NONLINEAR FRAME ANALYSIS AND PHASELESS RECONSTRUCTION

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ABSTRACT. Frame design for phaseless reconstruction is now part of the broader problem of nonlinear recon- struction and is an emerging topic in harmonic analysis. The problem of phaseless reconstruction can be simply stated as follows. Given the magnitudes of the coefficients of an output of a linear redundant system (frame), we want to reconstruct the unknown input. This problem has first occurred in X-ray crystallography starting from the early 20th century. The same nonlinear reconstruction problem shows up in speech processing, particularly in speech recognition.

In this lecture we shall cover existing analysis results as well as algorithms for signal recovery including: necessary and sufficient conditions for injectivity, Lipschitz bounds of the nonlinear map and its left inverses, stochastic performance bounds, convex relaxation algorithms for inversion, least-squares inversion algorithms.

1. INTRODUCTION

This lecture notes concerns the problem of finite dimensional vector reconstruction from magnitudes of frame coefficients. While the problem can be stated in the more general context of infinite dimensional Hilbert spaces, in these lectures we focus exclusively on the finite dimensional case. In this case any spanning set is a frame. Specifically let $H = \mathbb{C}^n$ denote the n dimensional complex Hilbert space and let $\mathcal{F} = \{f_1, \cdots, f_m\}$ be a set of $m \geq n$ vectors that span H. Fix a real linear space V, that is also subset of H, $V \subset H$. Our problem is to study when a vector $x \in V$ can be reconstructed from magnitudes of its frame coefficients $\{|\langle x, f_k\rangle|, 1 \leq k \leq m\}$, and how to do so efficiently. This setup covers both the real case and the complex case as studied before in literature: in the real case $\mathcal{F} \subset V = \mathbb{R}^n$; in the complex case $V = H = \mathbb{C}^n$. Note we assume V is a real linear space which may not be closed under multiplication with complex scalars.

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Consider the following additional notations. Let

$$T : H \to \mathbb{C}^m \ , \ (T(x))_k = \langle x, f_k \rangle \ , \ 1 \leq k \leq m \tag{1.1}$$

denote the frame *analysis map*. Its adjoint is called the *synthesis map* and is defined by

$$T^* : \mathbb{C}^m \to H \ , \ T^*(c) = \sum_{k=1}^{m} c_k f_k \tag{1.2}$$

We define now the main nonlinear function we discussed in this paper $x \mapsto \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m}$. For two vectors $x, y \in H$, consider the equivalence relation $x \sim y$ if and only if there is a constant c of magnitude 1 so that $x = cy$. Thus $x \sim y$ if and only if $x = e^{i\varphi} y$ for some real φ. Let $\hat{H} = H/\sim$ denote the quotient space. Note the nonlinear α is well defined on $\hat{H}$ since $\alpha(cx) = \alpha(x)$ for all scalars c with $|c| = 1$. We let α denote the quotient map

$$\alpha : \hat{H} \to \mathbb{R}^m \ , \ (\alpha(x))_k = |\langle x, f_k \rangle| \ , \ 1 \leq k \leq m \tag{1.3}$$

For purposes that will become clear later let us define also the map

$$\beta : \hat{H} \to \mathbb{R}^m \ , \ (\beta(x))_k = |\langle x, f_k \rangle|^2 \ , \ 1 \leq k \leq m \tag{1.4}$$

For the subspace V denote by $\hat{V}$ the set of equivalence classes $\hat{V} = \{\hat{x} \ , \ x \in V\}$.

Definition 1.1. *The frame $\mathcal{F}$ is called a* phase retrievable frame with respect to a set V *if the restriction $\alpha|_{\hat{V}}$ is injective.*

In this paper we study the following problems:

(1) Find necessary and sufficient conditions for $\alpha|_{\hat{V}}$ to be a one-to-one (injective) map;
(2) Study Lipschitz properties of maps α, β and their inverses;
(3) Study robustness guarantees (such as Cramer-Rao Lower Bounds) for any inversion algorithm;
(4) Recovery using convex algorithms (e.g. PhaseLift and PhaseCut);
(5) Recovery using iterative least-squares algorithms.

2. Geometry of $\hat{H}$ and $\mathcal{S}^{p,q}$ spaces

2.1. $\hat{H}$**.** Recall $\hat{H} = \hat{\mathbb{C}^n} = \mathbb{C}^n/\sim = \mathbb{C}^n/T^1$ where $T^1 = \{z \in \mathbb{C} \ , \ |z| = 1\}$. Algebraically $\hat{\mathbb{C}^n}$ is a homogeneous space being invariant to multiplications by positive real scalars. In particular any $x \in \hat{\mathbb{C}^n} \setminus \{0\}$ has a unique decomposition $x = rp$, where $r = \|x\| > 0$ and $p \in \mathbb{CP}^{n-1}$ is in the projective space $\mathbb{CP}^{n-1} = \mathbb{P}(\mathbb{C}^n)$. Thus topologically

$$\hat{\mathbb{C}^n} = \{0\} \cup \left((0, \infty) \times \mathbb{CP}^{n-1}\right)$$

The subset

$$\mathring{\hat{\mathbb{C}^n}} = \hat{\mathbb{C}^n} \setminus \{0\} = (0, \infty) \times \mathbb{CP}^{n-1}$$

is a real analytic manifold.

Now consider the set $\hat{V}$ of equivalence classes associated to vectors in V. Similar to $\hat{H}$ it admits the following decomposition

$$\hat{V} = \{0\} \cup ((0,\infty) \times \mathbb{P}(V))$$

where $\mathbb{P}(V) = \{ \{zx \ , \ z \in \mathbb{C}\} \ , \ x \in V, x \neq 0\}$ denote the projective space associated to V. The interior subset

$$\mathring{\hat{V}} = \hat{V} \setminus \{0\} = (0,\infty) \times \mathbb{P}(V)$$

is a real analytic manifold of (real) dimension $1 + \dim_{\mathbb{R}} \mathbb{P}(V)$.

Two important cases are as follows:

- Real case. $V = \mathbb{R}^n$ embedded as $x \in \mathbb{R}^n \mapsto x + i0 \in \mathbb{C}^n = H$. Then two vectors $x, y \in V$ are $\sim$ equivalent if and only if $x = y$ or $x = -y$. Similarly, the projective space $\mathbb{P}(V)$ is difeomorphically equivalent to the real projective space $\mathbb{RP}^{n-1}$ which is of dimension $n-1$. Thus

$$\dim_{\mathbb{R}}(\mathring{\hat{V}}) = n$$

- Complex case. $V = \mathbb{C}^n$ which has real dimension $2n$. Then the projective space $\mathbb{P}(V) = \mathbb{CP}^{n-1}$ has real dimension $2n-2$ (it is also a Khäler manifold) and thus

$$\dim_{\mathbb{R}}(\mathring{\hat{V}}) = 2n - 1$$

2.2. $\mathcal{S}^{p,q}$**.** Consider now $Sym(H) = \{T : \mathbb{C}^n \to \mathbb{C}^n \ , \ T = T^*\}$ the real vector space of self-adjoint operators over $H = \mathbb{C}^n$ endowed with the Hilbert-Schmidt scalar product $\langle T, S \rangle_{HS} = trace(TS)$. We also use the notation $Sym(\mathcal{V})$ for the real vector space of symmetric operators over a vector space $\mathcal{V}$. In both cases symmetric means the operator T satisfies $\langle Tx, y \rangle = \langle x, Ty \rangle$ for every x, y in the underlying vector space (H or $\mathcal{V}$, respectively). T^* means the adjoint operator of T, and therefore the transpose conjugate of T, when T is a matrix. When T is a an operator acting on a real vector space, T^T denotes its adjoint. For two vectors $x, y \in \mathbb{C}^n$ we denote

$$[\![x, y]\!] = \frac{1}{2}(xy^* + yx^*) \in Sym(\mathbb{C}^n) \tag{2.5}$$

their symmetric outer product. On $Sym(H)$ and $B(H) = \mathbb{C}^{n\times n}$ we consider the class of p-norms defined by the p-norm of the vector of singular values:

$$\|T\|_p = \left\{ \begin{array}{ll} \max_{1\leq k\leq n} \sigma_k(T) & for \quad p = \infty \\ (\sum_{k=1}^n \sigma_k^p)^{1/p} & for \quad 1 \leq p < \infty \end{array} \right. \tag{2.6}$$

where $\sigma_k = \sqrt{\lambda_k(T^*T)}$, $1 \leq k \leq n$, are the singular values of T, with $\lambda_k(S)$, $1 \leq k \leq n$, denoting eigenvalues of S.

Fix two integers $p, q \geq 0$ and set

(2.7)

$$\mathcal{S}^{p,q}(H) = \{T \in Sym(H) \ , \ T \text{ has at most } p \text{ positive eigenvalues and at most } q \text{ negative eigenvalues}\}$$

(2.8)
$$\mathring{\mathcal{S}}^{p,q}(H) = \{T \in Sym(H)\ ,\ T \text{ has exactly } p \text{ positive eigenvalues and exactly } q \text{ negative eigenvalues}\}$$

For instance $\mathring{\mathcal{S}}^{0,0}(H) = \mathcal{S}^{0,0}(H) = \{0\}$ and $\mathring{\mathcal{S}}^{1,0}(H)$ is the set of all non-negative rank one operators. When there is no confusion we shall drop the underlying vector space H from notation.

The following basic properties can be found in [Ba13], Lemma 3.6 (the last statement is a special instance of the Witt's decomposition theorem):

Lemma 2.1.

(1) *For any $p_1 \leq p_2$ and $q_1 \leq q_2$, $\mathcal{S}^{p_1,q_1} \subset \mathcal{S}^{p_2,q_2}$;*

(2) *For any nonnegative integers p, q the following disjoint decomposition holds true*

$$\mathcal{S}^{p,q} = \cup_{r=0}^{p} \cup_{s=0}^{q} \mathring{\mathcal{S}}^{r,s} \tag{2.9}$$

where by convention $\mathring{\mathcal{S}}^{p,q} = \emptyset$ for $p + q > n$.

(3) *For any $p, q \geq 0$,*

$$-\mathcal{S}^{p,q} = \mathcal{S}^{q,p} \tag{2.10}$$

(4) *For any linear operator $T : H \to H$ (symmetric or not, invertible or not) and nonnegative integers p, q,*

$$T^*\mathcal{S}^{p,q}T \subset \mathcal{S}^{p,q} \tag{2.11}$$

(5) *For any nonnegative integers p, q, r, s,*

$$\mathcal{S}^{p,q} + \mathcal{S}^{r,s} = \mathcal{S}^{p,q} - \mathcal{S}^{s,r} = \mathcal{S}^{p+r,q+s} \tag{2.12}$$

The spaces $\mathcal{S}^{1,0}$ and $\mathcal{S}^{1,1}$ play a special role in the following chapters. We summarize next their properties (see Lemmas 3.7 and 3.9 in [Ba13], and the comment after Lemma 9 in [BCMN13]).

Lemma 2.2 (Space $\mathcal{S}^{1,0}$)**.** *The following hold true:*

(1) $\mathring{\mathcal{S}}^{1,0} = \{xx^*\ ,\ x \in H, x \neq 0\}$;

(2) $\mathcal{S}^{1,0} = \{xx^*\ ,\ x \in H\} = \{0\} \cup \{xx^*\ ,\ x \in H, x \neq 0\}$;

(3) *The set $\mathring{\mathcal{S}}^{1,0}$ is a real analytic manifold in $Sym(n)$ of real dimension $2n - 1$. As a real manifold, its tangent space at $X = xx^*$ is given by*

$$T_X\mathring{\mathcal{S}}^{1,0} = \left\{ [\![x, y]\!] = \frac{1}{2}(xy^* + yx^*)\ ,\ y \in \mathbb{C}^n \right\}. \tag{2.13}$$

The $\mathbb{R}$-linear embedding $\mathbb{C}^n \mapsto T_X\mathring{\mathcal{S}}^{1,0}$ given by $y \mapsto [\![x, y]\!]$ has null space $\{iax\ ,\ a \in \mathbb{R}\}$.

Lemma 2.3 (Space $\mathcal{S}^{1,1}$)**.** *The following hold true:*

(1) $\mathcal{S}^{1,1} = \mathcal{S}^{1,0} - \mathcal{S}^{1,0} = \mathcal{S}^{1,0} + \mathcal{S}^{0,1} = \{[\![x, y]\!]\ ,\ x, y \in H\}$;

(2) *For any vectors $x, y, u, v \in H$,*

$$xx^* - yy^* = [\![x+y, x-y]\!] = [\![x-y, x+y]\!] \tag{2.14}$$

$$[\![u,v]\!] = \frac{1}{4}(u+v)(u+v)^* - \frac{1}{4}(u-v)(u-v)^* \tag{2.15}$$

Additionally, for any $T \in \mathcal{S}^{1,1}$ let $T = a_1\ e_1e_1^* - a_2e_2e_2^*$ be its spectral factorization with $a_1, a_2 \geq 0$ and $\langle e_i, e_j\rangle = \delta_{i,j}$. Then

$$T = [\![\sqrt{a_1}e_1 + \sqrt{a_2}e_2, \sqrt{a_1}e_1 - \sqrt{a_2}e_2]\!].$$

(3) *The set $\mathring{\mathcal{S}}^{1,1}$ is a real analytic manifold in Sym(n) of real dimension $4n-4$. Its tangent space at $X = [\![x,y]\!]$ is given by*

$$T_X\mathring{\mathcal{S}}^{1,1} = \{[\![x,u]\!] + [\![y,v]\!] = \frac{1}{2}(xu^* + ux^* + yv^* + vy^*)\ ,\ u,v \in \mathbb{C}^n\}. \tag{2.16}$$

The $\mathbb{R}$-linear embedding $\mathbb{C}^n \times \mathbb{C}^n \mapsto T_X\mathring{\mathcal{S}}^{1,1}$ given by $(u,v) \mapsto [\![x,u]\!] + [\![y,v]\!]$ has null space $\{a(ix,0) + b(0,iy) + c(y,-x) + d(iy,ix)\ ,\ a,b,c,d \in \mathbb{R}\}$.

(4) *Let $T = [\![u,v]\!] \in \mathcal{S}^{1,1}$. Then its eigenvalues and p-norms are:*

$$a_+ = \frac{1}{2}\left(real(\langle u,v\rangle) + \sqrt{\|u\|^2\|v\|^2 - (imag(\langle u,v\rangle))^2}\right) \geq 0 \tag{2.17}$$

$$a_- = \frac{1}{2}\left(real(\langle u,v\rangle) - \sqrt{\|u\|^2\|v\|^2 - (imag(\langle u,v\rangle))^2}\right) \leq 0 \tag{2.18}$$

$$\|T\|_1 = \sqrt{\|u\|^2\|v\|^2 - (imag(\langle u,v\rangle))^2} \tag{2.19}$$

$$\|T\|_2 = \sqrt{\frac{1}{2}\left(\|u\|^2\|v\|^2 + (real(\langle u,v\rangle))^2 - (imag(\langle u,v\rangle))^2\right)} \tag{2.20}$$

$$\|T\|_\infty = \frac{1}{2}\left(|real(\langle u,v\rangle)| + \sqrt{\|u\|^2\|v\|^2 - (imag(\langle u,v\rangle))^2}\right) \tag{2.21}$$

(5) *Let $T = xx^* - yy^* \in \mathcal{S}^{1,1}$. Then its eigenvalues and p-norms are:*

$$a_+ = \frac{1}{2}\left(\|x\|^2 - \|y\|^2 + \sqrt{(\|x\|^2 + \|y\|^2)^2 - 4|\langle x,y\rangle|^2}\right) \tag{2.22}$$

$$a_- = \frac{1}{2}\left(\|x\|^2 - \|y\|^2 - \sqrt{(\|x\|^2 + \|y\|^2)^2 - 4|\langle x,y\rangle|^2}\right) \tag{2.23}$$

$$\|T\|_1 = \sqrt{(\|x\|^2 + \|y\|^2)^2 - 4|\langle x,y\rangle|^2} \tag{2.24}$$

$$\|T\|_2 = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x,y\rangle|^2} \tag{2.25}$$

$$\|T\|_\infty = \frac{1}{2}\left(|\|x\|^2 - \|y\|^2| + \sqrt{(\|x\|^2 + \|y\|^2)^2 - 4|\langle x,y\rangle|^2}\right) \tag{2.26}$$

Note the above results hold true for the case of symmetric operators over real subspaces, say V. In particular the factorization at Lemma 2.3(a) implies:

$$\mathcal{S}^{1,1}(V) = \mathcal{S}^{1,0}(V) - \mathcal{S}^{1,0}(V) = \mathcal{S}^{1,0}(V) + \mathcal{S}^{0,1}(V) = \{[\![u,v]\!]\ ,\ u,v \in V\} \tag{2.27}$$

Minimally, the result holds for subsets $V \subset H$ that are closed under addition and substraction.

2.3. **Metrics.** The space $\hat{H} = \hat{\mathbb{C}^n}$ admits two classes of distances (metrics). The first class is the "natural metric" induced by the quotient space structure. The second metric is a matrix-norm induced distance.

Fix $1 \leq p \leq \infty$.

The *natural metric* denoted by $D_p : \hat{H} \times \hat{H} \to \mathbb{R}$ is defined by

$$D_p(\hat{x},\hat{y}) = \min_{\varphi \in [0,2\pi)} \|x - e^{i\varphi} y\|_p \tag{2.28}$$

where $x \in \hat{x}$ and $y \in \hat{y}$. In the case $p = 2$ the distance becomes

$$D_2(\hat{x},\hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x,y\rangle|}$$

By abuse of notation we use also $D_p(x,y) = D_p(\hat{x},\hat{y})$ since the distance does not depend ont he choice of representatiove.

The matrix-norm induced distance denoted by $d_p : \hat{H} \times \hat{H} \to \mathbb{R}$ is defined by

$$d_p(\hat{x},\hat{y}) = \|xx^* - yy^*\|_p \tag{2.29}$$

where again $x \in \hat{x}$ and $y \in \hat{y}$. In the case $p = 2$ we obtain

$$d_2(x,y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x,y\rangle|^2}$$

By abuse of notation we use also $d_p(x,y) = d_p(\hat{x},\hat{y})$ since the distance does not depend ont he choice of representatiove.

As analyzed in [BZ14], Proposition 2.4, D_p is not equivalent to d_p, however D_p is an equivalent distance to D_q and similarily, d_p is equivalent to d_q, for any $1 \leq p,q \leq q$ (see also [BZ15] for the last claim below):

Lemma 2.4.

(1) *For each $1 \leq p \leq \infty$, D_p and d_p are distances (metrics) on $\hat{H}$;*

(2) *$(D_p)_{1\leq p\leq\infty}$ are equivalent metrics, that is each D_p induces the same topology on $\hat{H}$ and, for every $1 \leq p,q \leq \infty$, the identity map $i : (\hat{H}, D_p) \to (\hat{H}, D_q)$, $i(x) = x$, is Lipschitz continuous with (upper) Lipschitz constant*

$$Lip^D_{p,q,n} = \max(1, n^{\frac{1}{q}-\frac{1}{p}})$$

(3) $(d_p)_{1\leq p\leq\infty}$ *are equivalent metrics, that is each* d_p *induces the same topology on* $\hat{H}$ *and, for every* $1\leq p,q\leq\infty$*, the identity map* $i:(\hat{H},d_p)\to(\hat{H},d_q)$*,* $i(x)=x$*, is Lipschitz continuous with (upper) Lipschitz constant*

$$Lip^{d}_{p,q,n}=\max(1,2^{\frac{1}{q}-\frac{1}{p}}).$$

(4) *The identity map* $i:(\hat{H},D_p)\to(\hat{H},d_p)$*,* $i(x)=x$ *is continuous but it is not Lipschitz continuous. The identity map* $i:(\hat{H},d_p)\to(\hat{H},D_p)$*,* $i(x)=x$ *is continuous but it is not Lipschitz continuous. Hence the induced topologies on* $(\hat{H},D_p)$ *and* $(\hat{H},d_p)$ *are the same, but the corresponding metrics are not Lipschitz equivalent.*

(5) *The metric space* $(\hat{H},d_p)$ *is isometrically isomorphic to* $\mathcal{S}^{1,0}$ *endowed with the p-norm. The isomorphism is given by the map*

$$\kappa_\beta:\hat{H}\to\mathcal{S}^{1,0}\ ,\ x\mapsto[\![x,x]\!]=xx^*.$$

(6) *The metric space* $(\hat{H},D_2)$ *is Lipschitz isomorphic (not isometric) with* $\mathcal{S}^{1,0}$ *endowed with the* 2*-norm. The bi-Lipschitz map*

$$\kappa_\alpha:\hat{H}\to\mathcal{S}^{1,0}\ ,\ x\mapsto kappa_\alpha(x)=\left\{\begin{array}{ccc}\frac{1}{\|x\|}xx^* & if & x\neq 0\\ 0 & & otherwise\end{array}\right.$$

has lower Lipschitz constant 1 *and upper Lipschitz constant* $\sqrt{2}$*.*

Note the Lipschitz bound $Lip^{D}_{p,q,n}$ is equal to the operator norm of the identity between $(\mathbb{C}^n,\|\cdot\|_p)$ and $(\mathbb{C}^n,\|\cdot\|_q)$: $Lip^{D}_{p,q,n}=\|I\|_{l^p(\mathbb{C}^n)\to l^q(\mathbb{C}^n)}$. Note also the equality $Lip^{d}_{p,q,n}=Lip^{D}_{p,q,2}$. A consequence of the last two claims in the above result is that while the identity map between $(\hat{H},D_p)$ and $(\hat{H},d_q)$ is not bi-Lipschitz, the map $x\mapsto\frac{1}{\sqrt{\|x\|}}x$ is bi-Lipschitz.

3. The Injectivity Problem

In this section we summarize existing results on the injectivity of the maps α and β. Our plan is to present the real and the complex case in a unified way.

Recall we denoted by V a real vector space which is subset of $H=\mathbb{C}^n$. The special two cases are $V=\mathbb{R}^n$ (the real case) and $V=\mathbb{C}^n$ (the complex case).

First we describe the realification of H and V. Consider the $\mathbb{R}$-linear map $\jmath:\mathbb{C}^n\to\mathbb{R}^{2n}$ defined by

$$\jmath(x)=\left[\begin{array}{c} real(x)\\ imag(x)\end{array}\right]$$

Let $\mathcal{V}=\jmath(V)$ be the embedding of V into $\mathbb{R}^{2n}$, and let Π denote the orthogonal projection (with respect to the real scalar product on $\mathbb{R}^{2n}$) onto $\mathcal{V}$. Let J denote the folowing orthogonal antisymmetric $2n\times 2n$ matrix

$$J=\left[\begin{array}{cc} 0 & -I_n\\ I_n & 0\end{array}\right] \tag{3.30}$$

where I_n denotes the identity matrix of order $n \times n$. Note the transpose $J^T = -J$, the square $J^2 = -I_{2n}$ and the inverse $J^{-1} = -J$.

Each vector f_k of the frame set $\mathcal{F} = \{f_1, \cdots, f_m\}$ gets mapped into a vector in $\mathbb{R}^{2n}$ denoted by φ_k, and a symmetric operator in $\mathcal{S}^{2,0}(\mathbb{R}^{2n})$ denoted by Φ_k:

$$\varphi_k = \jmath(f_k) = \left[\begin{array}{c} real(f_k) \\ imag(f_k) \end{array} \right] \ , \ \Phi_k = \varphi_k \varphi_k^T + J\varphi_k\varphi_k^T J^T \tag{3.31}$$

Note that when $f_k \neq 0$ the symmetric form Φ_k has rank 2 and belongs to $\mathring{\mathcal{S}}^{2,0}$. Its spectrum has two distinct eigenvalues: $\|\varphi_k\|^2 = \|f_k\|^2$ with multiplicity 2, and 0 with multiplicity $2n-2$. Furthermore, $\frac{1}{\|\varphi_k\|^2}\Phi_k$ is a rank 2 projection.

Let $\xi = \jmath(x)$ and $\eta = \jmath(y)$ denote the realifications of vectors $x, y \in \mathbb{C}^n$. Then a bit of algebra shows that

$$\begin{array}{rcl} \text{(3.32)} \qquad \langle x, f_k \rangle & = & \langle \xi, \varphi_k \rangle + i \langle \xi, J\varphi_k \rangle \\ \langle F_k, xx^* \rangle_{HS} = trace\,(F_k xx^*) = |\langle x, f_k\rangle|^2 & = & \langle \Phi_k \xi, \xi \rangle = trace\left(\Phi \xi \xi^T\right) = \langle \Phi_k, \xi\xi^T \rangle_{HS} \\ \langle F_k, [\![x,y]\!] \rangle_{HS} = trace\,(F_k [\![x,y]\!]) = real(\langle x, f_k\rangle \langle f_k, y\rangle) & = & \langle \Phi_k \xi, \eta \rangle = (trace(\Phi_k [\![\xi,\eta]\!]) = \langle \Phi_k, [\![\xi,\eta]\!] \rangle_{HS} \end{array}$$

where $F_k = [\![f_k, f_k]\!] = f_k f_k^* \in \mathcal{S}^{1,0}(H)$.

The following objects play an important role in subsequent theory:

$$R : \mathbb{C}^n \to Sym(\mathbb{C}^n) \quad , \quad R(x) = \sum_{k=1}^m |\langle x, f_k\rangle|^2 f_k f_k^* \ , \ x \in \mathbb{C}^n \tag{3.33}$$

$$\mathcal{R} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{R}(\xi) = \sum_{k=1}^m \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n} \tag{3.34}$$

$$\mathcal{S} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{S}(\xi) = \sum_{k:\Phi_k\xi \neq 0} \frac{1}{\langle \Phi_k \xi, \xi \rangle} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n} \tag{3.35}$$

$$\mathcal{Z} : \mathbb{R}^{2n} \to \mathbb{R}^{2n \times m} \quad , \quad \mathcal{Z}(\xi) = \left[\begin{array}{c|c|c} \Phi_1 \xi & \cdots & \Phi_m \xi \end{array} \right] \ , \ \xi \in \mathbb{R}^{2n} \tag{3.36}$$

Note $\mathcal{R} = \mathcal{Z}\mathcal{Z}^T$.

Following [BBCE07] we note that $|\langle x, f_k \rangle|^2)$ is the Hilbert-Schmidt scalar product between two rank 1 symmetric forms:

$$|\langle x, f_k \rangle|^2 = trace\,(F_k X) = \langle F_k, X \rangle_{HS}$$

where $X = xx^*$. Thus the nonlinear map β induces a linear map on the real vector space $Sym(\mathbb{C}^n)$ of symmetric forms over $\mathbb{C}^n$:

$$\mathbb{A} : Sym(\mathbb{C}^n) \to \mathbb{R}^m \ , \ \mathbb{A}(T) = (\langle T, F_k \rangle_{HS})_{1 \leq k \leq m} = (\langle T f_k, f_k \rangle)_{1 \leq k \leq m} \tag{3.37}$$

Similarly it induces a linear map on $Sym(\mathbb{R}^{2n})$ the space of symmetric forms over $\mathbb{R}^{2n} = \jmath(\mathbb{C}^n)$ that is denoted by $\mathcal{A}$:

$$\mathcal{A} : Sym(\mathbb{R}^{2n}) \to \mathbb{R}^m \ , \ \mathcal{A}(T) = (\langle T, \Phi_k \rangle_{HS})_{1 \leq k \leq m} = (\langle T\varphi_k, \varphi_k \rangle + \langle TJ\varphi_k, J\varphi_k \rangle)_{1 \leq k \leq m} \tag{3.38}$$

Now we are ready to state a necessary and sufficient condition for injectivity that works in both the real and the complex case:

Theorem 3.1 ([HMW11, BCMN13, Ba13])**.** *Let $H = \mathbb{C}^n$ and let V be a real vector space that is also a subset of H, $V \subset H$. Denote $\mathcal{V} = \jmath(V)$ the realification of V. Assume $\mathcal{F}$ is a frame for V. The following are equivalent:*

(1) *The frame $\mathcal{F}$ is phase retrievable with respect to V;*
(2) $\ker \mathbb{A} \cap (\mathcal{S}^{1,0}(V) - \mathcal{S}^{1,0}(V)) = \{0\}$*;*
(3) $\ker \mathbb{A} \cap \mathcal{S}^{1,1}(V) = \{0\}$*;*
(4) $\ker \mathbb{A} \cap (\mathcal{S}^{2,0}(V) \cup \mathcal{S}^{1,1}(V) \cup \mathcal{S}^{0,2}) = \{0\}$*;*
(5) *There do not exist vectors $u, v \in V$ with $[\![u, v]\!] \neq 0$ so that*

$$real\left(\langle u, f_k\rangle\langle f_k, v\rangle\right) = 0 \ , \ \forall\, 1 \leq k \leq m$$

(6) $\ker \mathcal{A} \cap (\mathcal{S}^{1,0}(\mathcal{V}) - \mathcal{S}^{1,0}(\mathcal{V})) = \{0\}$*;*
(7) $\ker \mathcal{A} \cap \mathcal{S}^{1,1}(\mathcal{V}) = \{0\}$*;*
(8) *There do not exist vectors $\xi, \eta \in \mathcal{V}$, with $[\![\xi, \eta]\!] \neq 0$ so that*

$$\langle \Phi_k \xi, \eta\rangle = 0 \ , \ \forall\, 1 \leq k \leq m$$

Proof.

(1) $\Leftrightarrow$ (2) It is immediate once we noticed that any element in the null space of $\mathbb{A}$ of the form $xx^* - yy^*$ means $\mathbb{A}(xx^*) = \mathbb{A}(yy^*)$ for some $x, y \in V$ with $\hat{x} \neq \hat{y}$.

(2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (5) are consequences of (2.27).

In (4) note that $\ker \mathbb{A} \cap \mathcal{S}^{2,0}(V) = \{0\} = \ker \mathbb{A} \cap \mathcal{S}^{0,2}(V)$ since $\mathcal{F}$ is frame for V. Thus (3) $\Leftrightarrow$ (4).

(6),(7) and (8) are simply restatements of (2),(3) and (4) using the realification framework. □

In case (4) above, note $\mathcal{S}^{2,0}(V) \cup \mathcal{S}^{1,1}(V) \cup \mathcal{S}^{0,2}$ is the set of all rank-2 symmetric operators in $Sym(V)$ (This case, in particular, has been proposed in [BCMN13]).

The above general injectivity result is next made more explicit in the cases $V = \mathbb{C}^n$ and $V = \mathbb{R}^n$.

Theorem 3.2 ([BCE06, Ba12])**.** *(The real case) Assume $\mathcal{F} \subset \mathbb{R}^n$. The following are equivalent:*

(1) *$\mathcal{F}$ is phase retrievable for $V = \mathbb{R}^n$;*
(2) *$R(x)$ is invertible for every $x \in \mathbb{R}^n$, $x \neq 0$;*
(3) *There do not exist vectors $u, v \in \mathbb{R}^n$ with $u \neq 0$ and $v \neq 0$ so that*

$$\langle u, f_k\rangle\langle f_k, v\rangle = 0 \ , \ \forall\, 1 \leq k \leq m$$

(4) *For any disjoint partition of the frame set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, either $\mathcal{F}_1$ spans $\mathbb{R}^n$ or $\mathcal{F}_2$ spans $\mathbb{R}^n$.*

Recall a set $\mathcal{F} \subset \mathbb{C}^n$ is called *full spark* if any subset of n vectors is linearly independent. Then an immediate corrolary of the above result is the following

Corollary 3.3 ([BCE06]). *Assume $\mathcal{F} \subset \mathbb{R}^n$. Then*

(1) *If $\mathcal{F}$ is phase retrievable for $\mathbb{R}^n$ then $m \geq 2n-1$;*
(2) *If $m = 2n-1$, then $\mathcal{F}$ is phase retrievable if and only if $\mathcal{F}$ is full spark;*

Proof

Indeed, the first claim follows from Theorem 3.2(4): If $m \leq 2n-2$ then there is a partition of $\mathcal{F}$ into two subsets each of cardinal less than or equal to $n-1$. Thus neither set can span $\mathbb{R}^n$. Contradition.

The second claim is immediate from same statement as above. □

A more careful analysis of Theorem 3.2(4) gives a receipe of constructing two non-similar vectors $x, y \in \mathbb{R}^n$ so that $\alpha(x) = \alpha(y)$. Indeed, if $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ so that $\dim \operatorname{span}(\mathcal{F}_1) < n$ and $\dim \operatorname{span}(\mathcal{F}_2) < n$ then there are non-zero vectors $u, v \in \mathbb{R}^n$ with $\langle u, f_k \rangle = 0$ for all $k \in I$ and $\langle v, f_k \rangle = 0$ for all $k \in I^c$. Here I is the index set of frame vectors in $\mathcal{F}_1$ and I^c denotes its complement in $\{1, \cdots, m\}$. Set $x = u + v$ and $y = u - v$. Then $|\langle x, f_k \rangle| = |\langle v, f_k \rangle| = |\langle v, f_k \rangle|$ for all $k \in I$, and $|\langle x, f_k \rangle| = |\langle u, f_k \rangle| = |\langle y, f_k \rangle|$ for all $k \in I^c$. Thus $\alpha(x) = \alpha(y)$, but $x \neq y$ and $x \neq -y$.

Theorem 3.4 ([BCMN13, Ba13]). *(The complex case) The following are equivalent:*

(1) *$\mathcal{F}$ is phase retrievable for $H = \mathbb{C}^n$;*
(2) $\operatorname{rank}(\mathcal{Z}(\xi)) = 2n-1$ *for all* $\xi \in \mathbb{R}^{2n}$, $\xi \neq 0$;
(3) $\dim \ker \mathcal{R}(\xi) = 1$ *for all* $\xi \in \mathbb{R}^{2n}$, $\xi \neq 0$;
(4) *There do not exist $\xi, \eta \in \mathbb{R}^{2n}$, $\xi \neq 0$ and $\eta \neq 0$ so that $\langle J\xi, \eta \rangle = 0$ and*

$$\langle \Phi_k \xi, \eta \rangle = 0 \ , \ \forall 1 \leq k \leq m \tag{3.39}$$

In terms of cardinality, here is what we know:

Theorem 3.5 ([Mi67, HMW11, BH13, Ba13b, MV13, CEHV13, KE14, Viz15]).

(1) [HMW11] *If $\mathcal{F}$ is a phase retrievable frame for $\mathbb{C}^n$ then*

$$m \geq 4n - 2 - 2b + \begin{cases} 2 & \text{if } n \text{ odd and } b = 3 \bmod 4 \\ 1 & \text{if } n \text{ odd and } b = 2 \bmod 4 \\ 0 & \text{otherwise} \end{cases} \tag{3.40}$$

where $b = b(n)$ denotes the number of 1's in the binary expansion of $n-1$.
(2) [BH13] *For any positive integer n there is a frame with $m = 4n-4$ vectors so that $\mathcal{F}$ is phase retrievable for $\mathbb{C}^n$;*
(3) [CEHV13] *If $m \geq 4n-4$ then a (Zariski) generic frame is phase retrievable on $\mathbb{C}^n$;*
(4) [Ba13b] *The set of phase retrievable frames is open in $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$. In particular phase retrievable property is stable under small perturbations.*
(5) [CEHV13] *If $n = 2^k + 1$ and $m \leq 4m - 5$ then $\mathcal{F}$ cannot be phase retrievable for $\mathbb{C}^n$.*
(6) [Viz15] *For $n = 4$ there is a frame with $m = 11 < 4n - 4 = 12$ vectors that is phase retrievable.*

4. Finite Fourier Frames and z-Transforms

In this section we discuss the specific case of Finite Fourier Frames and sampling of the z-transform.

4.1. **The FFT.** Consider the case $H = \mathbb{C}^n$ and of Finite Fourier Transform (FFT) whose frame vectors are given by

$$f_k = \begin{bmatrix} 1 \\ e^{2\pi i k/m} \\ e^{2\pi i 2k/m} \\ \vdots \\ e^{2\pi i(n-1)k/m} \end{bmatrix} \ , \ 1 \leq m. \tag{4.41}$$

First we derive the *Patterson's Auto-Correlation Function* and then we show the reconstruction problem is equivalent with a spectral factorization problem. For a vector $x \in \mathbb{C}^n$, the map $\beta : \hat{H} \to \mathbb{R}^m$ is given by, for every $1 \leq k \leq m$,
(4.42)

$$(\beta(x))_k = |\sum_{t=1}^{n} e^{-2\pi i k t/m} x_t|^2 = \sum_{t_1,t_2=1}^{n} e^{-2\pi i k(t_1-t_2)/m} x_{t_1}\overline{x_{t_2}} = \sum_{\tau=-(n-1)}^{n-1} e^{-2\pi i k\tau/m} \sum_{\substack{1 \leq t_1,t_2 \leq n \\ t_1-t_2=\tau}} x_{t_1}\overline{x_{t_2}}.$$

Let us denote by $r = (r_\tau)_{-(n-1)\leq\tau\leq n-1}$ the *autcorrelation* of signal x,

$$r_\tau = \sum_{\substack{1 \leq t_1,t_2 \leq n \\ t_1-t_2=\tau}} x_{t_1}\overline{x_{t_2}} = \begin{cases} \sum_{t=1}^{n-\tau} x_{t+\tau}\overline{x_t} & if \quad 0 \leq \tau \leq n-1 \\ \sum_{t=1-\tau}^{n} x_{t+\tau}\overline{x_t} & if \quad -n+1 \leq \tau < 0 \end{cases} . \tag{4.43}$$

Statistically speaking we should call $r = (r_\tau)$ the *unnormalized sample auto-covariance*, but for simplicity we prefer to use the term autocorrelation instead. Note that r satisfies the symmetry relation

$$r_{-\tau} = \overline{r_\tau}$$

in other words, it has conjugate parity. Thus

$$(\beta(x))_k = \sum_{\tau=-(n-1)}^{n-1} e^{-2\pi i k\tau/m} r_\tau \ , \ 1 \leq k \leq m,$$

which is the finite Fourier transform of the sequence r. The transform is invertible provided $m \geq 2n-1$, where $2n-1$ is the cardinal of the set of distinct values of τ, $\{-(n-1), -(n-2), \cdots, -1, 0, 1, 2, \cdots, n-2, n-1\}$. In the following we assume $m \geq 2n-1$. The inverse finite Fourier transform gives:

$$r_\tau = \frac{1}{m}\sum_{k=1}^{m} e^{2\pi i k\tau/m}(\beta(x))_k \ , \ -(n-1) \leq \tau \leq n-1. \tag{4.44}$$

Thus the sample autocovariance can be computed directly from the magnitudes of frame coefficients, without need for phases. In the X-Ray Crystallography context $r = (r_\tau)$ representes the autocorrelation function of the electron density, known also as the Patterson function [Patt35].

Next we parametrize all possible vectors in $\mathbb{C}^n$ that have the same magnitudes of Fourier coefficients. We should that in general the map β is not injective. Nevertheless we can find the exact structure of $\beta^{-1}(c)$. These results can be found in literature. For instance [LaFrBa87] presents an image reconstruction algorithm based on spectral factorizations presented in Theorems 4.1, 4.3 below.

First we let $X(z)$ and $R(z)$ denote the z-transforms of vectors x and r, respectively:

$$X(z) = \sum_{k=1}^{n} z^{k-1}x_k \ , \ R(z) = \sum_{\tau=-n+1}^{n-1} z^\tau r_\tau$$

A direct computation shows

$$R(z) = \sum_{\tau=-n+1}^{n-1} \sum_{\substack{1 \leq t_1, t_2 \leq n \\ t_1 - t_2 = \tau}} z^{t_1 - 1}x_{t_1} z^{-(t_2-1)}\overline{x_{t_2}} = X(z)\bar{X}(\frac{1}{z})$$

where $\bar{X}(u) = \sum_{k=1}^{n} u^k \overline{x_k}$ is obtained by conjugating only the coefficients of polynomial $X(u)$. Thus we obtained that finding the vector x reduces to a factorization problem for the z-transform of the auto-covariance function. Note that $R(z)$ is a Laurent polynomial that satisfies:

$$\begin{aligned} R(\frac{1}{z}) &= \bar{R}(z) = \overline{R(\bar{z})} \\ R(e^{i\omega}) &= |X(e^{i\omega})|^2 \geq 0 \end{aligned}$$

Thus if $w \in \mathbb{C}$ is a zero of the polynomial $H(z) = z^{n-1}R(z)$, that is $R(w) = 0$, then so is $\frac{1}{\bar{w}}$, that means $R(\frac{1}{\bar{w}}) = 0$. In particular note $w \neq 0$. Let $\mathcal{W} = \{w_1, \cdots, w_{2n-2}\}$ denote the set of all $2n-2$ zeros of H, including multiplicities. On the other hand $H(z) = z^{n-1}R(z) = X(z)(z^{n-1}\bar{X}(\frac{1}{z}))$ shows that the set of zeros of polynomial $X(z)$ is a subset of $\mathcal{W}$. In fact we proved the following result

Theorem 4.1. *Consider the finite Fourier frame (4.41) with $m \geq 2n-1$. Given $c = \beta(x) \in \mathbb{R}^m$ compute the autocorrelation vector $r \in \mathbb{C}^{2n-1}$ by (4.44) and then the set $\mathcal{W}$ of $2n-2$ zeros of $H(z) = z^{n-1}R(z)$. The following hold true:*

(1) *The set* $\mathcal{W}_1$ *of* $n-1$ *zeros of* $X(z)$ *is a subset of* $\mathcal{W}$ *that satisfies the following properties: (i) if* $w \in \mathcal{W}_1$ *then* $\frac{1}{\bar{w}} \in \mathcal{W} \setminus \mathcal{W}_1$*; (ii) If* $|w| = 1$ *then* w *has even multiplicity.*

(2) *Let* $\mathcal{W}_2 \subset \mathcal{W}$ *be a subset of* $n-1$ *numbers (allowing for posible repetitions) that satisfies the following property: if* $w \in \mathcal{W}_2$ *then* $\frac{1}{\bar{w}} \in \mathcal{W} \setminus \mathcal{W}_2$*. Construct the polynomial*

$$\tilde{X}(z) = C \prod_{w \in \mathcal{W}_2} (z - w) = \sum_{k=1}^{n} z^{k-1} \tilde{x}_k \tag{4.45}$$

where C *is a constant so that* $\sum_{k=1}^{n} |\tilde{x}_k|^2 = r_0$*. Then* $\tilde{x} = (\tilde{x}_k)_{1 \le k \le n} \in \mathbb{C}^n$ *satisfies* $\beta(\tilde{x}) = \beta(x)$*.*

Note the constant C in 4.47) is given explicitly as follows. Let $y_1, \cdots, y_n \in \mathbb{C}$ be the coefficient of $\prod_{w \in \mathcal{W}_2} = \sum_{k=1}^{n} z^{k-1} y_k$. Then

$$C = e^{i\varphi} \sqrt{\frac{r_0}{|y_1|^2 + \cdots + |y_n|^2}}, \tag{4.46}$$

with $\varphi \in [0, 2\pi)$. Note we use the concept of *set* in a slightly more general way: we allow for repetitions of elements, and we keep track of elements when taking union of sets.

This theorem parametrizes all possible classes $\hat{\tilde{x}} \in \mathbb{C}^n$ of vectors that have the same magnitudes of Fourier coefficients: each class is associated with a distinct subset $\mathcal{W}_2 \subset \mathcal{W}$ that satisfies the property in part (2) of the theorem. For distinct (simple) zeros of $H(z)$ the possible number of subsets $\mathcal{W}_2$ is 2^{n-1}. When repetitions occur, the number is smaller. Specifically the number if given by the following result.

Proposition 4.2. *Consider the setup in Theorem 4.1. Let* d_1 *denote the number of distinct zeros in* $\mathcal{W}$ *of magnitude strictly larger than* 1*, say* $w_1, w_2, \cdots, w_{d_1}$*. Let* $n_1, n_2, \cdots, n_{d_1}$ *denote their multiplicities. Let* d_0 *denote the number of distinct zeros on the unit circle (hence each of magnitude one). Denote by* $2p_1, \cdots, 2p_{d_0}$ *their (even) multiplicities. Then the number of distinct partitions of* $\mathcal{W}$ *that satisfy part 2 of Theorem 4.1, and hence the number of classes in* $\beta^{-1}(c)$ *is*

$$\prod_{k=1}^{d_1} \binom{2n_k}{n_k} \prod_{k=1}^{d_0} \binom{2p_k}{p_k}$$

On the other hand, when more information on x is known, such as x is a real vector and/or sparse, then fewer admissible partitions of the set $\mathcal{W}$ are possible. For instance if $x \in \mathbb{R}^n$ then if $w \in \mathcal{W}_1$ then so is $\bar{w} \in \mathcal{W}_1$, In this case the previous result takes the following form:

Theorem 4.3. *Consider the finite Fourier frame (4.41) with* $m \ge 2n-1$*. Given* $c = \beta(x) \in \mathbb{R}^m$ *for some* $x \in \mathbb{R}^n$*, compute the autocorrelation vector* $r \in \mathbb{C}^{2n-1}$ *by (4.44) and then the set* $\mathcal{W}$ *of* $2n-2$ *zeros of* $H(z) = z^{n-1}R(z)$*. The following hold true:*

(1) *The set* $\mathcal{W}_1$ *of* $n-1$ *zeros of* $X(z)$ *is a subset of* $\mathcal{W}$ *that satisfies the following properties: (i) if* $w \in \mathcal{W}_1$ *then* $\bar{w} \in \mathcal{W}_1$ *and* $\frac{1}{w}, \frac{1}{\bar{w}} \in \mathcal{W} \setminus \mathcal{W}_1$*; (ii) if* $|w| = 1$ *then* w *has even multiplicity;*

(2) *Let* $\mathcal{W}_2 \subset \mathcal{W}$ *be a subset of* $n-1$ *numbers (allowing for posible repetitions) that satisfies the following property: if* $w \in \mathcal{W}_2$ *then* $\bar{w} \in \mathcal{W}_2$ *and* $\frac{1}{w}, \frac{1}{\bar{w}} \in \mathcal{W} \setminus \mathcal{W}_2$. *Construct the polynomial*

$$\tilde{X}(z) = C \prod_{w \in \mathcal{W}_2} (z-w) = \sum_{k=1}^{n} z^{k-1} \tilde{x}_k \tag{4.47}$$

where C *is a constant so that* $\sum_{k=1}^{n} |\tilde{x}_k|^2 = r_0$. *Then* $\tilde{x} = (\tilde{x}_k)_{1 \leq k \leq n} \in \mathbb{R}^n$ *satisfies* $\beta(\tilde{x}) = \beta(x)$.

In this case, when all zeros of $H(z)$ are simple (have no multiplicity), let n_c and n_r denote the number of complex 4-tuples and real pairs of zeros:

$$\mathcal{W} = \{w_1, \overline{w_1}, \frac{1}{w_1}, \frac{1}{\overline{w_1}}, \cdots, w_{n_c}, \overline{w_{n_c}}, \frac{1}{w_{n_c}}, \frac{1}{\overline{w_{n_c}}}\} \cup \{p_1, \frac{1}{p_1}, \cdots, p_{n_r}, \frac{1}{p_{n_r}}\}.$$

Then the number of distinct classes in $\hat{\mathbb{R}^n}$ that have the same magnitudes of frame coefficients is $2^{n_c+n_r} = 2^{n-1}/2^{n_c}$ since $4n_c + 2n_r = 2n-2$. Thus the number of distinct classes is smaller than 2^{n-1} unless all zeros are real. When repetitions occur the number of distinct classes is smaller.

4.2. **The z-Transform.** One consequence of Theorems 4.1 and 4.3 is that increasing redundancy by increasing m does not solve the problem. Indeed, for any $m \geq 2n-1$ the multiplicity of solutions does not change. However the inversion problem can be solved if a redundancy is achieved by considering a different transform in addition to the FFT. Here we present the case of the z-transform sampled at special points. Results of this section appeared also in [BH13, Ba13b, Ba15]. Some of the techniques are borrowed from [Ja10].

Fix $z \in \mathbb{C}$ and define the frame vector

$$g^{(z)} = \begin{bmatrix} 1 \\ \bar{z} \\ \bar{z}^2 \\ \vdots \\ \bar{z}^{n-1} \end{bmatrix}.$$

For $z = e^{-2\pi i k/m}$ we obtain the same vectors as in (4.41). Fix $a > 0$ and define the following set of vectors

$$\mathcal{G} = \{g^{(z)}\ ,\ z \in \Lambda\}\ ,\ \Lambda = \Lambda_1 \cup \Lambda_2 \tag{4.48}$$

where

$$\Lambda_1 = \left\{e^{-2\pi i k/(2n-1)}\ ,\ 1 \leq k \leq 2n-3\right\} \tag{4.49}$$

and

$$\Lambda_2 = \left\{ \frac{sin\left(\frac{\pi}{2n-1}\right)}{sin(a)} e^{2\pi i(k-1)/(2n-1)} - e^{i\pi/(2n-1)} \frac{sin\left(\frac{\pi}{2n-1} - a\right)}{sin(a)}\ ,\ 1 \leq k \leq 2n-1 \right\}. \tag{4.50}$$

The construction has the following geometric picture: Λ is the set of $2n-1$-roots of unity except 1 and $e^{2\pi i/(2n-1)}$; Λ_2 is a set of $2n-1$ equidistant points on the circle of center $\gamma = -e^{i\pi/(2n-1)} sin(\frac{\pi 2n-1-a}{)}/sin(a)$ and radius $\rho = sin(\pi/(2n-1))/sin(a)$. Note this circle intersects the unit circle in exactly two points: 1 and $e^{2\pi i/(2n-1)}$, since it is the image of the real line under the conformal map, and Linear Fractional Transform, $t \mapsto \frac{e^{1a}t - e^{2\pi i/(2n-1)}}{e^{ia}t-1}$.

For any $z \in \mathbb{C}$, we denote

$$\beta(x)_z = |\langle x, g^{(z)}\rangle|^2 = |X(z)|^2.$$

For $z = \gamma + \rho e^{-i\theta}$, a point on the circle containing Λ_2, we obtain

$$X(z) = \sum_{k=0}^{n-1} x_{k+1}(\gamma + \rho e^{-i\theta})^k = \sum_{k=0}^{n-1} y_{k+1} e^{-ik\theta} = Y(e^{-i\theta})$$

where $(y_1, y_2, \cdots, y_n)$ is the linear transformation of $(x_1, x_2, \cdots, x_n)$ given by

$$y_{p+1} = \sum_{k=p}^{n-1} \left(\begin{array}{c} k \\ p \end{array} \right) \gamma^{k-p} \rho^p x_{k+1} \ , \ 0 \le p \le n-1$$

and $Y(z) = y_1 + y_2 z + \cdots + y_n z^{n-1}$ is the z-transform of y. Let

$$r_\tau^{(y)} = \sum_{\substack{1 \le t_1, t_2 \le n \\ t_1 - t_2 = \tau}} y_{t_1} \overline{y_{t_2}}$$

denote the autocorrelation of y. Thus for $z \in \Lambda_2$, $(\beta(x))_z = |Y(e^{-i\theta})|^2 = \sum_{\tau=-(n-1)}^{n-1} r_\tau^{(y)} e^{-i\tau\theta}$. A similar computation as in the previous subsection shows

$$r_\tau^{(y)} = \frac{1}{2n-1} \sum_{z=\gamma+\rho e^{2\pi ik/(2n-1)} \in \Lambda_2} e^{2\pi i k\tau/(2n-1)} (\beta(x))_z \ , \ -(n-1) \le \tau \le n-1.$$

Thus we can compute the z-transform of $r^{(y)}$ and obtain

$$R^{(y)}(z) := \sum_{\tau=-(n-1)}^{n-1} r_\tau^{(y)} z^\tau = Y(z)\bar{Y}(\frac{1}{z})$$

A careful computation relates $Y(z)$ to $X(z)$:

$$Y(z) = \sum_{p=0}^{n-1} y_{p+1} z^p = \sum_{p=0}^{n-1}\sum_{k=p}^{n-1} \left(\begin{array}{c} k \\ p \end{array} \right) \gamma^{k-p}\rho^p x_{k+1} z^p = \sum_{k=0}^{n-1} x_{k+1} \sum_{p=0}^{k} \left(\begin{array}{c} k \\ p \end{array} \right) \gamma^{k-p} (\rho z)^p = X(\gamma + \rho z).$$

Similarly we obtain $\bar{Y}(\frac{1}{z}) = \bar{X}(\bar{\gamma} + \frac{\rho}{z})$. Thus we get:

$$R^{(y)}(z) = X(\gamma + \rho z)\bar{X}(\bar{\gamma} + \frac{\rho}{z})$$

In particular for $z_0 = (1-\gamma)/\rho$ we obtain $z_0 = e^{i\psi}$, for some real ψ, and thus $R^{(y)}(z_0) = |X(1)|^2$. Similarly there is a real ψ_1 so that $z_1 = \gamma + \rho e^{-i\psi_1} = e^{2\pi i/(2n-1)}$. Thus $R^{(y)}(z_1) = |X(e^{2\pi i/(2n-1)})|^2$. Together with $(\beta(x))_z$ for $z \in \Lambda_1$ we obtain all values of $|X(e^{2\pi ik/(2n-1)})|^2$. By the computations of the previous section we get the Laurent polynomial $R^{(x)}(z) = X(z)\bar{X}(\frac{1}{z})$. Now we show that for special values of a the frame $\mathcal{G}$ is phase retrievable:

Theorem 4.4 ([BH13]). *Assume $\frac{a}{\pi}$ is irrational in $(0, \frac{1}{2})$. Then the frame $\mathcal{G} = \{g^{(z)}\ ,\ z \in \Lambda$ introduced in (4.48) of $4n-4$ vectors is phase retrievable.*

Proof The proof of this result comes from noticing the two sets of $2n-2$ zeros of $z^{n-1}R^{(y)}(z)$ and of $z^{n-1}R^{(x)}(z)$ can be uniquely partitioned to satisfy the corresponding symmetries. Specifically, we showed that from $(\beta(x))_z = |\langle x, g^{(z)}\rangle|^2$ with $z \in \Lambda$ we compute Laurent polynomials $R^{(x)}(z)$ and $R^{(y)}(z)$. Let $\mathcal{W}_1$ denote the set of $2n-2$ zeros of $H_x(z) = z^{n-1}R^{(x)}(z)$ and let $\mathcal{W}_2$ denote the set of $2n-2$ zeros of $H_y(z) = R^{(y)}(z)$. Set $\mathcal{W}^x = \emptyset$. Repeat the following steps $n-1$ times until $\mathcal{W}_1$ becomes empty.

(1) Pick a $w_0 \in \mathcal{W}_1$.
(2) If $\frac{w_0-\gamma}{\rho} \in \mathcal{W}_2$ then update:
  - $\mathcal{W}^{(x)} = \mathcal{W}^{(x)} \cup \{w_0\}$;
  - Remove one w_0 and one $\frac{1}{\bar{w_0}}$ from $\mathcal{W}_1$. Remove one $\frac{w_0-\gamma}{\rho}$ and one $\frac{\rho}{\overline{w_0}-\bar{\gamma}}$ from $\mathcal{W}_2$.

  Else update:
  - $\mathcal{W}^{(x)} = \mathcal{W}^{(x)} \cup \{\frac{1}{\overline{w_0}}\}$.
  - Remove one w_0 and one $\frac{1}{\bar{w_0}}$ from $\mathcal{W}_1$. Remove one $\frac{1}{\rho}(\frac{1}{\overline{w_0}} - \gamma)$ and one $\frac{\rho}{\frac{1}{w_0}-\bar{\gamma}}$ from $\mathcal{W}_2$.

At the end of the algorithm we obtain a set $\mathcal{W}^{(x)}$ of $n-1$ numbers. Then the polynomial $X(z)$ is obtained as

$$X(z) = C \prod_{w \in \mathcal{W}^{(x)}} (z-w)$$

where the normalization constant is as in Theorem 4.1, part 2. The only remaining item to prove is to show that when w_0 is a zero of $X(z)$ so that $\frac{1}{\overline{w_0}}$ is not a zero of $X(z)$, then $\frac{w_0-\gamma}{\rho} \in \mathcal{W}_2$ but $\frac{1}{\rho}(\frac{1}{\overline{w_0}} - \gamma) \notin \mathcal{W}_2$. First claim is immediate:

$$H_y(\frac{1}{\rho}(w_0-\gamma)) = (z^{n-1}\bar{X}(\bar{\gamma} + \frac{\rho}{z})|_{z=(w_0-\gamma)/\rho} X(w_0) = 0.$$

The second claim is proved by a special symmetry of zeros. See [BH13] for details.

5. Robustness of Reconstruction

In this section we analyze stability bounds for reconstruction. Specifically we analyze two types of margins:

- Deterministic, worst-case type bounds: These bounds are given by lower Lipschitz constant of the forward nonliner analysis map;
- Stochastic, average type bounds: Cramer-Rao Lower Bounds

5.1. **Bi-Lipschitzianity of the Nonlinear Analysis Maps.** In section 2 we introduced two metrics on $\hat{H}$. As the following theorem shows, the nonlinear maps α and β are bi-Lipschitz with respect to the corresponding metric:

Theorem 5.1. [Ba12, EM12, BCMN13, Ba13, BW13, BZ14, BZ15] *Let $\mathcal{F}$ be a phase retrievable frame for V, a real linear space, subset of $H = \mathbb{C}^n$. Then:*

(1) *The nonlinear map $\alpha : (\hat{V}, D_2) \to (\mathbb{R}^m, \|\|_2)$ is bi-Lipschitz. Thus there are positive constants $0 < A_0 \le B_0 < \infty$ so that*

$$\sqrt{A_0} D_2(x,y) \le \|\alpha(x) - \alpha(y)\|_2 \le \sqrt{B_0} D_2(x,y) \ , \ \forall x,y \in V \tag{5.51}$$

(2) *The nonlinear map $\beta : (\hat{V}, d_1) \to (\mathbb{R}^m, \|\|_2)$ is bi-Lipschitz. Thus there are positive constants $0 < a_0 \le b_0 < \infty$ so that*

$$\sqrt{a_0} d_1(x,y) \le \|\beta(x) - \beta(y)\|_2 \le \sqrt{b_0} d_1(x,y) \ , \ \forall x,y \in V \tag{5.52}$$

The converse is also true: If either (5.51) or (5.52) holds true for all $x, y \in V$ then $\mathcal{F}$ is phase retrievable for V.

The choice of distance D_2 and d_1 in the statement of this theorem is only for convenience reasons. Any other distance D_p instead of D_2, and d_q instead of d_1 would work. The Lipschitz constants would be different, of course.

On the other hand, if α satisfies (5.51) or β satisfies (5.52) then $\mathcal{F}$ is phase retrievable for V. Thus, in effect, we obtained a necessary and sufficient condition for phase retrievability. We state this condition now:

Theorem 5.2. *Let $\mathcal{F} \subset H = \mathbb{C}^n$ and let V be a real vector space, subset of H. Denote by $\mathcal{V} = \jmath(V) \subset \mathbb{R}^{2n}$ the realification of V, and let Π denote the projection onto $\mathcal{V}$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is phase retrievable for V;*
(2) *There is a constant $a_0 > 0$ so that*

$$\Pi \mathcal{R}(\xi) \Pi \ge a_0 \Pi P_{J\xi}^{\perp} \Pi \ , \ \forall \xi \in \mathcal{V}, \|\xi\| = 1 \tag{5.53}$$

where $P_{J\xi}^{\perp} = I_{2n} - P_{J\xi} = I_{2n} - J\xi\xi^T J^T$ is the orthogonal projection onto the orthogonal complement to $J\xi$;
(3) *There is $a_0 > 0$ so that for all $\xi, \eta \in \mathbb{R}^{2n}$,*

$$\sum_{k=1}^{m} |\langle \Pi \Phi_k \Pi \xi, \eta \rangle|^2 \ge a_0 \left(\|\Pi\xi\|^2 \|\Pi\eta\|^2 - |\langle J\Pi\xi, \Pi\eta \rangle|^2 \right) \tag{5.54}$$

Note the same constant a_0 can be chosen in (5.52) and (5.53) and (5.54).

The lower bounds computation is fairly subtle. In fact there is a distinction between local bounds and global bounds. Specifically for every $z \in V$ we define the following:

The *type I local lower Lipschitz bounds* are defined as:

$$A(z) = \lim_{r\to 0} \inf_{x,y\in V, D_2(x,z)<r, D_2(y,z)<r} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \tag{5.55}$$

$$a(z) = \lim_{r\to 0} \inf_{x,y\in V, d_1(x,z)<r, d_1(y,z)<r} \frac{\|\beta(x)-\beta(y)\|_2^2}{d_1(x,y)^2} \tag{5.56}$$

The *type II local lower Lipschitz bounds* are defined by:

$$\tilde{A}(z) = \lim_{r\to 0} \inf_{y\in V, D_2(y,z)<r} \frac{\|\alpha(z)-\alpha(y)\|_2^2}{D_2(z,y)^2} \tag{5.57}$$

$$\tilde{a}(z) = \lim_{r\to 0} \inf_{y\in V, d_1(y,z)<r} \frac{\|\beta(z)-\beta(y)\|_2^2}{d_1(z,y)^2} \tag{5.58}$$

Similarly the *type I local upper Lipschitz bounds* are defined as:

$$B(z) = \lim_{r\to 0} \sup_{x,y\in V, D_2(x,z)<r, D_2(y,z)<r} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \tag{5.59}$$

$$b(z) = \lim_{r\to 0} \sup_{x,y\in V, d_1(x,z)<r, d_1(y,z)<r} \frac{\|\beta(x)-\beta(y)\|_2^2}{d_1(x,y)^2} \tag{5.60}$$

and the *type II local upper Lipschitz bounds* are defined by:

$$\tilde{B}(z) = \lim_{r\to 0} \sup_{y\in V, D_2(y,z)<r} \frac{\|\alpha(z)-\alpha(y)\|_2^2}{D_2(z,y)^2} \tag{5.61}$$

$$\tilde{b}(z) = \lim_{r\to 0} \sup_{y\in V, d_1(y,z)<r} \frac{\|\beta(z)-\beta(y)\|_2^2}{d_1(z,y)^2} \tag{5.62}$$

The *global lower bounds* are defined by:

$$A_0 = \inf_{x,y\in V, D_2(x,y)>0} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \tag{5.63}$$

$$a_0 = \inf_{x,y\in V, d_1(x,y)>0} \frac{\|\beta(x)-\beta(y)\|_2^2}{d_1(x,y)^2} \tag{5.64}$$

whereas the *global upper bounds* are defined by:

$$B_0 = \sup_{x,y\in V, D_2(x,y)>0} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \tag{5.65}$$

$$b_0 = \sup_{x,y\in V, d_1(x,y)>0} \frac{\|\beta(x)-\beta(y)\|_2^2}{d_1(x,y)^2} \tag{5.66}$$

and represent the square of the corresponding Lipschitz constants.

Due to homogeneity $A_0 = A(0)$, $B_0 = B(0)$, $a_0 = a(0)$, $b_0 = b(0)$. On the other hand, for $z \neq 0$, $A(z) = A(\frac{z}{\|z\|})$, $B(z) = B(\frac{z}{\|z\|})$, $a(z) = a(\frac{z}{\|z\|})$, $b(z) = b(\frac{z}{\|z\|})$.

The exact expressions for these constants is summarized by the following results. For any $I \subset \{1,2,\cdots,m\}$ let $\mathcal{F}[I] = \{f_k\ ,\ k \in I\}$ denote the frame subset indexed by I. Let also $\sigma_1^2[I]$ and $\sigma_n^2[I]$ denote the upper and the lower frame bound of set $\mathcal{F}[I]$, respectively. Thus:

$$\sigma_1^2[I] = \lambda_{max}\left(\sum_{k\in I} f_k f_k^*\right)$$

$$\sigma_n^2[I] = \lambda_{min}\left(\sum_{k\in I} f_k f_k^*\right)$$

As usual, I^c denotes the complement of index set I, that is $I^c = \{1,\cdots,m\} \setminus I$.

Theorem 5.3 ([BW13]). *(The real case) Assume $\mathcal{F} \subset \mathbb{R}^n$ is a phase retrievable frame for $\mathbb{R}^n$. Let A and B denote its optimal lower and upper frame bound, respectively.Then:*

(1) *For every $0 \neq x \in \mathbb{R}^n$, $A(x) = \sigma_n^2(supp(\alpha(x))$, where $supp(\alpha(x)) = \{k\ ,\ \langle x, f_k\rangle \neq 0\}$;*
(2) *For every $x \in \mathbb{R}^n$, $\tilde{A}(x) = A$;*
(3) $A_0 = A(0) = \min_I(\sigma_n^2[I] + \sigma_n^2[I^c])$;
(4) *For every $x \in \mathbb{R}^n$, $B(x) = \tilde{B}(x) = B$;*
(5) $B_0 = B(0) = \tilde{B}(0) = B$, *the optimal upper frame bound;*
(6) *For every $0 \neq x \in \mathbb{R}^n$, $a(x) = \tilde{a}(x) = \lambda_{min}(R(x))/\|x\|^2$;*
(7) $a_0 = a(0) = \tilde{a}(0) = min_{\|x\|=1}\lambda_{min}(R(x))$;
(8) *For every $0 \neq x \in \mathbb{R}^n$, $b(x) = \tilde{b}(x) = \lambda_{max}(R(x))/\|x\|^2$;*
(9) $b_0 = b(0) = \tilde{b}(0) = \max_{\|x\|=1}\lambda_{max}(R(x))$;
(10) *a_0 is the largest constant so that*

$$R(x) \geq a_0\|x\|^2 I_n\ ,\ \forall\, x \in \mathbb{R}^n$$

or, equivalently,

$$\sum_{k=1}^m |\langle x, f_k\rangle|^2 |\langle y, f_k\rangle|^2 \geq a_0\|x\|^2\|y\|^2\ ,\ \forall\, x, y \in \mathbb{R}^n$$

(11) *b_0 is the 4^{th} power of the frame analysis operator norm $T : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_4)$,*

$$b_0 = \|T\|_{B(l^2,l^4)}^4 = \max_{\|x\|_2=1}\sum_{k=1}^m |\langle x, f_k\rangle|^4$$

The complex case is more subtle. The following result presents some of the local and global Lipschitz bounds.

Theorem 5.4 ([BZ15]). *(The complex case) Assume $\mathcal{F}$ is phase retrievable for $H = \mathbb{C}^n$ and A, B are its optimal frame bounds. Then:*

(1) *For every $0 \neq z \in \mathbb{C}^n$, $A(z) = \lambda_{2n-1}(\mathcal{S}(\jmath(z)))$ (the next to the smallest eigenvalue);*
(2) $A_0 = A(0) > 0$;

(3) *For every* $z \in \mathbb{C}^n$, $\tilde{A}(z) = \lambda_{2n-1}\left(\mathcal{S}(\jmath(z)) + \sum_{k:\langle z,f_k\rangle=0}\Phi_k\right)$ *(the next to the smallest eigenvalue);*
(4) $\tilde{A}(0) = A$, *the optimal lower frame bound;*
(5) *For every* $z \in \mathbb{C}^n$, $B(z) = \tilde{B}(z) = \lambda_1\left(\mathcal{S}(\jmath(z)) + \sum_{k:\langle z,f_k\rangle=0}\Phi_k\right)$ *(the largest eigenvalue);*
(6) $B_0 = B(0) = \tilde{B}(0) = B$, *the optimal upper frame bound;*
(7) *For every* $0 \neq z \in \mathbb{C}^n$, $a(z) = \tilde{a}(z) = \lambda_{2n-1}(\mathcal{R}(\jmath(z)))/\|z\|^2$ *(the next to the smallest eigenvalue);*
(8) *For every* $0 \neq z \in \mathbb{C}^n$, $b(z) = \tilde{b}(z) = \lambda_1(\mathcal{R}(\jmath(z)))/\|z\|^2$ *(the largest eigenvalue);*
(9) a_0 *is the largest constant to that*

$$\mathcal{R}(\xi) \geq a_0(I - J\xi\xi^T J^T)\ ,\ \forall\,\xi \in \mathbb{R}^{2n}, \|\xi\| = 1$$

or, equivalently

$$\sum_{k=1}^{m} |\langle \Phi_k\xi, \eta\rangle|^2 \geq a_0\left(\|\xi\|^2\|\eta\|^2 - |\langle J\xi, \eta\rangle|^2\right)\ ,\ \forall\,\xi,\eta \in \mathbb{R}^{2n}$$

(10) $b(0) = \tilde{b}(0) = b_0$ *is the* 4^{th} *power of the frame analysis operator norm* $T : (\mathbb{C}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_4)$,

$$b_0 = \|T\|^4_{B(l^2,l^4)} = \max_{\|x\|_2=1}\sum_{k=1}^{m}|\langle x, f_k\rangle|^4$$

(11) $\tilde{a}(0)$ *is given by*

$$\tilde{a}(0) = \min_{\|z\|=1}\sum_{k=1}^{m}|\langle z, f_k\rangle|^4$$

The results presented so far show that both α and β admit left inverses that are Lipschitz continuous. One remaining problem is to know if these left inverses can be extended to Lipschitz maps over the entire $\mathbb{R}^m$. The following two results provide a positive answer (see [BZ14, BZ15] for the construction):

Theorem 5.5 ([BZ15])**.** *Assume* $\mathcal{F} \subset H = \mathbb{C}^n$ *is a phase retrievable frame for* $\mathbb{C}^n$*. Let* $\sqrt{A_0}$ *be the lower Lipschitz constant of the map* $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \|\cdot\|_2)$*. Then there is a Lipschitz map* $\omega : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, D_2)$ *so that: (i)* $\omega(\alpha(x)) = x$ *for all* $x \in \hat{H}$*, and (ii) its Lipschitz constant is* $Lip(\omega) \leq \frac{4+3\sqrt{2}}{\sqrt{A_0}}$.

Theorem 5.6 ([BZ14])**.** *Assume* $\mathcal{F} \subset H = \mathbb{C}^n$ *is a phase retrievable frame for* $\mathbb{C}^n$*. Let* $\sqrt{a_0}$ *be the lower Lipschitz constant of the map* $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \|\cdot\|_2)$*. Then there is a Lipschitz map* $\psi : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, d_1)$ *so that: (i)* $\psi(\beta(x)) = x$ *for all* $x \in \hat{H}$*, and (ii) its Lipschitz constant is* $Lip(\psi) \leq \frac{4+3\sqrt{2}}{\sqrt{a_0}}$.

5.2. **Cramer-Rao Lower Bounds.** Consider the following measurement process:

$$y_k = |\langle x, f_k\rangle|^2 + \nu_k \ , \ 1 \leq k \leq m \tag{5.67}$$

where $\mathcal{F} = \{f_1, \cdots, f_m\} \subset H = \mathbb{C}^n$ is a phase retrievable frame for V, a real linear space, subset of H, and $x \in V$. We further assume that $\nu = (\nu_1, \cdots, \nu_m)$ is a sample of a normal random variable of zero mean and variance $\sigma^2 I_m$. We would like to find a lower bound on the variance of any unbiased estimator for x. To make the problem identifiable we make an additional assumption. Let $z_0 \in V$ be a fixed vector. Define

$$V_{z_0} = \{x \in V \ , \ \langle x, z_0\rangle > 0\} \tag{5.68}$$

where the scalar product is the one from H. Set $E_{z_0} = span_{\mathbb{R}}(V_{z_0})$ the real vector space spanned by V_{z_0}.

To make (5.67) identifiable we assume $x \in V_{z_0}$.

Thus any unbiased estimator is a map $\psi : \mathbb{R}^m \to E_{z_0}$ so that $\mathbb{E}[\psi(\beta(x) + \nu)] = x$ for all $x \in V_{z_0}$. Here the expectation is taken with respect to the noise random variable.

For the process (5.67) one can compute the Fisher information matrix $I(x)$. Following [BCMN13] and [Ba13] we obtain:

$$I(x) = \frac{4}{\sigma^2}\mathcal{R}(\xi) = \frac{4}{\sigma^2}\sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k \tag{5.69}$$

where $\xi = \jmath(x) \in \mathbb{R}^{2n}$. In general $I(x)$ has rank at most $2n-1$ because $J\xi$ is always in its kernel. A careful analysis of the estimation process shows that the CRLB (Cramer-Rao Lower Bound) for the estimation problem (5.67) is given by $(\Pi_{z_0} I(x) \Pi_{z_0})^\dagger$ where Π_{z_0} is the orthogonal projection onto $\mathcal{V}_{z_0} = \jmath(E_{z_0})$ in $\mathbb{R}^{2n}$ and upper script $\dagger$ denotes the Moore-Penrose pseudo-inverse. Thus, the covariance of any unbiased estimator $\psi : \mathbb{R}^m \to E_{z_0}$ is bounded as follows:

$$Cov[\psi] \geq \frac{\sigma^2}{4}\left(\Pi_{z_0}\mathcal{R}(\xi)\Pi_{z_0}\right)^\dagger \tag{5.70}$$

In the real case, $\mathcal{F} \subset V = \mathbb{R}^n \subset \mathbb{C}^n$, and the Fisher information matrix takes the form

$$I(x) = \frac{4}{\sigma^2}\left[\begin{array}{cc} R(x) & 0 \\ 0 & 0 \end{array}\right]$$

Restricting to the real component of the estimator, the CRLB becomes:

$$Cov[\psi] \geq \frac{\sigma^2}{4} R(x)^{-1}$$

In the complex case $\mathcal{F} \subset V = H = \mathbb{C}^n$, $\Pi_{z_0} = I_{2n} - J\psi_0\psi_0^T J^T$ with $\psi_0 = \jmath(z_0)$ and the CRLB becomes:

$$Cov[\psi] \geq \frac{\sigma^2}{4}(\Pi_{z_0}\mathcal{R}(\xi)\Pi_{z_0})^\dagger$$

6. Reconstruction Algorithms

We present two types of reconstruction algorithms:

- Rank 1 matrix recovery: PhaseLift;
- Iterative algorithm: Least-Square Optimization

Throughout this section we assume $\mathcal{F}$ is a phase retrievable frame for $H = \mathbb{C}^n$.

6.1. **Rank 1 Matrix Recovery.** Consider the noiseless case $y = \beta(x)$. The main idea is embodied in the following feasibility problem:

$$\mathrm{find}_{\text{subject to:}\mathbb{A}(X)=y, X=X^*\geq 0, rank(X)=1} X$$

Except for $rank(X) = 1$ the optimization problem is convex. However the rank constraint destroys the convexity property. Once a solution X is found, the vector x can be easily obtained from the factorization: $X = xx^*$.

The feasibility problem admits at most a unique solution and so does the following optimization problem:

$$\min_{\mathbb{A}(X)=y, X=X^*\geq 0} rank(X) \tag{6.71}$$

which is still non-convex. The insight provided by the matrix completion theory and exploited in [CSV12, CESV12] is to replace $rank(X)$ by $trace(X)$ which is convex. Thus one obtains:

$$\text{(PhaseLift)} \quad \min_{\mathbb{A}(X)=y, X=X^*\geq 0} trace(X) \tag{6.72}$$

which is a convex optimization problem (a semi-definite program: SDP). In [CL12] the authors proved that for random frames, with high probability the problem (6.72) has the same solution as the problem (6.71):

Theorem 6.1. *Assume each vector f_k is drawn independently from $\mathcal{N}(0, I_n/2) + i\mathcal{N}(0, I_n/2)$, or each vector is drawn independently from the uniform distribution on the complex sphere of radius $\sqrt{n}$. Then there are universal constants $c_0, c_1, \gamma > 0$ so that for $m \geq c_0 n$, for every $x \in \mathbb{C}^n$ the problem (6.72) has the same solution as (6.71) with probability at least $1 - c_1 e^{-\gamma n}$.*

The PhaseLift algorithm is also robust to noise. Consider the measurement

$$y = \beta(x) + \nu$$

for some $\nu \in \mathbb{R}^m$ noise vector. Consider the following modified optimization problem:

$$\min_{X=X^*\geq 0} \|\mathbb{A}(X) - y\|_1 \tag{6.73}$$

In [CL12] the following result has been shown:

Theorem 6.2. *Consider the same stochastic process for the random frame $\mathcal{F}$. There is a universal constant $C_0 > 0$ so that for all $x \in C^n$ the solution to (6.73) obeys*

$$\|X - xx^*\|_2 \leq C_0 \frac{\|\nu\|_1}{m}$$

For the Gaussian model this holds with the same probability as in the noiseless case, whereas the probability of failure is exponentially small in n in the uniform model. The principal eigenvector x^0 of X (normalized by the square root of the principal eigenvalue) obeys

$$D_2(x^0, x) \leq C_0 \min(\|x\|_2, \frac{\|\nu\|_1}{m\|x\|_2}).$$

6.2. **An Iterative Algorithm.** Consider the measurement process

$$y_k = |\langle x, f_k\rangle|^2 + \nu_k \ , \ 1 \leq k \leq m$$

The Least-Squares criterion:

$$\min_{x\in\mathbb{C}^n} \sum_{k=1}^{m} ||\langle x, f_k\rangle|^2 - y_k|^2$$

can be understood as the Maximum Likelihood Estimator (MLE) when the noise vector $\nu \in \mathbb{R}^m$ is normal distributed with zero mean and covariance $\sigma^2 I_m$. However the optimization problem is not convex and has many local minima.

The iterative algorithm described next tries to find the global minimum using a regularization term. Consider the following optimization criterion:

(6.74)

$$J(u, v; \lambda, \mu) = \sum_{k=1}^{m} \left| \frac{1}{2}(\langle u, f_k\rangle\langle f_k, v\rangle + \langle v, f_k\rangle\langle f_k, u\rangle) - y_k \right|^2 + \lambda\|u\|_2^2 + \mu\|u - v\|_2^2 + \lambda\|v\|_2^2$$

The Iterative Regularized Least-Squares (IRLS) algorithm presented in [Ba13] works as follows.

Fix a stopping criterion, such as a tolerance ε, a desired level of signal-to-noise-ratio snr, or/and a maximum number of steps T. Fix an initialization parameter $\rho \in (0, 1)$, a learning rate $\gamma \in (0, 1)$ and a saturation parameter $\mu_{min} > 0$.

Step 1. Initialization. Compute the principal eigenvector of $R_y = \sum_{k=1}^{m} y_k f_k f_k^*$ using e.g. the power method. Let (e_1, a_1) be the eigen-pair with $e_1 \in \mathbb{C}^n$ and $a_1 \in \mathbb{R}$. If $a_1 \leq 0$ then set $x = 0$ and exit. Otherwise initialize:

$$x^0 = \sqrt{\frac{(1-\rho)a_1}{\sum_{k=1}^{m} |\langle e_1, f_k\rangle|^4}} e_1 \tag{6.75}$$

$$\lambda_0 = \rho a_1 \tag{6.76}$$

$$\mu_0 = \rho a_1 \tag{6.77}$$

$$t = 0 \tag{6.78}$$

Step 2. Iteration. Perform:

2.1 Solve the least-square problem:

$$x^{t+1} = \mathrm{argmin}_u J(u, x^t; \lambda_t, \mu_t)$$

using the conjugate gradient method.

2.2 Update:

$$\lambda_{t+1} = \gamma\lambda_t \ , \ \mu_t = max(\gamma\mu_t, \mu_{min}) \ , \ t = t+1$$

Step 3. Stopping. Repeat Step 2 until:

- The error criterion is achieved: $J(x^t, x^t; 0, 0) < \varepsilon$; or
- The desired signal-to-noise-ratio is reached: $\frac{\|x^t\|^2}{J(x^t,x^t;0,0)} > snr$; or
- The maximum number of iterations is reached: $t > T$.

The final estimate can be x^T, or the best estimate obtained in the iteration path: $x^{est} = x^{t_0}$ where $t_0 = \mathrm{argmin}_t J(x^t, x^t; 0, 0)$.

The initialization is performed as in (6.75) for the following reason. Consider the modified criterion:

$$H(x;\lambda) = J(x,x;\lambda,0) = \|\beta(x) - y\|_2^2 + \lambda\|x\|_2^2 = \sum_{k=1}^{m} |\langle x, f_k\rangle|^4 + \langle(\lambda I_n - R_y)x, x\rangle + \|y\|_2^2$$

In general this function is not convex in x, except for large values of λ. Specifically for $\lambda > a_1$, the largest eigenvalue of R_y, $x \mapsto H(x;\lambda)$ is convex and has a unique global minimum at $x = 0$. For $a_1 - \varepsilon < \lambda < a_1$ the criterion is no longer convex, but the global minimum stays in a neighborhood of the origin. Neglecting the 4^{th} order terms, the critical points are given by the eigenvectors of R_y. Choosing $\lambda = \rho a_1$ and $x = se_1$, the optimal value of s for $s \mapsto H(se_1; \rho a_1)$ is given in (6.75).

The path of iterates $(x^t)_{t\geq 0}$ can be thought of as trying to approximate the measured vector y with a linear transformation of a rank 2, $\mathbb{A}([\![x^{t-1}, x^t]\!])$. The parameter μ penalizes the negative eigenvalue of $[\![x^{t-1}, x^t]\!]$; the larger the value of μ_t the smaller the iteration step $\|x^{t+1} - x^t\|$ and the smaller the deviation from a rank 1 of $outpx^{t+1}x^t$; the smaller the parameter μ_t the larger in magnitude the negative eigenvalue of $[\![x^{t+1}, x^t]\!]$. This fact explains why in the noisy case the iterates first decrease the matching error $J(x^t, x^t; 0, 0)$ up to some t_0 and then they start to increase the matching error: the rank 2 self-adjoint operator $T = [\![x^{t+1}, x^t]\!]$ decreases the matching error $\|\mathbb{A}(T) - y\|_2$ instead of the rank-1 self-adjoint operator $[\![x^t, x^t]\!]$.

At any point on the path, if the value of criterion J is smaller than the value reached at the true value x, then we can offer convergence guarantees. Specifically in [Ba13] the following result has been proved:

Theorem 6.3 ([Ba13]Theorem 5.6)**.** *Fix $0 \neq z_0 \in \mathbb{C}^n$. Assume the frame $\mathcal{F}$ is so that $\ker \mathbb{A} \cap \mathcal{S}^{2,1} = \{0\}$. Then there is a constant $A_3 > 0$ that depends of $\mathcal{F}$ so that for every $x \in \Omega_{z_0}$ and $\nu \in \mathbb{C}^n$ that produce $y = \beta(x) + \nu$ if there are $u, v \in \mathbb{C}^n$ so that $J(u, v; \lambda, \mu) < J(x, x; \lambda, \mu)$ then*

$$\|[\![u, v]\!] - xx^*\|_1 \leq \frac{4\lambda}{A_3} + \frac{2\|\nu\|_2}{\sqrt{A_3}} \tag{6.79}$$

Moreover, let $[\![u,v]\!] = a_+e_+e_+^* + a_-e_-e_-^*$ *be its spectral factorization with* $a_+ \geq 0 \geq a_-$ *and* $\|e_+\| = \|e_-\| = 1$*. Set* $\tilde{x} = \sqrt{a_+}e_+$*. Then*

$$D_2(x,\tilde{x})^2 \leq \frac{4\lambda}{A_3} + \frac{2\|\nu\|_2}{\sqrt{A_3}} + \frac{\|\nu\|_2^2}{4\mu} + \frac{\lambda\|x\|_2^2}{2\mu} \tag{6.80}$$

The kernel requirement on $\mathbb{A}$ is satisfied for generic frames when $m \geq 6n$. In particular it implies the frame is phase retrievable for $\mathbb{C}^n$.

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