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STABILITY OF FRAMES WHICH GIVE PHASE RETRIEVAL

RADU BALAN

Abstract. In this paper we study the property of phase retrievability by redundant sysems of vectors under perturbations of the frame set. Specifically we show that if a set $\mathcal{F}$ of m vectors in the complex Hilbert space of dimension n allows for vector reconstruction from magnitudes of its coefficients, then there is a perturbation bound ρ so that any frame set within ρ from $\mathcal{F}$ has the same property. In particular this proves a recent construction for the case $m = 4n - 4$ is stable under perturbations. Additionally we provide estimates of the stability radius.

1. INTRODUCTION

The *phase retrieval* problem presents itself in many applications is physics and engineering. Recent papers on this topic (see [7, 14, 5, 6, 1, 11, 31]) present a full list of examples ranging from X-Ray crystallography to audio and image signal processing, classification with deep networks, quantum information theory, and fiber optics data transmission.

In this paper we consider the complex case, namely the Hilbert space $H = \mathbb{C}^n$ endowed with the usual Euclidian scalar product $\langle x, y\rangle = \sum_{k=1}^n x_k\overline{y_k}$. On H we consider the equivalence relation $\sim$ between two vectors $x, y \in H$ defined as follows; the vectors x and y are similar $x \sim y$ if and only if there is a complex constant z of unit magnitude, $|z| = 1$, so that $y = zx$. Let $\hat{H} = H/\sim$ be the quotient space. Thus an equivalence class (a *ray*) has the form $\hat{x} = \{e^{i\varphi}x\ ,\ \varphi \in [0, 2\pi)\}$. A subset $\mathcal{F} = \{f_i; i \in I\} \subset H$ of the Hilbert space H (regardless whether it is finite dimensional or not) is called *frame* if there exist two positive constants

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$0 < A \leq B < \infty$ (called *frame bounds*) so that for any vector $x \in H$,

$$A \left\| x \right\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq B \left\| x \right\|^2 \tag{1}$$

In the finite dimensional case considered in this paper, the frame condition simply reduces to $\mathcal{F}$ being a spanning set. Specifically $\mathcal{F} = \{f_1, \ldots, f_m\}$ is frame for H if and only if $H = span(\mathcal{F})$. Obviously $m \geq n$ must hold. When we can choose $A = B$ the frame is called *tight*. If furthermore $A = B = 1$ then $\mathcal{F}$ is said a *Parseval frame*. Consider the following nonlinear map

$$\beta : \hat{H} \rightarrow \mathbb{R}^m \ , \ (\beta(\hat{x}))_k = |\langle x, f_k \rangle|^2 \ , \ 1 \leq k \leq m \tag{2}$$

which is well defined on the classes $\hat{x}$ since $|\langle x, f_k\rangle|^2 = |\langle y, f_k\rangle|^2$ when $x \sim y$.

Definition 1. The frame $\mathcal{F} = \{f_1, \ldots, f_m\}$ is called *phase retrievable* (or we say it is a *frame that gives phase retrieval*) if the nonlinear map β is injective.

Notice that any signal $x \in H$ is uniquely defined by the magnitudes of its frame coefficients $\beta(x)$ up to a global phase factor, if and only if $\mathcal{F}$ is phase retrievable. The main result of this paper states that the phase retrievable property is stable under small perturbations of the frame set. Specifically we show

Theorem 1.1. *Assume $\mathcal{F} = \{f_1, \ldots, f_m\}$ is a phase retrievable frame for the complex Hilbert space $\mathbb{C}^n$. Then there is a $\rho > 0$ so that any set $\mathcal{F}' = \{f'_1, \ldots, f'_m\}$ with $\|f_k - f'_k\| < \rho$, $1 \leq k \leq m$, is also a phase retrievable frame.*

Let a_0 be the lower constant introduced in Lemma 3.2, and let B denote the upper frame bound. The following choice for ρ

$$\rho = \min\left(\frac{1}{\sqrt{m}}, \frac{a_0}{2\sqrt{2}(3B+2)^{3/2}}\right) \tag{3}$$

provides a lower constant $a_0(\mathcal{F}')$ uniformly bounded below by $\frac{a_0}{2}$, for such each perturbed frame $\mathcal{F}'$.

We prove this theorem in section 3. The proof is based on a recent necessary and sufficent condition obtained independently in [11] and [6]. The exact form of this result is slightly different than the equivalent results stated in the aforementioned papers. Consequently we will provide an additional proof. The authors of the recent paper [13] proved a similar stability result but for real frames. In fact they showed a more general result that covers the case of arbitrary projections instead of just rank-one projections considered here. Their argument can be applied to the complex case as well (cf. [17]). However in this paper we additionally provide an estimate of the stability radius.

An interesting problem on phase retrievable frames that is still open is to show the existence (and compute its value) of a cardinal number $m^*(\mathbb{C}^n)$ that has the following properties:

(A) For any $m \geq m^*(\mathbb{C}^n)$ the set of phase retrievable frames is generic (i.e. open and dense) with respect to the norm topology;
(B) If $\mathcal{F}$ is a phase retrievable frame of m vectors, then $m \geq m^*(\mathbb{C}^n)$.

Clearly (B) is equivalent to the following minimality property:

(C) If $m < m^*(\mathbb{C}^n)$ there is no frame $\mathcal{F}$ of m vectors that is also phase retrievable.

On the other hand denote by $m^0(\mathbb{C}^n)$ the smallest cardinal of a phase retrievable frame for $\mathbb{C}^n$. Obviously $m^0(\mathbb{C}^n) \geq n$. If $m^*(\mathbb{C}^n)$ exists then $m^*(\mathbb{C}^n) = m^0(\mathbb{C}^n)$ by property (C).

It is possible that the cardinal $m^*(\mathbb{C}^n)$ may exist only for some integers n. On the other hand $m^0(\mathbb{C}^n)$ always exists for any integer n. Here we prove that for any $m \geq m^0(\mathbb{C}^n)$ the set of phase retrievable frames is open in $(\mathbb{C}^n)^m$.

The current state-of-the-art on this problem is summarized by the following statements:

(1) (see [19]) If $m \geq 4n-4$ then any generical (with respect to the Zariski topology) frame is phase retrievable for $\mathbb{C}^n$;
(2) (see [12]) For any $n \geq 2$, $m^0(\mathbb{C}^n) \leq 4n-4$;
(3) (see [25]) For any $n \geq 2$,

$$m^0(\mathbb{C}^n) \geq 4n-2-2\beta + \begin{cases} 2 & if\ n\ odd\ and\ \beta = 3\,mod\,4 \\ 1 & if\ n\ odd\ and\ \beta = 2\,mod\,4 \\ 0 & otherwise \end{cases};$$

where $\beta = \beta(n)$ is the number of 1's in the binary expansion of $n-1$.
(4) (see [30]) For $n = 4$, $m^0(\mathbb{C}^4) = 11 = 4n-5$;
(5) (see [19]) For $n = 2^p+1$, $M^*(\mathbb{C}^n)$ exists and $m^*(\mathbb{C}^n) = m^0(\mathbb{C}^n) = 4n-4$.

Hence, if the critical cardinal $m^*(\mathbb{C}^n)$ exists, we know $4n - O(log(n)) \leq m^*(\mathbb{C}^n) \leq 4n-4$. The authors of [11] conjectured that $m^0(\mathbb{C}^n) = 4n-4$. This fails for $n = 4$ but holds for $n = 2^p+1$. In the case $m = 4n-4$, Bodmann and Hammen constructed [12] a phase retrievable frame. In section 4 we review their construction and we show it is stable under small perturbations. Our result is independent of [19] which can be used to obtain a similar conclusion. Here we additionally obtain a stability bound for phase retrievability. Note also additional constructions by [20] and [22].

The corresponding problem for the real case is completely solved. In fact [7] gives a geometric condition equivalent to a frame being phase retrievable in $\mathbb{R}^n$. That condition (namely, for any partition of the frame set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, at least one of $\mathcal{F}_1$ or $\mathcal{F}_2$ must span $\mathbb{R}^n$) is stable under small perturbations. Thus $m^*(\mathbb{R}^n)$ exists and $m^*(\mathbb{R}^n) = m^0(\mathbb{R}^n) = 2n-1$ is the critical cardinal.

2. Notations

In this section we recall some notations we introduced in [6] that is used in the following sections. Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a frame in $\mathbb{C}^n$. Let $\mathbf{j} : \mathbb{C}^n \to \mathbb{R}^{2n}$ denote the embedding

$$\mathbf{j}(x) = \begin{bmatrix} real(x) \\ imag(x) \end{bmatrix} \tag{4}$$

which is a norm-preserving isomorphism between $\mathbb{C}^n$ seen as a real vector space endowed with the real inner product $\langle x, y \rangle_{\mathbb{R}} = real(\langle x, y \rangle)$ and $\mathbb{R}^{2n}$:

$$\langle x, y \rangle_{\mathbb{R}} = real(\langle x, y \rangle) = \langle \mathbf{j}(x), \mathbf{j}(y) \rangle. \tag{5}$$

For two vectors $u, v \in \mathbb{R}^{2n}$, $[\![u, v]\!]$ denotes the symmetric outer poduct

$$[\![u, v]\!] = \frac{1}{2}(uv^T + vu^T). \tag{6}$$

and similarly for two vector $x, y \in \mathbb{C}^n$ denote by $[\![x, y]\!]$ their symmetric outer product defined by

$$[\![x, y]\!] = \frac{1}{2}(xy^* + yx^*) \tag{7}$$

For each n-vector $f_k \in \mathbb{C}^n$ we denote by φ_k the $2n$ real vector, and by Φ_k the symmetric nonnegative rank-2, $2n \times 2n$ matrix defined respectively by

$$\varphi_k = \mathbf{j}(f_k) = \begin{bmatrix} \mathrm{real}(f_k) \\ \mathrm{imag}(f_k) \end{bmatrix} \ , \ \Phi_k = [\![\varphi_k, \varphi_k]\!] + [\![J\varphi_k, J\varphi_k]\!] = \varphi_k {\varphi_k}^T + J\varphi_k {\varphi_k}^T J^T \tag{8}$$

where

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

and I the identity matrix of size n. Note the following key relations:

$$real(\langle x, f_k \rangle) = \langle \xi, \varphi_k \rangle \tag{9}$$

$$|\langle x, f_k \rangle|^2 = \langle \Phi_k \xi, \xi \rangle \tag{10}$$

$$real(\langle x, f_k \rangle \langle f_k, y \rangle) = \langle \Phi_k \xi, \eta \rangle \tag{11}$$

where $\xi = \mathbf{j}(x)$ and $\eta = \mathbf{j}(y)$. For every $\xi \in \mathbb{R}^{2n}$ set

$$R(\xi) = \sum_{k=1}^{m} [\![\Phi_k \xi, \Phi_k \xi]\!] = \Phi_k \xi \xi^T \Phi_k. \tag{12}$$

Let $\mathcal{S}^{1,0}$ and $\mathcal{S}^{1,1}$ denote the following spaces of symmetric operators over a Hilbert space H (real or complex)

$$\begin{aligned} \mathcal{S}^{1,0}(H) &= \{T \in Sym(H)\ ,\ rank(T) \le 1\ ,\ \lambda_{max}(T) \ge 0 = \lambda_{min}(T)\} \\ \mathcal{S}^{1,1}(H) &= \{T \in Sym(H)\ ,\ rank(T) \le 2\ ,\ \lambda_{max} \ge 0 \ge \lambda_{min}\} \end{aligned}$$

where $Sym(H)$ denotes the set of self-adjoint operators (matrices) on H, $Sp(T)$ denotes the spectrum (i.e. the set of eigenvalues) of T, and $\lambda_{max}, \lambda_{min}$ denote the largest, and smallest eigenvalue of the corresponding operator. Note

$$\mathcal{S}^{1,0}(H) = \{T = [\![x, x]\!] = xx^*\ ,\ x \in H\}.$$

Lemma 3.2 justifies the definition of $\mathcal{S}^{1,1}$. For the frame $\mathcal{F} = \{f_1, \ldots, f_m\}$ we let $\mathcal{A}$ denote the linear operator

$$\mathcal{A} : Sym(H) \to \mathbb{R}^m\ ,\ (\mathcal{A}(T))_k = \langle T f_k, f_k \rangle = trace\,(T[\![f_k, f_k]\!]) \tag{13}$$

Note the frame condition (1) reads equivalently:

$$A \left\| [\![x, x]\!] \right\|_1 \le \left\| \mathcal{A}([\![x, x]\!]) \right\|^2 \le B \left\| [\![x, x]\!] \right\|_1 \tag{14}$$

where $\|T\|_1 = \sum_{k=1}^{dim(H)} |\lambda_k(T)|$ denotes the nuclear norm of operator T, that is the sum of its singular values, or the sum of magnitudes of its eigenalues when T is symmetric. In the case of finite frames the upper bound is always true (for an appropriate B). The lower bound in (1) or (14) is equivalent to the spanning condition $\text{span}(\mathcal{F}) = H$. In turn this spanning condition can be restated in terms of a null space condition for $\mathcal{A}$. Specifically let $\ker(\mathcal{A}) = \{T \in Sym(H)\ ,\ \mathcal{A}(T) = 0\}$ denote the kernel of the linear operator $\mathcal{A}$. Then $\text{span}(\mathcal{F}) = H$ (and therefore $\mathcal{F}$ is frame) if and only if

$$\ker(\mathcal{A}) \cap \mathcal{S}^{1,0} = \{0\} \tag{15}$$

3. Stability of Phase Retrievable Frames

We start by presenting two lemmas regarding the objects we introduced earlier.

Lemma 3.1. *For any real or complex Hilbert space H:*

(1) *As sets, $\mathcal{S}^{1,1}(H) = \mathcal{S}^{1,0}(H) - \mathcal{S}^{1,0}(H)$. Explicitly this means:*

$$\forall\, T \in \mathcal{S}^{1,1}\ \exists\, T_1, T_2 \in \mathcal{S}^{1,0}\ s.t.\ T = T_1 - T_2.$$

Conversely: $\forall\, T_1, T_2 \in \mathcal{S}^{1,0}\ ,\ T_1 - T_2 \in \mathcal{S}^{1,1}$.

(2) *For any* $T \in \mathcal{S}^{1,1}(H)$ *there are* $u, v \in H$ *so that* $T = [\![u, v]\!]$*;*
(3) *For any* $u, v \in H$, $[\![u, v]\!] \in \mathcal{S}^{1,1}(H)$*;*
(4) $\mathcal{S}^{1,1}(H) = \{T = [\![u, v]\!]\ ,\ u, v \in H\}$.

The proof of this lemma can be found in [6] Lemmas 3.7, 3.8.

Lemma 3.2. *Consider the case* $H = \mathbb{C}^n$*. The following are equivalent:*

(1) *The nonlinear map* β *is injective;*
(2) $\ker(\mathcal{A}) \cap \mathcal{S}^{1,1} = \{0\}$
(3) *There is a constant* $a_0 > 0$ *so that*

$$\sum_{k=1}^{m} \left| |\langle x, f_k\rangle|^2 - |\langle y, f_k\rangle|^2 \right|^2 \geq a_0 \left(\|x - y\|^2 \|x + y\|^2 - 4(imag(\langle x, y\rangle))^2 \right) \tag{16}$$

for any $x, y \in \mathbb{C}^n$*;*

(4) *There is a constant* $a_0 > 0$ *so that for all* $\xi \in \mathbb{R}^{2n}$, $\lambda_{2n-1}(R(\xi)) \geq a_0 \|\xi\|^2$ *(here,* $\lambda_{2n-1}(T)$ *denotes the* $2n-1^{st}$ *largest eigenvalue of* T*, which is also the second smallest eigenvalue);*
(5) *There is a constant* $a_1 > 0$ *so that for any* $\xi \in \mathbb{R}^{2n}$,

$$L_1(\xi) := R(\xi) + J[\![\xi, \xi]\!]J^T = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k + J\xi\xi^T J^T \geq a_1 \|\xi\|^2 1_{\mathbb{R}^{2n}} \tag{17}$$

where the inequality is between symmetric operators;

(6) *There are constants* $a \geq a_0 > 0$ *so that for any* $\xi \in \mathbb{R}^{2n}$,

$$L_a(\xi) := R(\xi) + aJ\xi\xi^T J^T = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k + aJ\xi\xi^T J^T \geq a_0 \|\xi\|^2 1_{\mathbb{R}^{2n}} \tag{18}$$

where the inequality is between symmetric operators;

(7) *For every* $0 \neq \xi \in \mathbb{R}^{2n}$, $\dim\ker R(\xi) = 1$ *;*
(8) *For every* $0 \neq \xi \in \mathbb{R}^{2n}$, $rank(R(\xi)) = 2n - 1$ *;*
(9) *For every* $0 \neq \xi \in \mathbb{R}^{2n}$, $\ker(R(\xi)) = \{c\, J\xi\ ,\ c \in \mathbb{R}\}$*;*
(10) *There is a constant* $a_0 > 0$ *so that for all* $\xi \in \mathbb{R}^{2n}$,

$$R(\xi) \geq a_0 \|\xi\|^2 (1 - P_{J\xi}) \tag{19}$$

where $P_{J\xi} = \frac{1}{\|\xi\|^2} J\xi\xi^T J^T$ *is the orthogonal projection onto* $span(J\xi)$.

Remark. Note the constants a_0 in (iii), (iv), (vi) and (x) can be chosen to be equal to one another. Hence the optimal (i.e. the largest) a_0 is given by

$$a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(R(\xi)) \tag{20}$$

The constant a_1 at (v) can be chosen as $a_1 = min(1, a_0)$.

Proof of Lemma 3.2

Claims (i),(ii),(iv),(vii)-(x) have been shown before - Theorem 2.2 and Theorem 3.1 in [6] (and references there in), and Theorem 4 in [11].

(x)→(v),(vi): Claim (v) follows from (x) by adding $\|\xi\|^2 P_{J\xi}$ on both sides. Claim (vi) follows by adding $a\|\xi\|^2 P_{J\xi}$ on both sides.

(v), (vi)→(vii): Note $J\xi \in ker(R(\xi)$. Either one of (v) or (vi) implies that $J\xi$ is the only independent vector in the kernel of $R(\xi)$.

Claim (iii) follows from Theorem 3.1 (2) of [6], where we set $u = x - y$ and $v = x + y$ and by remarking $imag(\langle u, v\rangle) = 2\, imag(\langle x, y\rangle)$ and $real(\langle u, f_k\rangle\langle f_k, v\rangle) = |\langle x, f_k\rangle|^2 - |\langle y, f_k\rangle|^2$. □

Condition (vi) admits a form that depends essentially only on one (unkown) constant a_0. Note that if (18) holds for some a with $a \geq a_0$ it follows that (18) holds true for any a with $a \geq a_0$, and vice-versa. Consequently it is sufficient to find an upper bound for a_0 that can be directly computed. Such an estimate is given by the following lemma:

Lemma 3.3. *The largest eigenvalue of $R(\xi)$ satisfies*

$$\max_{\|\xi\|=1} \lambda_{max}(R(\xi)) = \|T\|_{2,4}^4 \tag{21}$$

where $\|T\|_{2,4}$ represents the mixed $l^2 - l^4$ norm of the frame analysis operator, $x \mapsto T(x) = (\langle x, f_k\rangle)_{k=1}^m$ and

$$\|T\|_{2,4}^4 = \sup_{x\in H:\|x\|=1} \sum_{k=1}^m |\langle x, f_k\rangle|^4.$$

An upper bound for this mixed norm is provided by

$$\|T\|_{2,4}^4 \leq \left(\max_{1\leq k\leq m} \|f_k\|^2\right) B \leq B^2 \tag{22}$$

where B is the frame upper bound. In particular

$$a_0 \leq \|T\|_{2,4}^4 \leq \left(\max_{1\leq k\leq m} \|f_k\|^2\right) B \leq B^2 \tag{23}$$

and the constant a in (18) can be chosen as $\|T\|_{2,4}^4$ or $\max_{1\leq k\leq m} \|f_k\|^2 B$, or even B^2.

Proof of Lemma 3.3

A similar result appeared in [10] in the context of real frames. The estimate holds true in the complex case as well, as we prove here. Let $\xi = \mathbf{j}(x)$ and $\eta = \mathbf{j}(y)$

for some $x, y \in H$ with $\|x\| = \|y\| = 1$.

$$\begin{aligned}\lambda_{max}(R(\xi)) &= \max_{\|\eta\|=1} \langle R(\xi)\eta, \eta\rangle = \max_{\|\eta\|=1} \sum_{k=1}^{m} |\langle \Phi_k \xi, \eta\rangle|^2 \\ &= \max_{\|y\|=1} \sum_{k=1}^{m} |real(\langle x, f_k\rangle \langle f_k, y\rangle)|^2 = \sum_{k=1}^{m} |\langle x, f_k\rangle|^4\end{aligned}$$

where the last equality is a consequence of the Cauchy-Schwarz inequality. Thus

$$\sup_{\|\xi\|=1} \lambda_{max}(R(\xi)) = \max_{\|x\|=1} \sum_{k=1}^{m} |\langle x, f_k\rangle|^4 = \|T\|_{2,4}^4$$

which shows (21). Finally, the upper bound (22) is obtained as follows

$$\begin{aligned}\|T\|_{2,4}^4 &= \max_{\|x\|=1} \sum_{k=1}^{m} |\langle x, f_k\rangle|^4 \leq \left(\max_{\|x\|=1} |\langle x, f_k\rangle|^2\right) \left(\max_{\|x\|=1} \sum_{k=1}^{m} |\langle x, f_k\rangle|^2\right) \\ &= \left(\max_{1\leq k\leq m} \|f_k\|^2\right) B \leq B^2.\end{aligned}$$

□

Using the estimate (23), conditions (v) and (vi) of Lemma 3.2 can be equivalently restated as, for every $\xi \in \mathbb{R}^{2n}$:

$$L_{B^2}(\xi) := R(\xi) + B^2 J\xi\xi^T J^T = \sum_{k=1}^{m} \Phi_k \xi\xi^T \Phi_k + B^2 J\xi\xi^T J^T \geq a_0 \|\xi\|^2 1_{\mathbb{R}^{2n}}. \tag{24}$$

Recall two frames $\mathcal{F} = \{f_1, \ldots, f_m\}$ and $\mathcal{G} = \{g_1, \ldots, g_m\}$ for the same Hilbert space H are said *equivalent* if there is an invertible operator $T : H \to H$ so that $g_k = T f_k$, for all $1 \leq k \leq m$ (see [2, 24]). The property of being phase retrievable is invariant among equivalent frames, as the following lemma shows.

Lemma 3.4. *Assume $\mathcal{F} = \{f_1, \ldots, f_m\}$ is a phase retrievable frame for H. Then*

1. *For any invertible operator $T : H \to H$ and non-zero scalars $z_1, \ldots, z_m \in K$, the frame $\mathcal{G} = \{g_1, \ldots, g_m\}$ defined by $g_k = z_k T f_k$, $1 \leq k \leq m$, is also phase retrievable;*
2. *For any invertible operator $T : H \to H$, the equivalent frame $\mathcal{G} = \{g_1, \ldots, g_m\}$ defined by $g_k = T f_k$, $1 \leq k \leq m$ is also phase retrievable;*
3. *The canonical dual frame $\tilde{\mathcal{F}} = \{\tilde{f}_1, \ldots, \tilde{f}_m\}$ is also phase retrievable, where $\tilde{f}_k = S^{-1} f_k$, $1 \leq k \leq m$;*

(4) *The associated Parseval frame* $\mathcal{F}^{\#} = \{f_1^{\#}, \ldots, f_m^{\#}\}$ *is also phase retrievable, where* $f_k^{\#} = S^{-1/2} f_k$, $1 \leq k \leq m$;

(5) *Any finite set of vectors* $\mathcal{G} \subset H$ *so that* $\mathcal{F} \subset \mathcal{G}$ *is a phase retrievable frame;*

(6) *If* $\mathcal{G} \subset H$ *is not a phase retrievable frame then any subset* $\mathcal{H} \subset \mathcal{G}$ *is also not a phase retrievable frame. Additionally, if* $\mathcal{G} \subset H$ *is a phase retrievable frame then any finite set* $\mathcal{H} \supset \mathcal{G}$ *is also a phase retrievable frame;*

Proof of Lemma 3.4

(i) Note that each $z_k \neq 0$ and hence $\mathcal{G}$ is also frame. Let $\beta_{\mathcal{G}} : \hat{H} \to \mathbb{R}^m$ be the nonlinear map associated to $\mathcal{G}$, $(\beta_{\mathcal{G}}(x))_k = |\langle x, g_k \rangle|^2$. If $x, \in \hat{H}$ are so that $\beta_{\mathcal{G}}(x) = \beta_{\mathcal{G}}(y)$ then $\beta(T^*x) = \beta(T^*y)$. Since $\mathcal{F}$ is phase retrievable it follows $T^*x = T^*y$ and hence $x = y$. (Note that any operator $R : H \to H$ lifts to a unique operator $R : \hat{H} \to \hat{H}$ that is denoted using the same letter).

(ii) Let $\beta_{\mathcal{G}} : \hat{H} \to \mathbb{R}^m$ be the nonlinear map assiciated to $\mathcal{G}$. Note $(\beta_{\mathcal{G}}(x))_k = |\langle x, g_k \rangle|^2 = |\langle T^*x, f_k \rangle|^2 = (\beta(T^*x))_k$. Hence $\beta_{\mathcal{G}}$ is injective if and only if β is injective.

(iii)-(iv) follows from (ii). Claims (v) and (vi) are obvious. □

Remark. The claim (vi) in previous Lemma states that subsets of frames that do not give phase retrieval are not phase retrievable, and sets that include a phase retrievable frame are also phase retrievable frames. For spanning sets the equivalent statements to these two properties are also true: a subset of an incomplet set is incomplete, whereas a set that includes a spanning set is spanning. Spanning sets have an additional property: For every finite dimensional Hilbert space H there is a critical threshold $s(H) = dim(H)$ so that: (1) Every spanning set is of cardinal greater than or equal to $s(H)$; (2) If a set is of cardinal less than $s(H)$ then it cannot be spanning; (3) For every spanning set of cardinal strictly larger than $s(H)$ there is a subset of cardinal exactly $s(H)$ that is spanning.

Now we ask whether a similar threshold exists for phase retrievable frames. The natural candidate is $m^0(H)$ since it is the minimum cardinal of a phase retrievable frame in H. However, as the following example shows, the property (3) for spanning sets does not hold for phase retrievable frames. A similar construction was considered by [18]. The example is for $H = \mathbb{R}^n$ where we know $m^0(\mathbb{R}^n) = 2n - 1$, but the conclusion applies equally well to the complex case.

To summarize: If a phase retrievable frame $\mathcal{F}$ has cardinal $m > m^0(\mathbb{C}^n)$ then it might not contain a subset of cardinal $m^0(\mathbb{C}^n)$ that is also phase retrievable.

Example 3.5. *Consider $H = \mathbb{R}^3$ and the frame with $m = 6$ vectors:*

(25)
$$f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, f_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, f_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, f_6 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The associated rank-1 operators $F_k = f_k f_k^T$, $1 \leq k \leq 6$, belong to the linear space of symmetric 3×3 matrices $Sym(\mathbb{R}^3)$. Note the $Sym(\mathbb{R}^3)$ is a real vectors space of dimension 6. The Gram matrix $G^{(2)}$ associated to $\{F_1, \ldots, F_6\}$ is a 6×6 symmetric matrix of entries $G^{(2)}_{k,l} = \langle F_k, F_l \rangle = |\langle f_k, f_l \rangle|^2$, which are the square of the entries of Gram matrix associated to $\mathcal{F}$. Explicitely $G^{(2)}$ is given by

$$G^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 1 & 1 & 4 \end{bmatrix} \tag{26}$$

Its determinant is $det(G^{(2)}) = 8$. Hence $\{F_1, \ldots, F_6\}$ is a basis for $Sym(\mathbb{R}^3)$ and thus $ker(\mathcal{A}) = \{0\}$ which implies $\mathcal{F}$ is a phase retrievable frame. On the other hand consider any subset $\mathcal{G}$ of 5 vectors of $\mathcal{F}$. It is easy to check $\mathcal{G}$ is a frame for $\mathbb{R}^3$. However for each $\mathcal{G}$ there is a subset of 3 elements that is not linearly independent, hence cannot span $\mathbb{R}^3$. This fact together with Corollary 2.6 from [7] *proves that $\mathcal{G}$ is not phase retrievable. Thus we constructed a frame $\mathcal{F}$ of 6 vectors so that any subset of cardinal $m^0(\mathbb{R}^3) = 5$ is not phase retrievable.*

We are now ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1

Assume $\mathcal{F}$ is a phase retrievable frame. Then equation (18) is satisfied for some $a_0 > 0$. Let B be the upper frame bound for $\mathcal{F}$. Then set ρ as in (3). We will show that (18) is satisfied for any set $\mathcal{F}' = \{f'_1, \ldots, f'_m\}$ with $\|f_k - f'_k\| < \rho$. Let $0 < A \leq B < \infty$ be the frame bounds of $\mathcal{F}$, set $a = B \max_{1 \leq k \leq m} \|f_k\|^2$, and let $L'_a(\xi)$ denote the right hand side in (18) associated to $\mathcal{F}'$ for this particular a:

$$L'_a(\xi) = \sum_{k=1}^{m} \Phi'_k \xi \xi^T \Phi'_k + aJ\xi\xi^T J^T.$$

We compute

$$\begin{aligned}
|\langle L_a(\xi)\eta,\eta\rangle - \langle L_a^{'}(\xi)\eta,\eta\rangle| &\leq \sum_{k=1}^{m} \left|\, |\langle \Phi_k\xi,\eta\rangle|^2 - |\langle \Phi_k'\xi,\eta\rangle|^2 \,\right| \\
&\leq \sum_{k=1}^{m} \left(|\langle \Phi_k\xi,\eta\rangle| + |\langle \Phi_k'\xi,\eta\rangle|\right) |\langle(\Phi_k-\Phi_k')\xi,\eta\rangle| \\
&\leq \left(\sum_{k=1}^{m} |\langle \Phi_k\xi,\eta\rangle| + \sum_{k=1}^{m} |\langle \Phi_k'\xi,\eta\rangle|\right) \max_{1\leq k\leq m} |\langle(\Phi_k-\Phi_k')\xi,\eta\rangle|.
\end{aligned}$$

Fix $\xi \in \mathbb{R}^{2n}$. Then

$$\max_{\|\eta\|=1} \sum_{k=1}^{m} |\langle \Phi_k\xi,\eta\rangle| \leq \left(\sum_{k=1}^{m} \langle \Phi_k\xi,\xi\rangle\right)^{1/2} \max_{\|\eta\|=1} \left(\sum_{k=1}^{m} \langle \Phi_k\eta,\eta\rangle\right)^{1/2} \leq B\,\|\xi\|\,.$$

Thus for any $\xi,\eta \in \mathbb{R}^{2n}$,

$$\sum_{k=1}^{m} |\langle \Phi_k\xi,\eta\rangle| \leq B\,\|\xi\|\,\|\eta\| \;\;,\;\; \sum_{k=1}^{m} |\langle \Phi_k'\xi,\eta\rangle| \leq B'\,\|\xi\|\,\|\eta\|$$

where B' is the upper frame bound of $\mathcal{F}'$. On the other hand we bound

$$|\langle(\Phi_k-\Phi_k')\xi,\eta\rangle| \leq \|\Phi_k-\Phi_k'\|\,\|\xi\|\,\|\eta\|\,.$$

According to Lemma 3.14 (4) from [6], $\|\Phi_k-\Phi_k'\| = \|F_k-F_k'\|$, where $F_k = f_kf_k^*$ and $F_k' = f_k'f_k'^*$. Note $F_k - F_k' \in \mathcal{S}^{1,1}(\mathbb{C}^n)$ and $F_k - F_k' = [\![f_k - f_k', f_k + f_k']\!]$. Now using Lemma 3.8 (1) from [6], we obtain

$$\begin{aligned}
\|F_k-F_k'\| &\leq \|F_k-F_k'\|_1 \\
&= \sqrt{\|f_k-f_k'\|^2\,\|f_k+f_k'\|^2 - (imag(\langle f_k-f_k', f_k+f_k'\rangle))^2} \\
&\leq \|f_k-f_k'\|\,\|f_k+f_k'\|
\end{aligned}$$

where $\|T\|_1$ is the nuclear norm (the sum of its singular values) of T. Next notice $\|f_k+f_k'\| \leq \|f_k\|+\|f_k'\| \leq \sqrt{B}+\sqrt{B'} \leq \sqrt{2(B+B')}$. Putting all these estimates together we obtain:

$$|\langle L_a(\xi)\eta,\eta\rangle - \langle L_a^{'}(\xi)\eta,\eta\rangle| \leq \sqrt{2}(B+B')^{3/2}\left(\max_{1\leq k\leq m}\|f_k-f_k'\|\right)\|\xi\|^2\,\|\eta\|^2\,.$$

Thus

$$L_a^{'}(\xi) \geq (a_0 - \sqrt{2}(B+B')^{3/2}\rho)\,\|\xi\|^2\,1_{\mathbb{R}^{2n}}\,.$$

Finally we obtain an estimate of B' in terms of B, ρ and m. Let $\delta_k = f'_k - f_k$. Then

$$\sum_{k=1}^m |\langle x, f'_k\rangle|^2 = \sum_{k=1}^m |\langle x, f_k\rangle + \langle x, \delta_k\rangle|^2 \leq 2\left(\sum_{k=1}^m |\langle x, f_k\rangle|^2 + \sum_{k=1}^m |\langle x, \delta_k\rangle|^2\right)$$

$$= 2(B + m\max_k \|\delta_k\|^2)\,\|x\|^2 . \qquad (27)$$

Since $\rho \leq \frac{1}{\sqrt{m}}$ from (3) we obtain $B' = \sup_{\|x\|=1}\sum_{k=1}^m |\langle x, f'_k\rangle|^2 \leq 2(B+1)$. This bound implies that

$$L'_a(\xi) \geq \frac{a_0}{2}\,\|\xi\|^2\, 1_{\mathbb{R}^{2n}}$$

and hence $\mathcal{F}'$ is a frame which gives phase retrieval. □

4. The case $m = 4n - 4$

This section comments on the recent construction by Bodmann and Hammen [12] of a $4n-4$ phase retrievable frame in $\mathbb{C}^n$. Their construction is as follows. Fix $a \in \mathbb{R}\setminus\pi\mathbb{Q}$, an irrational multiple of π. The frame set $\mathcal{F}$ is given by a union of two sets, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2^{(a)}$, where $\mathcal{F}_1$ contains the following $2n-3$ vectors:

$$\mathcal{F}_1 = \{f_k^{(1)}\ ,\ 1 \leq k \leq 2n-3\} \qquad (28)$$

where:

$$f_k^{(1)} = \begin{bmatrix} 1 & e^{2\pi ik/(2n-1)} & e^{2\pi ik2/(2n-1)} & \cdots & e^{2\pi ik(n-1)/(2n-1)} \end{bmatrix}^T$$

and $\mathcal{F}_2^{(a)}$ contains the following $2n-1$ vectors:

$$\mathcal{F}_2^{(a)} = \{f_k^{(2)} = \begin{bmatrix} 1 & \overline{z_k} & \overline{z_k}^2 & \cdots & \overline{z_k}^{n-1} \end{bmatrix}^T\ ,\ 1 \leq k \leq 2n-1\} \qquad (29)$$

where

$$z_k = \frac{sin\left(\frac{\pi}{2n-1}\right)}{sin(a)} e^{2\pi i\frac{k-1}{2n-1}} - e^{i\frac{\pi}{2n-1}}\frac{sin\left(\frac{\pi}{2n-1} - a\right)}{sin(a)} \qquad (30)$$

The proof that $\mathcal{F}$ is a phase retrievable frame is based on a result by P. Jaming from [26]. Our Theorem 1.1 proves that, in fact, $\mathcal{F}$ remains phase retrievable for a small perturbation. Since $f_k^{(2)}$ depends continuously on a, it follows that the set $\mathbb{R}\setminus\pi\mathbb{Q}$ can be replaced by a much larger set of real numbers that includes most of rational multiples of π. Going through the proof of Theorem 2.3 in [12], and in particular of Lemma 2.2, the only requirement on a is that, any set of $2(n-1)$ complex numbers cannot be simultaneoulsy symmetric with respect to the real line and to a line of angle $a - \frac{\pi}{2n-1}$ passing through cot $\frac{\pi}{2n-1}$. This phenomenon happens for any n when a is an irrational multiple of π. However, for a fixed n, only finitely many values of a may allow such a symmetry. In fact when such a symmetric set of $2(n-1)$ complex numbers exists, $a = \frac{\pi}{2n-1} + \pi\frac{p}{q}$ for some $q \leq 2(n-1)$. Thus the frame set above $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2^{(a)}$ remains phase retrievable for all values of a except a finite set included in $\{\frac{\pi}{2n-1} + \pi\frac{p}{q}\ ,\ -2q \leq p \leq 2q \leq 4(n-1)\}$.

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Department of Mathematics, CSCAMM and Norbert Wiener Center, University of Maryland, College Park, MD 20742, USA

E-mail address: rvbalan@math.umd.edu