ then $A \in \mathcal{C}$.*

Consequently $\mathcal{C}$ is a C^* algebra.

Proof. (1) Fix an $\varepsilon > 0$. Set $N = \max(N(A, \frac{\varepsilon}{4}), N(B, \frac{\varepsilon}{4}))$ with $N(A, \frac{\varepsilon}{4}), N(B, \frac{\varepsilon}{4})$ as in Definition 7.6. Thus for all i, we have

$$\sum_{j \in I \setminus B_N(i)} \left| \langle (A+B)\delta_i, \delta_j \rangle \right|^2 \leqslant 2 \left(\sum_{j \in I \setminus B_{N(A, \frac{\varepsilon}{4})}(i)} \left| \langle A\delta_i, \delta_j \rangle \right|^2 + \sum_{j \in I \setminus B_{N(B, \frac{\varepsilon}{4})}(i)} \left| \langle B\delta_i, \delta_j \rangle \right|^2 \right) < \varepsilon.$$

This proves $A + B$ is non-expansive.

Setting $N = N(A, \frac{\varepsilon}{|c|})$ yields

$$\sum_{j \in I \setminus B_N(i)} \left| \langle cA\delta_i, \delta_j \rangle \right|^2 < \varepsilon$$

for all $i \in I$, which proves cA is non-expansive.

(2) Fix $\varepsilon > 0$. Let $\varepsilon_B = \frac{\varepsilon}{4\|A\|^2}$ and set $N_B = N(B, \varepsilon_B)$. Let $\varepsilon_A = \frac{\varepsilon}{4\|B\|^2 D(N_B)}$, where $D(N_B) = \sup_i |B_{N_B}(i)|$ (the upper bound on the number of points of I in a ball of radius N_B); set $N_A = N(A, \varepsilon_A)$. Let $N = N_A + N_B$ and fix $i \in I$. We first note

$$B\delta_i = \sum_{l \in I} \langle B\delta_i, \delta_l \rangle \delta_l = v + \sum_{l \in B_{N_B}(i)} \langle B\delta_i, \delta_l \rangle \delta_l$$

for some vector v with $\|v\|^2 < \varepsilon_B$. Now

$$\sum_{j \in I \setminus B_N(i)} \left| \langle AB\delta_i, \delta_j \rangle \right|^2 = \sum_{j \in I \setminus B_N(i)} \left| \langle Av, \delta_j \rangle + \sum_{l \in B_{N_B}(i)} \langle B\delta_i, \delta_l \rangle \langle A\delta_l, \delta_j \rangle \right|^2$$

$$\leqslant 2 \sum_{j \in I} \left| \langle Av, \delta_j \rangle \right|^2 + 2 \sum_{j \in I \setminus B_N(i)} \left| \sum_{l \in B_{N_B}(i)} \langle B\delta_i, \delta_l \rangle \langle A\delta_l, \delta_j \rangle \right|^2$$

$$\leqslant 2\|A\|^2 \varepsilon_B + 2 \sum_{j \in I \setminus B_N(i)} \left(D(N_B) \sum_{l \in B_{N_B}(i)} \left| \langle B\delta_i, \delta_l \rangle \right|^2 \left| \langle A\delta_l, \delta_j \rangle \right|^2 \right)$$

$$= \frac{\varepsilon}{2} + 2D(N_B) \sum_{l \in B_{N_B}(i)} \left| \langle B\delta_i, \delta_l \rangle \right|^2 \sum_{j \in I \setminus B_N(i)} \left| \langle A\delta_l, \delta_j \rangle \right|^2.$$

Now note that $I \setminus B_N(i) \subset I \setminus B_{N_A}(l)$ for any $l \in B_{N_B}(i)$. Thus

$$\sum_{j \in I \setminus B_N(i)} \left|\langle A\delta_l, \delta_j \rangle\right|^2 \leqslant \sum_{j \in I \setminus B_{N_A}(l)} \left|\langle A\delta_l, \delta_j \rangle\right|^2 < \varepsilon_A$$

and therefore:

$$\sum_{j \in I \setminus B_N(i)} \left|\langle AB\delta_i, \delta_j \rangle\right|^2 \leqslant \frac{\varepsilon}{2} + 2D(N_B) \sum_{l \in B_{N_B}(i)} \left|\langle B\delta_i, \delta_l \rangle\right|^2 \varepsilon_A$$

$$\leqslant \frac{\varepsilon}{2} + 2D(N_B)\|B\delta_i\|^2 \varepsilon_A$$

$$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(3) Let $\varepsilon > 0$ be given. Then there is $K \in \mathcal{J}$ so that for all $k \in K$, A_k is non-expansive and $\|A - A_k\|^2 < \frac{\varepsilon}{4}$. Let $N_\varepsilon = N(A_k, \frac{\varepsilon}{4})$ for some fixed $k \in K$. Then for every $i \in I$,

$$\sum_{j \in I \setminus B_{N_\varepsilon}(i)} \left|\langle A\delta_i, \delta_j \rangle\right|^2 = \sum_{j \in I \setminus B_{N_\varepsilon}(i)} \left|\langle (A - A_k)\delta_i, \delta_j \rangle + \langle A_k\delta_i, \delta_j \rangle\right|^2$$

$$\leqslant 2 \sum_{j \in I} \left|\langle (A - A_k)\delta_i, \delta_j \rangle\right|^2 + 2 \sum_{j \in I \setminus B_{N_\varepsilon}(i)} \left|\langle A_k\delta_i, \delta_j \rangle\right|^2$$

$$\leqslant 2\|A - A_j\|^2 + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Definition 7.8. We shall say that a frame $\mathcal{F}$ is non-expansive if its associated Gram projection is non-expansive.

Using elementary holomorphic functional calculus (see [29, §149]) we can obtain the following.

Proposition 7.9. *Let C be a C^* algebra acting on a Hilbert space and let an operator $A \in C$. If the range of A is closed then the orthogonal projection onto the range of A and the orthogonal projection onto the range of A^* are both in C.*

This result has a couple of consequences: it gives a simpler sufficient (but not necessary) condition for a frame to be non-expansive (Corollary 7.10 below) and it plays a key role in the proof of Theorem 7.14.

Corollary 7.10. *For any frame $\mathcal{F} \in \mathcal{F}[I]$, if its Gram operator $G : l^2(I) \to l^2(I)$, $G(c) = \{\sum_{j \in I} \langle f_j, f_i \rangle c_j\}_{i \in I}$ is non-expansive, then the $\mathcal{F}$ is non-expansive, as are the associated Parseval frame and the canonical dual frame.*

Proof. If G is non-expansive, $G \in \mathcal{C}$. Since $\mathcal{F}$ is frame, the range of G is closed. Thus the associated Gram projection, by Proposition 7.9, is also in $\mathcal{C}$, and thus $\mathcal{F}$ is non-expansive. Since $\mathcal{F}$, the associated Parseval frame $\mathcal{F}^\# = \{S^{-1/2} f_i\}$ and the canonical dual frame $\tilde{\mathcal{F}} = \{S^{-1} f_i\}$ all have the same associated Gram projection, they are all non-expansive. $\quad \square$

Remark 7.11. Corollary 7.10 is merely a sufficient condition as the following construction demonstrates. Let S be a self-adjoint operator that is not non-expansive. It follows that the invertible operator $G = S + 2\|S\|I$ is also not non-expansive. In this case, the frame $\mathcal{G} = \{g_i = G^{1/2}\delta_i\}$ is a Riesz basis and hence is non-expansive (since the corresponding projection for a Riesz basis is the identity). However, the frame $\mathcal{G}$ has a non-expansive Gram operator G.

7.3. The measure function and supersets

In this subsection we show that condition (4) of Definition 5.5 can be extended to non-orthogonal superframes that are non-expansive. In particular we obtain a density-type result.

The main result that allows us to develop the theory is the tracial property of the extended measure $\overline{m}$ on $\mathcal{C}$ (Lemma 7.13). The result will hold when the quasi distance d and the decomposition $I = \bigcup_n I_n$ have the following compatibility which essentially says that the boundary (with respect to d) of subsets $(I_n)_{n \geqslant 0}$ are asymptotically smaller than their interior:

Definition 7.12. The collection $(I, d, (I_n)_n)$ is called a *uniform metric index set* if the quasi-distance d has finite upper density and for all $R > 0$,

$$\lim_{n \to \infty} \frac{|\bigcup_{j \in I \setminus I_n} B_R(j) \cap I_n|}{|I_n|} = 0. \tag{27}$$

Lemma 7.13. *Assume $(I, d, (I_n)_n)$ is a uniform metric index set. Then for any two non-expansive operators $T_1, T_2 \in \mathcal{C}$,*

$$\overline{m}(T_1 T_2) = \overline{m}(T_2 T_1). \tag{28}$$

Proof. Equation (28) is equivalent to

$$\lim_{n \to \infty} \frac{1}{|I_n|} b_n(T_1 T_2 - T_2 T_1) = \lim_{n \to \infty} \frac{1}{|I_n|} \big(b_n(T_1 T_2) - b_n(T_2 T_1)\big) = 0. \tag{29}$$

Recall that $T \in \mathcal{C}$ implies that both T and T^* are non-expansive. Since $\{\delta_i\}_{i \in I}$ is an orthonormal basis:

$$\frac{1}{|I_n|} b_n(T_1 T_2) = \frac{1}{|I_n|} \sum_{i \in I_n} \sum_{j \in I} \langle T_2 \delta_i, \delta_j \rangle \langle T_1 \delta_j, \delta_i \rangle.$$

Using the corresponding expansion for $\frac{1}{|I_n|} b_n(T_2 T_1)$ and subtracting from the above, we get

$$\frac{1}{|I_n|} b_n(T_1 T_2 - T_2 T_1)$$

$$= \frac{1}{|I_n|} \sum_{i \in I_n} \sum_{j \notin I_n} \langle T_1 \delta_j, \delta_i \rangle \langle T_2 \delta_i, \delta_j \rangle - \frac{1}{|I_n|} \sum_{i \notin I_n} \sum_{j \in I_n} \langle T_1 \delta_j, \delta_i \rangle \langle T_2 \delta_i, \delta_j \rangle. \tag{30}$$

We shall show that the right-hand side of (30) has limit 0 as $n \to \infty$ which will establish the result. We apply Cauchy–Schwarz to the first term on the right-hand side of (30) and obtain

$$\left| \frac{1}{|I_n|} \sum_{i \in I_n} \sum_{j \notin I_n} \langle T_1 \delta_j, \delta_i \rangle \langle T_2 \delta_i, \delta_j \rangle \right|^2 \tag{31}$$

$$\leqslant \left(\frac{1}{|I_n|} \sum_{i \in I_n} \sum_{j \in I \setminus I_n} |\langle T_1^* \delta_i, \delta_j \rangle|^2 \right) \left(\frac{1}{|I_n|} \sum_{i \in I_n} \sum_{j \in I \setminus I_n} |\langle T_2 \delta_i, \delta_j \rangle|^2 \right). \tag{32}$$

Fix $\varepsilon > 0$. Let N be a radius in the definition of non-expansiveness that works for T_1, T_2, T_1^*, T_2^* simultaneously. Write $I_n = J_n \cup D_n$ where $D_n = I_n \cap (\bigcup_{j \in I \setminus I_n} B_N(j))$ is the set of points of I_n that are within distance N of the boundary, and $J_n = I_n \setminus D_n$ is the rest. Decomposing the sums over $i \in I_n$ into the sums over D_n and J_n, we have that (31) is bounded above by

$$\left(\varepsilon + \frac{1}{|I_n|} \sum_{i \in D_n} \sum_{j \in I} |\langle T_1^* \delta_i, \delta_j \rangle|^2 \right) \left(\varepsilon + \frac{1}{|I_n|} \sum_{i \in D_n} \sum_{j \in I} |\langle T_2 \delta_i, \delta_j \rangle|^2 \right)$$

$$\leqslant \left(\varepsilon + \frac{|D_n|}{|I_n|} \|T_1\|^2 \right) \left(\varepsilon + \frac{|D_n|}{|I_n|} \|T_2\|^2 \right).$$

A similar inequality is obtained for the second term in (30) and thus

$$\left| \frac{1}{|I_n|} b_n(T_1 T_2 - T_2 T_1) \right| \leqslant 2 \left(\varepsilon + \frac{|D_n|}{|I_n|} A \right),$$

where $A = \max(\|T_1\|^2, \|T_2\|^2)$. Using the asymptotic assumption (27) we obtain

$$\lim_{n \to \infty} \left| \frac{1}{|I_n|} b_n(T_1 T_2 - T_2 T_1) \right| \leqslant 3\varepsilon.$$

Since ε was arbitrary, we obtain (29). $\quad\square$

We now prove that frame measure functions are linear on supersets of non-expansive frames:

Theorem 7.14. *Assume $(I, d, (I_n)_n)$ is a uniform metric index set and $m : \mathcal{F}[I] \to \mathcal{C}^*(M)$ a frame measure function. Suppose $(\mathcal{F}_1, \mathcal{F}_2)$ is a superframe of two non-expansive frames. Then $\mathcal{F}_1 \oplus \mathcal{F}_2$ is non-expansive and*

$$m(\mathcal{F}_1 \oplus \mathcal{F}_2) = m(\mathcal{F}_1) + m(\mathcal{F}_2). \tag{33}$$

Proof. We first show that $\mathcal{F}_1 \oplus \mathcal{F}_2$ is non-expansive. Let P_1, P_2 denote the associated Gram projections to the two frames $\mathcal{F}_1$ and $\mathcal{F}_2$. The definition of non-expansive frames gives $P_1, P_2 \in \mathcal{C}$. Since $\mathcal{F}_1 \oplus \mathcal{F}_2$ is a frame, we have by Proposition A.2 that $P_1 + P_2$ has closed range and thus by Proposition 7.9, the projection onto the range of $P_1 + P_2$, which is the associated Gram projection for $\mathcal{F}_1 \oplus \mathcal{F}_2$, is also non-expansive.

Let P be the associated Gram projection for $\mathcal{F}_1 \oplus \mathcal{F}_2$, i.e. P is the projection onto the range of $P_1 + P_2$, that is $P = P_1 \vee P_2$. The statement (33) is equivalent to proving

$$\tilde{m}(P) = \tilde{m}(P_1) + \tilde{m}(P_2). \tag{34}$$

Consider $A = P_1 - P_1 P_2$. The superframe condition amounts (equivalently) to the condition that $\|P_1 P_2\| < 1$. Hence, when restricted to $\operatorname{Ran} P_1$, $A = 1 - P_1 P_2$ is invertible, hence its range is $\operatorname{Ran} P_1$. Therefore $\operatorname{Ran} A$ is closed, and equals $\operatorname{Ran} P_1$. On the other hand any $x \in l^2(I)$ admits a unique decomposition $x = x_1 + x_2 + x'$, where $x_1 \in \operatorname{Ran} P_1$, $x_2 \in \operatorname{Ran} P_2$, and $x' \in \operatorname{Ran}(1 - P)$. Then $\|Ax\| = \|Ax_1\| \geqslant (1 - \|P_1 P_2\|)\|x_1\|$. Hence $\ker A = \ker(P - P_2)$ which implies $(\ker A)^\perp = \operatorname{Ran}(P - P_2)$. Since A is in $\mathcal{C}$, the partial isometry V of the polar decomposition $A = V(A^*A)^{1/2}$ belongs to $\mathcal{C}$; this again follows from the use of standard holomorphic functional calculus arguments (as in [29]). Furthermore V has initial space $\operatorname{Ran}(P - P_2)$, and final space $\operatorname{Ran} P_1$, that is $VV^* = P_1$, and $V^*V = P - P_2$. Since $\tilde{m}$ is tracial on $\mathcal{C}$, it follows $\tilde{m}(P_1) = \tilde{m}(VV^*) = \tilde{m}(V^*V) = \tilde{m}(P - P_2)$. But $P = (P - P_2) + P_2$ is an orthogonal decomposition of P, therefore $\tilde{m}(P) = \tilde{m}(P - P_2) + \tilde{m}(P_2)$, which together with the previous relation proves (34) and the theorem. $\quad\square$

The following corollary immediately follows using induction.

Corollary 7.15. *Assume* $(\mathcal{F}_1, \ldots, \mathcal{F}_D)$ *is a superframe of non-expansive frames. Then* $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_D$ *is non-expansive and*

$$m(\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_D) = m(\mathcal{F}_1) + \cdots + m(\mathcal{F}_D). \tag{35}$$

8. Measure functions and the index set

In this section we study how different frame indexing and finite averaging methods affect the measure function and the property of non-expansiveness. Because all measure functions contain a copy of the ultrafilter measure function μ (cf. Corollary 5.24) we shall consider only the case of the ultrafilter frame measure function μ, and comment on the extension of these results to arbitrary frame measure functions.

Assume I and J are countable index sets, and $a : I \to J$ is a bijection. Assume also $(I_n)_n$ and $(J_n)_n$ are nested sequences of finite subsets covering I, respectively J. Our goal is to establish how equivalence classes of frames in $\mathcal{F}[I]$ are related to equivalence classes of frames in $\mathcal{F}[J]$. More generally, we will examine the correspondence of operators between $B(l^2(I))$ and $B(l^2(J))$ and the preservation of the non-expansiveness property.

First we note that the map a induces a mapping on frames:

$$a_* : \mathcal{F}[J] \to \mathcal{F}[I], \quad a_*(\mathcal{F}) = \{f_{a(i)}; \ i \in I\} \tag{36}$$

and a mapping on operators:

$$a_* : B(l^2(J)) \to B(l^2(I)), \quad \langle a_*(T)\delta_{i_1}, \delta_{i_2} \rangle = \langle T\epsilon_{a(i_1)}, \epsilon_{a(i_2)} \rangle \tag{37}$$

where $(\delta_i)_i$ and $(\epsilon_j)_j$ are the canonical bases of $l^2(I)$ and $l^2(J)$, respectively.

We are interested in the following tasks:

(1) *Measure preservation.* Find conditions on a so that for all operators $T \in B(l^2(J))$, the ultrafilter frame measure functions for T and $a_*(T)$ are equal.
(2) *Non-expansiveness preservation.* Assuming that (I, d) and (J, e) are quasi-metric index sets, find conditions on a so that for all operators $T \in B(l^2(J))$, T is non-expansive if and only if

$a_*(T)$ is non-expansive. In particular we obtain that $\mathcal{F} \in \mathcal{F}[J]$ is non-expansive if and only if $a_*(\mathcal{F})$ is non-expansive.

We address each of these in the subsequent two sections.

8.1. Measure preserving indexing

The following gives a condition for a that preserves the value of the measure function.

Proposition 8.1. *If the map $a : I \to J$ satisfies the following property*:

$$\lim_n \frac{|a(I_n) \cap J_n|}{|I_n|} = \lim_n \frac{|J_n|}{|I_n|} = 1 \tag{38}$$

then $\mu(T) = \mu(a_*(T))$ for all $T \in B(l^2(J))$. Explicitly this means:

$$p\text{-}\lim_n \frac{1}{|J_n|} \sum_{j \in J_n} T_{j,j} = p\text{-}\lim_n \frac{1}{|I_n|} \sum_{i \in I_n} T_{a(i),a(i)} \tag{39}$$

for all $p \in \mathbf{N}^*$.

Proof. Since T is bounded, it follows $|T_{j,j}| \leqslant r := \|T\|$ for all j. First we have:

$$\frac{1}{|J_n|} \sum_{j \in J_n} T_{j,j} - \frac{1}{|I_n|} \sum_{i \in I_n} T_{a(i),a(i)}$$

$$= \frac{1}{|J_n|} \sum_{j \in J_n \backslash a(I_n)} T_{j,j} + \frac{|I_n| - |J_n|}{|I_n| \cdot |J_n|} \sum_{j \in J_n \cap a(I_n)} T_{j,j} - \frac{1}{|I_n|} \sum_{j \in a(I_n) \backslash J_n} T_{j,j}. \tag{40}$$

Upper bounding each term, we get:

$$\left| \frac{1}{|J_n|} \sum_{j \in J_n} T_{j,j} - \frac{1}{|I_n|} \sum_{i \in I_n} T_{a(i),a(i)} \right|$$

$$\leqslant r \frac{|J_n \backslash a(I_n)|}{|J_n|} + r \frac{||I_n| - |J_n|| \cdot |J_n \cap a(I_n)|}{|I_n| \cdot |J_n|} + r \frac{|a(I_n) \backslash J_n|}{|I_n|}. \tag{41}$$

Condition (38) implies now that each term tends to zero as n goes to infinity. Hence we get

$$\lim_n \left[\frac{1}{|J_n|} \sum_{j \in J_n} T_{j,j} - \frac{1}{|I_n|} \sum_{i \in I_n} T_{a(i),a(i)} \right] = 0$$

which implies (39). $\square$

Remark 8.2. The same condition (39) guarantees the preservation of equivalence classes of frames, that is for all $\mathcal{F}^1, \mathcal{F}^2 \in \mathcal{F}[J]$

$$\mathcal{F}^1 \approx_J \mathcal{F}^2 \quad \text{if and only if} \quad a_*(\mathcal{F}^1) \approx_I a_*(\mathcal{F}^2).$$

Thus, in general, an arbitrary frame measure function on $\mathcal{F}[I]$, $m : \mathcal{F}[I] \to \mathcal{C}^*(M)$, induces a measure function on $\mathcal{F}[J]$, $a^*(m) : \mathcal{F}[J] \to \mathcal{C}^*(M)$ via $a^*(m)(\mathcal{F}) = m(a_*(\mathcal{F}))$.

8.2. Indexing preserving non-expansiveness

Now we examine when non-expansive operators are pulledback through a_* into non-expansive operators. We use the same setting as before where now (I, d) and (J, e) are assumed to be quasi-metric index sets and $a : I \to J$ is the bijection. We have the following result.

Proposition 8.3. *Suppose there exists a function* $r : [0, \infty) \to [0, \infty)$ *such that*

$$\forall j_1, j_2 \in J \quad d\big(a^{-1}(j_1), a^{-1}(j_2)\big) < r\big(e(j_1, j_2)\big). \tag{42}$$

Then if $T \in B(l^2(J))$ *is non-expansive, then* $a_*(T)$ *is non-expansive in* $B(l^2(I))$.

Proof. Assume that T is non-expansive and choose an arbitrary $\varepsilon > 0$. Set $N = N_\varepsilon$ from the non-expansive definition for T, then

$$\sum_{i' \in I, \, d(i,i') > r(N)} \big|\langle a_*(T)\delta_i, \delta_{i'} \rangle\big|^2 = \sum_{j' \in J, \, d(i,a^{-1}(j)) > r(N)} \big|\langle T\epsilon_{a(i)}, \epsilon_{j'} \rangle\big|^2$$

$$\leqslant \sum_{j' \in J, \, e(a(i),j') > N} \big|\langle T\epsilon_{a(i)}, \epsilon_{j'} \rangle\big|^2 < \varepsilon.$$

A similar argument holds for T^* and thus $a_*(T)$ is non-expansive. $\quad\square$

Remark 8.4. An immediate consequence of this result is that if $\mathcal{F} \in \mathcal{F}[J]$ is non-expansive then $a_*(\mathcal{F})$ is non-expansive as well.

Remark 8.5. If the two quasi-metric spaces (I, d) and (J, e) satisfy the assumption of Proposition 8.3, then one can always choose a continuous and monotonically increasing r in (42).

8.3. A consequence

Now we can put together Theorem 7.14, and Propositions 8.1, 8.3, and obtain the following.

Theorem 8.6. *Assume* $(I, d, (I_n)_n)$ *is a uniform metric index set and* $(J, e, (J_n)_n)$ *is so that* (J, e) *is a quasi-metric index set. Assume* $a : I \to J$ *is a bijection that satisfies*

$$\lim_n \frac{|a(I_n) \cap J_n|}{|I_n|} = \lim_n \frac{|J_n|}{|I_n|} = 1 \tag{43}$$

and there exists a function $r : [0, \infty) \to [0, \infty)$ such that

$$\forall j_1, j_2 \in J \quad d\big(a^{-1}(j_1), a^{-1}(j_2)\big) < r\big(e(j_1, j_2)\big). \tag{44}$$

Assume $\mathcal{F}^1 \in \mathcal{F}[I]$ is non-expansive with respect to the quasi-metric index set (I, d) and $\mathcal{F}^2 \in \mathcal{F}[J]$ is non-expansive with respect to the quasi-metric index set (J, e).

Then, if $\mathcal{F} = \{f_i^1 \oplus f_{a(i)}^2; \ i \in I\}$ is frame (that is, $(\mathcal{F}^1, a_*(\mathcal{F}^2))$ is a superframe) then $\mathcal{F}$ is nonexpansive with respect to (I, d) and

$$\mu(\mathcal{F})(p) = \mu\big(\mathcal{F}^1\big)(p) + \mu\big(\mathcal{F}^2\big)(p), \quad \forall p \in \mathbf{N}^*. \tag{45}$$

Explicitly, for every free ultrafilter $p \in \mathbf{N}^*$,

$$\mu(\mathcal{F})(p) = p\text{-}\lim \frac{1}{|I_n|} \sum_{i \in I_n} \langle f_i^1, \tilde{f}_i^1 \rangle + p\text{-}\lim \frac{1}{|J_n|} \sum_{j \in J_n} \langle f_j^2, \tilde{f}_j^2 \rangle. \tag{46}$$

This statement can be straightforwardly extended to a finite collection of frames that form a superframe.

One can replace the free ultrafilter frame measure function μ by any other frame measure function m on $\mathcal{F}[I]$; consequently, in this case we have:

$$m(\mathcal{F})(x) = m\big(\mathcal{F}^1\big)(x) + a^*(m)\big(\mathcal{F}^2\big)(x). \tag{47}$$

9. Application to Gabor frames and superframes

In this section, we apply our results to Gabor frames and superframes. We begin with some added notation and preliminaries.

For a function $g \in L^2(\mathbf{R}^m)$, a point $\lambda = (t, \omega) \in \mathbf{R}^m \times \mathbf{R}^m$, and a phase $\varphi_\lambda \in \mathbf{R}$ denote by $g_\lambda(x) = e^{i\varphi_\lambda} e^{2\pi i \langle \omega, x \rangle} g(x - t)$ the λ-time–frequency shift of g.

Definition 9.1. Given a function $g \in L^2(\mathbf{R}^m)$ and a set of time–frequency shifts $\Lambda \subset \mathbf{R}^m \times \mathbf{R}^m$, and phases $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ define the *Gabor set* $\mathcal{G}(g, \Lambda) = \{g_\lambda\}_{\lambda \in \Lambda}$. A *Gabor frame* is a Gabor set that is a frame sequence.

For ease of notation we will omit the explicit mention of the phase system $\{\varphi_\lambda\}_\lambda$.

We define $Q_n(c) = \{\lambda \in \mathbf{R}^{2m} \mid \|\lambda - c\|_\infty \leq \frac{n}{2}\}$ to be the box inside $\mathbf{R}^m \times \mathbf{R}^m$ centered at $c \in \mathbf{R}^{2m}$ and of size length n.

Given a Gabor set $\mathcal{G}(g, \Lambda)$, the most natural way of indexing is given by the set Λ itself. Thus $(\Lambda, \|\cdot\|_\infty)$ becomes a quasi-metric index set. Note that $\|\cdot\|_\infty$ may not be a distance because we allow repetitions of the same time–frequency point in Λ.

We need to define the nested sequence of finite subsets $(\Lambda_n)_n$. Fix a center $O \in \mathbf{R}^{2m}$ (not necessarily the origin). It turns out that the natural choice of $\Lambda_n = Q_n(O) \cap \Lambda$ is not suitable for measuring Gabor frames. To fix this issue we instead replace $Q_n(O)$ by a "skewed" tile $M Q_n(O)$, where M is a suitable $2m \times 2m$ invertible matrix. We can do this either by simply defining $\Lambda_n = (M Q_n(O)) \cap \Lambda$, or by changing the distance in $\mathbf{R}^{2m}$ and replacing $\|x\|_\infty$ by

$\|x\|_{M,\infty} := \|M^{-1}x\|_\infty$. The two approaches are equivalent. However for simplicity of computations we will adopt the former approach, namely we keep the $\|\cdot\|_\infty$ distance in $\mathbf{R}^{2m}$ and define $\Lambda_n = (MQ_n(O)) \cap \Lambda$.

We will compute the free ultrafilter frame measure function of $\mathcal{G}(g, \Lambda)$ with respect to partition $(\Lambda_n)_n$. We will show that $(\Lambda, \|\cdot\|_\infty, (\Lambda_n)_n)$ is a uniform metric index set, and $\mathcal{G}(g, \Lambda)$ is non-expansive. Next we compute the frame measure function from Gabor superframes and obtain a necessary density type condition.

9.1. Free ultrafilter frame measure function of Gabor frames

Let us consider a Gabor frame $\mathcal{G}(g, \Lambda)$. Then the upper and lower Beurling densities of Λ, $D_B^+(\Lambda)$, and $D_B^-(\Lambda)$, satisfy (see the historical note [24] of this result)

$$1 \leqslant D_B^-(\Lambda) \leqslant D_B^+(\Lambda) < \infty,$$

where

$$D_B^+(\Lambda) = \limsup_n \sup_{c \in \mathbf{R}^{2m}} \frac{|\Lambda \cap Q_n(c)|}{n^{2m}}, \qquad D_B^-(\Lambda) = \liminf_n \inf_{c \in \mathbf{R}^{2m}} \frac{|\Lambda \cap Q_n(c)|}{n^{2m}}.$$

In particular this means there is a size $L_0 > 0$ and an integer $U_0 \geqslant 1$ so that every box of side length L_0 in $\mathbf{R}^{2m}$ contains at least one point of Λ and at most U_0 points of Λ. Fix a point $O \in \mathbf{R}^{2m}$, an invertible matrix M in $\mathbf{R}^{2m \times 2m}$ and let $\Lambda_n = \Lambda \cap MQ_n(O)$ as before. For any length R, the box $Q_n(O)$ is covered by at most $(\frac{n}{R} + 1)^{2m}$ boxes of side length R, and includes at least $(\frac{n}{R} - 1)^{2m}$ disjoint boxes of side length R. For the skewed box $MQ_n(O)$ the situation is the following. There are two numbers $c_1(M)$ and $c_2(M)$ depending on the matrix M so that, at most $c_1(M)(\frac{n}{R})^{2m} + c_2(M)(\frac{n}{R})^{2m-1}$ boxes are needed to cover $MQ_n(O)$, and at least $c_1(M)(\frac{n}{R})^{2m} - c_2(M)(\frac{n}{R})^{2m-1}$ disjoint boxes of side length R are included inside $MQ_n(O)$. With this set up we have the following.

Theorem 9.2. *The collection $(\Lambda, \|\cdot\|_\infty, (\Lambda_n)_n)$ is a uniform metric index set.*

Proof. $(\Lambda, \|\cdot\|_\infty)$ has finite upper density since every ball of radius R contains at most $(\frac{2R}{L_0} + 1)^{2m}$ boxes of side length L_0, and every box of side length L_0 has at most U_0 points. The second condition (27) is proved as follows. On the one hand for large n, each Λ_n has the cardinal bounded by

$$c_1(M)\left(\frac{n}{L_0}\right)^{2m} - c_2(M)\left(\frac{n}{L_0}\right)^{2m-1} \leqslant |\Lambda_n| \leqslant \left(c_1(M)\left(\frac{n}{L_0}\right)^{2m} + c_2(M)\left(\frac{n}{L_0}\right)^{2m-1}\right)U_0.$$

On the other hand,

$$\bigcup_{j \in \Lambda \setminus \Lambda_n} B_R(j) \cap \Lambda_n = \left(M\left(Q_n(O) \setminus Q_{n-R}(O)\right)\right) \cap \Lambda.$$

Hence

$$\left| \bigcup_{j \in \Lambda \setminus \Lambda_n} B_R(j) \cap \Lambda_n \right| \leqslant \left(\left(c_1(M) \left(\frac{n}{L_0} \right)^{2m} + c_2(M) \left(\frac{n}{L_0} \right)^{2m-1} \right) \right.$$
$$\left. - \left(c_1(M) \left(\frac{n-2R}{L_0} \right)^{2m} - c_2(M) \left(\frac{n-2R}{L_0} \right)^{2m-1} \right) \right) U_0.$$

Putting these two estimates together we obtain

$$\lim_{n \to \infty} \frac{|\bigcup_{j \in \Lambda \setminus \Lambda_n} B_R(j) \cap \Lambda_n|}{|\Lambda_n|} = 0. \qquad \square$$

Consider a Gabor frame $\mathcal{G}(g, \Lambda)$ for $L^2(\mathbf{R}^m)$. Fix a point $O \in \mathbf{R}^{2m}$, an invertible matrix $M \in \mathbf{R}^{2m \times 2m}$, and set $\Lambda_n = \Lambda \cap M Q_n(O)$ as before. For any free ultrafilter $p \in \mathbf{N}^*$, the set Λ has density:

$$D(\Lambda; p, M) = p\text{-lim} \frac{|\Lambda_n|}{\text{vol}(M Q_n(O))} = p\text{-lim} \frac{|\Lambda \cap (M Q_n(O))|}{\det(M) n^{2m}}. \qquad (48)$$

We recall a fundamental result obtained in [2,3].

Theorem 9.3. *(See [2].) Assume $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbf{R}^m)$ and $\{\tilde{g}_\lambda;\ \lambda \in \Lambda\}$ is its canonical dual frame. Then for any free ultrafilter $p \in \mathbf{N}^*$,*

$$p\text{-lim} \frac{1}{|\Lambda_n|} \sum_{\lambda \in \Lambda_n} \langle g_\lambda, \tilde{g}_\lambda \rangle = \frac{1}{D(\Lambda; p, M)}. \qquad (49)$$

The fact that we use skewed boxes instead of regular boxes does not affect the result. As we mentioned earlier, we can change the metric to account for the skewness and directly apply the results of [2,3].

This fundamental relation gives us a simple way to compute the free ultrafilter frame measure function of irregular Gabor frames (compare to [3, Theorem 3]):

Theorem 9.4. *For any Gabor frame $\mathcal{G}(g, \Lambda)$ and indexing $(\Lambda, (\Lambda_n)_n)$ as before, the free ultrafilter frame measure function is*

$$\mu(\mathcal{G})(p) = \frac{1}{D(\Lambda; p, M)}, \quad \forall p \in \mathbf{N}^*. \qquad (50)$$

Remark 9.5. If Λ has uniform density D_0 (that is $D_B^-(\Lambda) = D^+(\Lambda) = D_0$) then $\mu(\mathcal{G}) = \frac{1}{D_0} 1_{\mathbf{N}^*}$, that is, the measure function of $\mathcal{G}$ is the constant function $\frac{1}{D_0}$, independent of the matrix M. In fact, for any measure function $m : \mathcal{F}[\Lambda] \to \mathcal{C}^*(W)$ the measure of $\mathcal{G}$ is $m(\mathcal{G}) = \frac{1}{D_0} 1_W$.

For $\Lambda = A\mathbf{Z}^{2m}$ for some invertible matrix A, then $D_0 = \frac{1}{\det(A)}$ regardless of matrix M, and thus $m(\mathcal{G}) = (\det(A)) 1_W$. In particular, for $\Lambda = \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m$, $D_0 = \frac{1}{(\alpha\beta)^m}$ and $m(\mathcal{G}) = (\alpha\beta)^m 1_W$.

9.2. Non-expansiveness of Gabor frames

Consider now a Gabor frame $\mathcal{G}(\gamma, \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m)$, where $0 < \alpha, \beta < 1$ and $\gamma(x) = \exp(-\|x\|_2^2)$. The choice of α, β will be irrelevant, but as an example the reader may think of the case $\alpha = \beta = \frac{1}{2}$. Let $\tilde{\gamma}$ denote its canonical dual frame generator. Let E denote the upper frame bound of $\mathcal{G}(\tilde{\gamma}, \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m)$. For two functions $f, h \in L^2(\mathbf{R}^m)$, we denote by

$$V_f h : \mathbf{R}^{2m} \to \mathbf{C}, \qquad V_f h(\lambda) = \langle h, f_\lambda \rangle$$

the windowed Fourier transform of h with respect to f. The modulation spaces M^p, $1 \leqslant p \leqslant 2$, are defined by (see [21])

$$M^p = \left\{ f \in L^2(\mathbf{R}^m) \mid V_\gamma f \in L^p(\mathbf{R}^{2m}) \right\}, \quad \|f\|_{M^p} := \|V_\gamma f\|_{L^p}.$$

In particular $\gamma, \tilde{\gamma}$ are both in M^1. Note $M^2 = L^2$ as sets, and the norms are equivalent. The Wiener amalgam space $W(C, l^p)$ is defined by

$$W(C, l^p) = \left\{ f; \ f : \mathbf{R}^b \to \mathbf{C}, \ f \text{ continuous}, \ \|f\|_{W(C,l^p)}^p := \sum_{k \in \mathbf{Z}^b} \sup_{x \in Q_1(k)} |f(x)|^p < \infty \right\}.$$

The following result is proved in [9, Proposition A.3]. For all $f \in L^2(\mathbf{R}^m)$, $V_\gamma f \in W(C, l^2)$ and

$$\|V_\gamma f\|_{W(C,l^2)} \leqslant C \|\gamma\|_{M^1} \|f\|_2 \tag{51}$$

where the constant C can be chosen as $C = 3^{m/2}$. We can now prove the following.

Theorem 9.6. *Assume $\mathcal{G}(g, \Lambda)$ is a Gabor frame in $L^2(\mathbf{R}^m)$. Then $\mathcal{G}(g, \Lambda)$ is non-expansive with respect to the quasi-metric index set $(\Lambda, \|\cdot\|_\infty)$.*

Proof. We will show the Gram operator of $\mathcal{G}$ is non-expansive, and then the conclusion follows from Corollary 7.10.

We start with the following decomposition:

$$\langle g_{\lambda_1}, g_{\lambda_2} \rangle = \sum_{k, j \in \mathbf{Z}^m} \langle g_{\lambda_1}, \gamma_{\alpha k, \beta j} \rangle \langle \tilde{\gamma}_{\alpha k, \beta j}, g_{\lambda_2} \rangle = (AB)_{\lambda_1, \lambda_2},$$

where $A : l^2(\alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m) \to l^2(\Lambda)$, $B : l^2(\Lambda) \to l^2(\alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m)$, are defined through $A_{\lambda, (\alpha k, \beta j)} = \langle g_\lambda, \gamma_{\alpha k, \beta j} \rangle$, $B_{(\alpha k, \beta j), \lambda} = \langle \tilde{\gamma}_{\alpha k, \beta j}, g_\lambda \rangle$. A and B are bounded operators since they are compositions of analysis and synthesis operators associated to frames $\mathcal{G}(g, \Lambda)$, $\mathcal{G}(\gamma, \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m)$ and $\mathcal{G}(\tilde{\gamma}, \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m)$. Note

$$|A_{\lambda, (\alpha k, \beta j)}| = |V_\gamma g((\alpha k, \beta j) - \lambda)|, \qquad |B_{(\alpha k, \beta j), \lambda}| = |V_{\tilde{\gamma}} g(\lambda - (\alpha k, \beta j))|.$$

Consider the map $a : \Lambda \to \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m$, $a(\lambda) = (\zeta_k \lfloor \frac{\lambda_k}{\zeta_k} \rfloor)_{1 \leqslant k \leqslant 2m}$, where $\lambda = (\lambda_k)_{1 \leqslant k \leqslant 2m}$, $\zeta_k = \alpha$ for $1 \leqslant k \leqslant m$, $\zeta_k = \beta$ for $m + 1 \leqslant k \leqslant 2m$, and $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. Thus $\|a(\lambda) - \lambda\|_\infty < 1$.

Recall that $V_\gamma g$ and $V_{\tilde{\gamma}} g$ are both in $W(C, l^2)$. Combining this fact to the fact that every box of size length L_0 has at most U_0 points (see previous subsection), we obtain that, for every $\rho > 0$ there are $N_A(\rho), N_B(\rho) > 0$ so that

$$\forall r \in \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m, \qquad \sum_{\lambda \in \Lambda \backslash Q_{N_A}(r)} \left| V_\gamma g(r - \lambda) \right|^2 < \rho; \tag{52}$$

$$\forall \lambda \in \Lambda, \qquad \sum_{k, j \in \mathbf{Z}^m, \, \|(\alpha k, \beta j) - a(\lambda)\|_\infty > N_B(\rho)} \left| V_{\tilde{\gamma}} g\big(\lambda - (\alpha k, \beta j)\big) \right|^2 < \rho. \tag{53}$$

Fix $\varepsilon > 0$. We will find $N = N_\varepsilon > 0$ so that for all $\lambda \in \Lambda$,

$$\sum_{v \in \Lambda \backslash B_N(\lambda)} \left| \langle g_v, g_\lambda \rangle \right|^2 < \varepsilon. \tag{54}$$

Since the Gram operator is symmetric, this will conclude the proof.

The remainder of the proof mirrors the argument used in Theorem 7.7 that shows that non-expansiveness is preserved under multiplication.

Let $\varepsilon_B = \frac{\varepsilon}{4\|A\|^2}$ and $N_B = N_B(\varepsilon_B)$ as in (53), $\varepsilon_A = \frac{\varepsilon}{4E\|g\|^2} (\frac{\alpha\beta}{(2N_B+1)^2})^m$ and $N_A = N_A(\varepsilon_A)$ the associated integer that satisfies (52). Set $N = N_A + N_B + 1$. We prove this choice satisfies (54).

Let $(\delta_\lambda)_\lambda$ denote the sequence whose entries are zero except for the λth entry which is one. Thus $\{\delta_\lambda; \ \lambda \in \Lambda\}$ is the canonical orthonormal basis of $l^2(\Lambda)$. Note for all $v, \lambda \in \Lambda$, $(AB)_{\lambda,v} = \langle AB\delta_v, \delta_\lambda \rangle$.

Fix a $\eta \in \Lambda$. Let $v, w \in l^2(\alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m)$ denote the vectors of $B\delta_\eta = v + w$, where all entries of $v = (v_{\alpha k, \beta j})$ vanish for $\|(\alpha k, \beta j) - a(\eta)\|_\infty < N_B$, and all entries of $w = (w_{\alpha k, \beta j})$ vanish for $\|(\alpha k, \beta j) - a(\eta)\|_\infty \geqslant N_B$. By (53) we obtain $\|v\|_{l^2}^2 < \varepsilon_B$, and hence $\|Av\|_{l^2}^2 \leqslant \frac{\varepsilon}{4}$. Now we have:

$$T := \sum_{\lambda \in \Lambda \backslash B_N(\eta)} \left| (AB)_{\lambda,\eta} \right|^2 = \sum_{\lambda \in \Lambda \backslash B_N(\eta)} \left| \langle Av, \delta_\lambda \rangle + \sum_{\substack{r \in \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m \\ \|r - a(\eta)\|_\infty < N_B}} A_{\lambda,r} B_{r,\eta} \right|^2,$$

$$T \leqslant 2 \sum_{\lambda \in \Lambda} \left| \langle Av, \delta_\lambda \rangle \right|^2 + 2 \sum_{\lambda \in \Lambda \backslash B_N(\eta)} \left| \sum_{\substack{r \in \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m \\ \|r - a(\eta)\|_\infty < N_B}} A_{\lambda,r} B_{r,\eta} \right|^2 \tag{55}$$

$$\leqslant \frac{\varepsilon}{2} + 2 \sum_{\lambda \in \Lambda \backslash B_N(\eta)} \left(\sum_{\substack{r \in \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m \\ \|r - a(\eta)\|_\infty < N_B}} 1 \right) \left(\sum_{\substack{r \in \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m \\ \|r - a(\eta)\|_\infty < N_B}} |A_{\lambda,r} B_{r,\eta}|^2 \right) \tag{56}$$

$$\leqslant \frac{\varepsilon}{2} + 2 \left(\frac{2N_B^2}{\alpha\beta} \right)^m \sum_{\substack{r \in \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m \\ \|r - a(\eta)\|_\infty < N_B}} |B_{r,\eta}|^2 \sum_{\lambda \in \Lambda \backslash B_N(\eta)} |A_{\lambda,r}|^2 \tag{57}$$

$$\leqslant \frac{\varepsilon}{2} + 2 \left(\frac{(2N_B + 1)^2}{\alpha\beta} \right)^m E\|g\|^2 \varepsilon_A = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{58}$$

where the last inequality follows from $\Lambda \setminus B_N(\eta) \subset \Lambda \setminus Q_{N_A}(r)$, for all $r \in \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m$ with $\|r - a(\eta)\|_\infty < N_B$, and (52). This proves (54) and thus the statement. $\square$

Remark 9.7. In terminology of [2], (52) means $(\mathcal{G}(g, \Lambda), a, \mathcal{G}(\gamma, \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m))$ has l^2-column decay, whereas (53) means that $(\mathcal{G}(g, \Lambda), a, \mathcal{G}(\tilde{\gamma}, \alpha \mathbf{Z}^m \times \beta \mathbf{Z}^m))$ has l^2-row decay.

Using the terminology from [2], Theorem 9.6 states that $(\mathcal{G}(g, \Lambda), a)$ is l^2-self-localized, and l^2-localized with respect to its canonical dual frame.

9.3. Measure functions of Gabor superframes

Consider now two Gabor frames $\mathcal{G}(g, \Lambda)$ and $\mathcal{H}(h, \Sigma)$ in $L^2(\mathbf{R}^m)$. Assume there is a bijection $a : \Lambda \to \Sigma$ so that $(\mathcal{G}(g, \Lambda), \mathcal{H}(h, \Sigma))$ is a superframe, that is

$$\mathcal{F} = \{g_\lambda \oplus h_{a(\lambda)}; \ \lambda \in \Lambda\} \tag{59}$$

is frame for $L^2(\mathbf{R}^m) \oplus L^2(\mathbf{R}^m)$. Note $\mathcal{F} \in \mathcal{F}[\Lambda]$.

Proposition 9.8. *Assume $(\mathcal{G}(g, \Lambda), \mathcal{H}(h, \Sigma))$ is a Gabor superframe with respect to the correspondence $a : \Lambda \to \Sigma$. Assume there are invertible matrices $M^1, M^2 \in \mathbf{R}^{2m \times 2m}$ so that the map a satisfies*

$$\lim_n \frac{|a^{-1}(\Sigma \cap (M^2 Q_n(O))) \cap (M^1 Q_n(O))|}{|\Lambda \cap (M^1 Q_n(O))|} = \lim_n \frac{|\Sigma \cap (M^2 Q_n(O))|}{|\Lambda \cap (M^1 Q_n(O))|} = 1 \tag{60}$$

and there exists a function $r : [0, \infty) \to [0, \infty)$ such that for all $\sigma_1, \sigma_2 \in \Sigma$,

$$\|a^{-1}(\sigma_1) - a^{-1}(\sigma_2)\| \leqslant r(\|\sigma_1 - \sigma_2\|). \tag{61}$$

Then the direct sum frame $\mathcal{F}$ defined in (59) has the free ultrafilter frame measure:

$$\mu(\mathcal{F})(p) = \frac{1}{D(\Lambda; p, M^1)} + \frac{1}{D(\Sigma; p, M^2)}, \quad \forall p \in \mathbf{N}^*. \tag{62}$$

In particular, the following is a necessary condition:

$$\limsup_n \left(\frac{\det(M^1)}{|\Lambda \cap (M^1 Q_n(O))|} + \frac{\det(M^2)}{|\Sigma \cap (M^2 Q_n(O))|} \right) n^{2m} \leqslant 1. \tag{63}$$

Proof. Note (60) and (61) imply that a satisfies (38) and (42). Now (62) follows from Theorems 8.6, 9.4, and 9.6. Equation (63) is obtained from (62), and (48), and the fact that for any frame $\mathcal{F}$, $\mu(\mathcal{F})(p) \leqslant 1$ for all p. $\square$

Remark 9.9. Let $L_\Sigma > 0$ be such that any box of side length L_Σ in $\mathbf{R}^{2m}$ contains at least one point of Σ. Then condition (61) can be replaced equivalently by the following boundedness condition:

$$\exists R_0 > 0, \ \forall \sigma_1, \sigma_2 \in \Sigma \quad \|\sigma_1 - \sigma_2\| \leqslant \sqrt{2m} L_\Sigma \quad \Rightarrow \quad \|a^{-1}(\sigma_1) - a^{-1}(\sigma_2)\| \leqslant R_0. \tag{64}$$

Indeed, if (64) holds true then for any $N > 0$ there is a chain of $\frac{N}{\sqrt{2m}L_\Sigma}$ points in Σ so that the distance between any two adjacent points is at most $\sqrt{2m}L_\Sigma$. Using the triangle inequality it follows that (61) is satisfied with $r(u) = (1 + \frac{u}{\sqrt{2m}L_\Sigma})R_0$.

Using induction one can immediately prove:

Theorem 9.10. *Assume $\mathcal{G}(g^k, \Lambda_k)$, $1 \leqslant k \leqslant d$, are Gabor frames in $L^2(\mathbf{R}^m)$ so that for maps $a_k : \Lambda_1 \to \Lambda_k$, $2 \leqslant k \leqslant d$, the set $\mathcal{F} = \{g_\lambda^1 \oplus g_{a_2(\lambda)}^2 \oplus \cdots \oplus g_{a_d(\lambda)}^d;\ \lambda \in \Lambda_1\}$ is frame for $L^2(\mathbf{R}^m) \oplus \cdots \oplus L^2(\mathbf{R}^m)$. Assume further that there are invertible matrices M^k, $1 \leqslant k \leqslant d$, such that all maps a_k satisfy*

$$\lim_n \frac{|a_k^{-1}(\Lambda_k \cap (M^k Q_n(O))) \cap (M^1 Q_n(O))|}{|\Lambda_1 \cap (M^1 Q_n(O))|} = \lim_n \frac{|\Lambda_k \cap (M^k Q_n(O))|}{|\Lambda_1 \cap (M^1 Q_n(O))|} = 1 \qquad (65)$$

and there exists a map $r : [0, \infty) \to [0, \infty)$ such that for all $\sigma_1, \sigma_2 \in \Lambda_k$,

$$\left\| a_k^{-1}(\sigma_1) - a_k^{-1}(\sigma_2) \right\| \leqslant r\left(\|\sigma_1 - \sigma_2\|\right). \qquad (66)$$

Then the free ultrafilter frame measure function of $\mathcal{F}$ is given by

$$\mu(\mathcal{F})(p) = \frac{1}{D(\Lambda_1; p, M^1)} + \cdots + \frac{1}{D(\Lambda_d; p, M^d)}, \qquad p \in \mathbf{N}^*. \qquad (67)$$

In particular it follows that necessarily

$$\frac{1}{D(\Lambda_1; p, M^1)} + \cdots + \frac{1}{D(\Lambda_d; p, M^d)} \leqslant 1, \qquad \forall p \in \mathbf{N}^*. \qquad (68)$$

In the special case of regular Gabor frames, $\Lambda_k = \{A_k n;\ n \in \mathbf{Z}^{2m}\}$, $1 \leqslant k \leqslant d$, we obtain that if $(\mathcal{G}(g_1; \Lambda_1), \ldots, \mathcal{G}(g_d; \Lambda_d))$ form a superframe with respect to the maps $a_k : \Lambda_1 \to \Lambda_k$, $a_k(A_1 n) = A_k n$, $2 \leqslant k \leqslant d$, then conditions (65) and (66) are satisfied with $M^k = A_k$, and we obtain immediately the following result which recovers and extends the result of [5].

Corollary 9.11. *Assume $g^1, \ldots, g^d \in L^2(\mathbf{R}^m)$ and $A_1, \ldots, A_d \in \mathbf{R}^{2m \times 2m}$ are so that $\mathcal{F} = \mathcal{G}(g^1, A_1\mathbf{Z}^{2m}) \oplus \cdots \oplus \mathcal{G}(g^d, A_d\mathbf{Z}^{2m})$ is frame for $L^2(\mathbf{R}^m) \oplus \cdots \oplus L^2(\mathbf{R}^m)$, then for any frame measure function $m : \mathcal{F}[\mathbf{Z}^{2m}] \to \mathcal{C}^*(W)$,*

$$m(\mathcal{F}) = \left(\det(A_1) + \cdots + \det(A_d)\right) 1_W. \qquad (69)$$

Consequently, as a necessary condition to have a superframe,

$$\det(A_1) + \cdots + \det(A_d) \leqslant 1. \qquad (70)$$

10. Redundancy

The word *redundancy* is often used to describe, qualitatively, the overcompleteness of frames. However, for frames with an infinite number of elements, there is no quantitative definition of redundancy. Here, we propose that the reciprocal of a frame measure function should be the quantitative definition of redundancy.

Definition 10.1. Given a measure function $m : \mathcal{F}[I] \to C^*(M)$, we define the *redundancy function* $R : \mathcal{F}[I] \to$ {functions from M to $\mathbf{R} \cup \infty$}, $R(\mathcal{F})(x) = (m(\mathcal{F})(x))^{-1}$. In the case when the measure function is the ultrafilter measure function, we call the redundancy function the *ultrafilter redundancy function*.

The rest of this section discusses the justification for this definition. We begin by listing a series of properties of the frame redundancy function, all of which mesh well with the qualitative notion of redundancy:

- We immediately have the desirable properties that for a frame, the redundancy function is greater than or equal to one with the redundancy function equal to one for any Riesz basis.
- By Theorem 9.4, for any Gabor frame $\mathcal{G}(g, \Lambda)$ and indexing $(\Lambda, (\Lambda_n)_n)$ as in Section 9, the ultrafilter redundancy function corresponds to the density of the time frequency shifts as follows:

$$R\big(\mathcal{G}(g, \Lambda)(p)\big) = D(\Lambda; p, M), \quad \text{for all free ultrafilters } p. \tag{71}$$

- This connection between redundancy and measure function extends to localized frames. Using the notation and results from [2] we have an explicit description of the ultrafilter redundancy function. Assume $\mathcal{F} \in \mathcal{F}[I]$ is a frame for H and $a : I \to \mathbf{Z}^d$ is a map so that $(\mathcal{F}, a, \mathcal{E})$ has both l^2-column and l^2-row decay (see [2] for definition), where $\mathcal{E} = \{e_k; \ k \in \mathbf{Z}^d\}$ is another frame for H. Set $I_n = a^{-1}(Q_n(0))$, where $Q_n(0)$ is the box of side length n centered at 0 in $\mathbf{Z}^d$, and consider the ultrafilter redundancy functions associated to $(I, (I_n)_n)$, respectively $(\mathbf{Z}^d, (Q_n(0))_n)$. Then Theorem 5 in [2] implies:

$$R(\mathcal{F})(p) = D(a; p)R(\mathcal{E})(p). \tag{72}$$

In particular, if $\mathcal{E}$ is a Riesz basis for H, then $R(\mathcal{E}) = 1$ and the previous equation turns simply into:

$$R(\mathcal{F})(p) = D(a; p). \tag{73}$$

- In these cases (Gabor and localized frames), the redundancy function is additive on unions of frames. Suppose $\mathcal{F}^1 \in \mathcal{F}[I]$ and $\mathcal{F}^2 \in \mathcal{F}[J]$ are two frames for same Hilbert space H, and that there are maps $a^1 : I \to \mathbf{Z}^d$ and $a^2 : J \to \mathbf{Z}^d$ so that $(\mathcal{F}^1, a^1, \mathcal{E})$ and $(\mathcal{F}^2, a^2, \mathcal{E})$ have both l^2-column and l^2-row decay, where $\mathcal{E}$ is a Riesz basis for H. Set $I_n = (a^1)^{-1}(Q_n(0))$, and $J_n = (a^2)^{-1}(Q_n(0))$. Consider the ultrafilter redundancy functions associated to $(I, (I_n)_n)$, $(J, (J_n)_n)$, $(I \cup J, (I_n \dot{\cup} J_n)_n)$ for frames $\mathcal{F}^1$, $\mathcal{F}^2$, and $\mathcal{F}^1 \dot{\cup} \mathcal{F}^2$, respectively. Here $\dot{\cup}$ denotes union with multiplicity. First it is immediate to check that $(\mathcal{F}^1 \dot{\cup} \mathcal{F}^2, a, \mathcal{E})$ has l^2-column and l^2-row decay, where $a : I \dot{\cup} J \to \mathbf{Z}^d$, $a(i) = a^1(i)$ for

$i \in I$, and $a(j) = a^2(j)$ for $j \in J$. Next note that $I_n \overset{.}{\cup} J_n = a^{-1}(Q_n(0))$. Then, applying (73) to $\mathcal{F}^1 \overset{.}{\cup} \mathcal{F}^2$ we obtain:

$$R(\mathcal{F}^1 \overset{.}{\cup} \mathcal{F}^2)(p) = D(a; p) = D(a^1; p) + D(a^2; p) = R(\mathcal{F}^1)(p) + R(\mathcal{F}^2)(p) \quad (74)$$

which proves additivity of the redundancy function. Equation (74) can be immediately extended to any finite number of frames.

In addition to the above properties, the redundancy function can be seen as an analogue of redundancy in the finite-dimensional case. In finite dimensions, the idea of redundancy is quantified. Here we have a frame $\mathcal{F} = \{f_j\}_{j \in J}$, consisting of $M = |J|$ vectors. If we let N be the dimension of the space spanned by the elements of $\mathcal{F}$, then the ratio $r = \frac{M}{N}$ is a natural quantity that is often referred to as the redundancy of the frame $\mathcal{F}$. Another way to arrive at the quantity r is as follows. Associated to $\mathcal{F}$ is the finite-dimensional Gram operator $G : l^2(J) \to l^2(J)$ defined entry-wise by $G_{i,j} = \langle f_i, f_j \rangle$. The ratio of the dimension of the space $l^2(J)$ (which is $|J|$) to the dimension of the range of G is also $r = \frac{M}{N}$. In other words the reciprocal of the redundancy, $\frac{1}{r}$, is the normalized trace of the associated Gram projection of the frame.

So what is the meaning of the quantity r? In this setting we have that a frame $\mathcal{F}$ is a basis if and only if $r = 1$. If $\mathcal{F}$ is the union of two bases on the same space then $r = 2$, however this is not the only type of frame that has $r = 2$; a basis of size n along with n additional copies of the first basis element also has $r = 2$. Thus the value of r does not reveal the whole story, but it does provide a one parameter classification of frames. One can then examine the set of frames with a given r and try and understand the variation in their characteristics (see [10,12]). One can also design frames with a particular value of r that maximizes certain channel capacity or energy considerations [30,31].

If one tries to use the finite-dimensional case as a road map for defining redundancy in infinite dimensions, one immediately encounters difficulty. In this case, we are considering a frame $\mathcal{F} = \{f_i\}_{i \in I}$ indexed by an infinite set I. A first attempt might be to view the frame as some sort of limit of frames composed of finite subsets. This is problematic for several reasons: the limit may not exist, the sequence of finite frames may not have a uniform lower frame sequence bound, and the finite frames may all be Riesz bases for their span whereas the limiting frame may not. A second approach is to directly consider the infinite-dimensional setting: the quantities M and N that correspond to the finite case are both generically infinite ($M = |I|$ and N being the dimension of the span of the f_i) and therefore the ratio $r = \frac{M}{N}$ is meaningless. Similarly, attempting to compare the dimension of $l^2(I)$ to the dimension of the range of the Gram operator of $\mathcal{F}$, yields a comparison of two infinite quantities.

By itself, comparing the dimensions of infinite-dimensional spaces is not completely hopeless. Those familiar with the study of von Neumann algebras will recall that the dimension function, introduced by von Neumann, provides a way of comparing certain infinite-dimensional subspaces of a fixed infinite-dimensional space. In this case, only subspaces that are ranges of projections in the algebra are considered; the dimension function of the subspace is then defined to be the normalized trace (which exists on a von Neumann algebra) of the projection. This connection has yielded many nice results about Gabor frames on regular lattices [16,19,20,26,28] (just to name a few); in these cases the regular lattice structure was enough to ensure that the Gramian had the necessary structure to allow the tools of von Neumann algebras to be useful. In general, however, this added structure is not available and we are further discouraged by the known fact that there

does not exist a dimension function that is finite and non-zero on all non-zero subspaces of a fixed infinite-dimensional space.

As mentioned earlier, in finite dimensions the reciprocal of the redundancy can be defined as the trace of the associated Gram projection to the given frame. The ultrafilter redundancy function can be seen as the infinite-dimensional analogue of this. To begin with, the ultrafilter frame measure function is determined by certain averages of $\langle \tilde{f}_i, f_i \rangle$, that is, certain averages of the diagonal elements of the corresponding Gram projection—a natural generalization of the normalized trace in finite dimensions which is the average of the diagonal elements of the Gram projection. The key structural feature of a trace is that the trace of AB and BA are equal for operators A and B. This feature is present for measure functions on the set of non-expansive operators (Lemma 7.13).

For these reasons, we feel our definition is the proper quantification of redundancy in the infinite setting. There remain unanswered questions about the redundancy function, an important one being if a frame has redundancy c, does there exist a subset of the frame that is a frame for the same space with redundancy 1 (or $1 + \epsilon$ for any $\varepsilon > 0$).

Acknowledgments

Appendix A. Supersets

We recall the notion of superframe (see [4,5,7]) (or disjoint frames, as used by D. Larson, see [22]). Let $\mathcal{F}_1, \ldots, \mathcal{F}_L \in \mathcal{F}[I]$, a finite number of frames indexed by I.

Definition A.1. We call $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ a *superframe* if

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_L := \left\{ f_i^1 \oplus \cdots \oplus f_i^L; \ i \in I \right\} \tag{A.1}$$

is a frame in $H_1 \oplus \cdots \oplus H_L$, the direct sum of Hilbert spaces spanned by $\mathcal{F}_1, \ldots, \mathcal{F}_L$, respectively.

An equivalent characterization of superframes is given by the following.

Theorem A.2. *(See [7].) The collection $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ is a superframe if and only if the following two conditions hold true*:

(1) *Each $\mathcal{F}_l$ is frame, $1 \leqslant l \leqslant L$;*
(2) *$E_k \cap (\sum_{l \neq l} E_l) = \{0\}$, for $1 \leqslant k \leqslant L$, and $\sum_{k=1}^{L} E_l$ is closed (where E_l is the range in $l^2(I)$ of the analysis operator associated to $\mathcal{F}_l$).*

In particular, the second condition above holds true when the ranges of E_l are mutually orthogonal. This special case is called *orthogonal in the sense of supersets* (or strongly disjoint, see [22]). More specifically we define the following.

Definition A.3. Two frames $\mathcal{F}_1 = \{f_i^1; i \in I\}$ and $\mathcal{F}_2 = \{f_i^2; \ i \in I\}$ indexed by I are said to be *orthogonal in the sense of supersets* if E_1, the range of analysis operator associated to $\mathcal{F}_1$, is orthogonal in $l^2(I)$ to E_2, the range of coefficients associated to $\mathcal{F}_2$. Equivalently,

$$\sum_{i \in I} \langle g, f_i^1 \rangle \langle f_i^2, h \rangle = 0 \ , \quad \forall g \in H_1 \ , \ \forall h \in H_2. \tag{A.2}$$

Remark A.4. Clearly if two frames $\mathcal{F}_1$, $\mathcal{F}_2$ are orthogonal in the sense of supersets, then $E_1 \cap E_s = \{0\}$ and $E_1 + E_2$ is closed, hence $(\mathcal{F}_1, \mathcal{F}_2)$ is a superframe. Note that in this case the range of the analysis operator associated to $\mathcal{F}_1 \oplus \mathcal{F}_2$ is exactly $E_1 \oplus E_2$, and the associated Gram projection P, is given by $P = P_1 + P_2$, the sum of the associated Gram projections of $\mathcal{F}_1$ and $\mathcal{F}_2$. In particular, the canonical dual of $\mathcal{F}_1 \oplus \mathcal{F}_2$ is the direct sum of the canonical duals of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Remark A.5. For any frame $\mathcal{F} \in \mathcal{F}[I]$, one can always construct $\mathcal{F}' \in \mathcal{F}[I]$ that is orthogonal to $\mathcal{F}$ in the sense of supersets, so that $\mathcal{F} \oplus \mathcal{F}'$ becomes a Riesz basic sequence. Let P be the associated Gram projection to $\mathcal{F}$. Then $Q = 1 - P$ is also an orthogonal projection in $l^2(I)$ (1 being the identity operator). Set $\mathcal{F}' = \{Q\delta_i; \ i \in I\}$. One can easily check that $\mathcal{F}'$ is a (Parseval) frame and that its associated Gram projection is Q; therefore $\mathcal{F}$ and $\mathcal{F}'$ are orthogonal in the sense of supersets. Furthermore, $\mathcal{F} \oplus \mathcal{F}'$ is a Riesz basis for $\overline{\text{span}\{\mathcal{F}\}} \oplus \text{Ran}\, Q$.

Appendix B. Ultrafilters

Consider the difference between the limit of a sequence and the liminf of a sequence. The liminf has the advantage that it is defined on all bounded sequences as opposed to the limit which is only defined on the relatively small set of sequences that have limits. However, unlike the limit, the liminf is not linear on its domain.

The existence of ultrafilters leads to linear functionals (Definition B.2) that achieve "the best of both worlds" in the sense that they are defined and linear on all bounded sequences (Proposition B.3).

Definition B.1. A collection p of subsets of M is called a *filter* if it satisfies the following properties:

(1) The empty set is not in p: $\emptyset \notin p$;
(2) If $A_1, A_2 \in p$, then $A_1 \cap A_2 \in p$;
(3) If $A \subset B \subset M$ with $A \in p$ then $B \in p$.

A filter p is an *ultrafilter* if it is 'maximal' in the following sense:

(4) For all $A \subset M$ either $A \in p$ or $(M \setminus A) \in p$ (but not both because of (1) and (2) above).

An ultrafilter that does not contain a finite set is called a *free* ultrafilter; the set of free ultrafilters will be denoted by M^*.

The existence of free ultrafilters is unintuitive and requires the axiom of choice. For our purposes we shall be concerned with the case $M = \mathbf{N}$, and $\mathbf{N}^*$ denotes the set of free ultrafilters.

The existence of ultrafilters allows us to define a family of limits on bounded sequences indexed by M:

Definition B.2. Let $\mathbf{x} = \{x_m\}_{m \in M}$ be a bounded sequence of complex numbers. Given an ultrafilter p on M, we say x *converges to $c \in \mathbf{C}$ with respect to the ultrafilter p* and write $c = p\text{-lim } \mathbf{x}$, if for any $\varepsilon > 0$ there is a set $A \in p$ such that $|x_m - c| < \varepsilon$ for all $m \in A$.

This notion of limit has the following consequences that can be found in any text about ultrafilters (see [25] for example):

Proposition B.3. *Let* $\mathbf{x} = \{x_m\}_{m \in M}$, $\mathbf{y} = \{y_m\}_{m \in M}$ *be bounded sequences of complex numbers and let p be a free ultrafilter.*

(1) *p-lim $\mathbf{x}$ exists and is unique.*
(2) *The function p-lim is linear, i.e. $p\text{-lim}(a\mathbf{x} + b\mathbf{y}) = a(p\text{-lim } \mathbf{x}) + b(p\text{-lim } \mathbf{y})$ for all scalars a, b.*
(3) *For $M = \mathbf{N}$, the value of p-lim $\mathbf{x}$ is an accumulation point of the set $x_1, x_2, \ldots$ Consequently, if the sequence $x_1, x_2, \ldots$ has a limit, then p-lim $\mathbf{x}$ is equal to that limit.*

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