Norms and embeddings of classes of positive semidefinite matrices

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$P(H)\Leftrightarrow St^1(H)$ represent the pure states. $\hat{H}\Leftrightarrow \mathcal{S}^{1,0}:=\{xx^*,x\in H\}$. Given a set of p.s.d. operators $\mathcal{F}=\{F_1,\cdots,F_m\}$ on H, consider two maps

$$\alpha: Sym(H)\to\mathbb{R}^m\ ,\ \alpha(T)=\left(\sqrt{tr(TF_k)}\right)_{1\leq k\leq m}$$

$$\beta: Sym(H)\to\mathbb{R}^m\ ,\ \beta(T)=(tr(TF_k))_{1\leq k\leq m}$$

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$P(H) \Leftrightarrow St^1(H)$ represent the pure states. $\hat{H} \Leftrightarrow \mathcal{S}^{1,0} := \{xx^*, x \in H\}$.
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Quantum Tomography: reconstruct $T \in St(H)$ from $\beta(T)$.
Phase Retrieval: estimate $x \in \hat{H}$ from $\alpha(xx^*)$ when $F_k = f_k f_k^*$.

Problem Statement

Today we shall discuss the following problem. Assume the maps

$$\alpha : \mathcal{S}^{r,0} \to \mathbb{R}^m \quad , \quad \alpha(T) = \left(\sqrt{\langle T, F_k \rangle}\right)_{1 \le k \le m}$$

$$\beta : \mathcal{S}^{r,0} \to \mathbb{R}^m \quad , \quad \beta(T) = (\langle T, F_k \rangle)_{1 \le k \le m}$$

are injective. Here

$$St^r(H) \subset \mathcal{S}^{r,0} := \{T = T^* \ge 0 \quad , \quad rank(T) \le r\}$$

We want to find bi-Lipschitz properties of these maps and understand if their left inverses can be extended to entire $\mathbb{R}^m$ are Lipschitz maps.

Metric Structures on $\hat{H}$ and $Sym(H)$

Norm Induced Metric

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance* on $Sym(H)$:

$$d_p : Sym(H) \times Sym(H) \to \mathbb{R} \ , \ d_p(X,Y) = \|X-Y\|_p,$$

the p-norm of the singular values.
On $\hat{H}$ it induces the metric

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ d_p(\hat{x},\hat{y}) = \|xx^* - yy^*\|_p$$

In the case $p=2$ we obtain

$$d_2(X,Y) = \|X-Y\|_F^2 \quad , \quad d_2(x,y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x,y\rangle|^2}$$

Metric Structures on $\hat{H}$ and $Sym(H)$

Natural Metric

The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ D_p(\hat{x},\hat{y}) = \min_{\varphi} \|x - e^{i\varphi} y\|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p=2$ we obtain

$$D_2(\hat{x},\hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x,y\rangle|}$$

On $Sym^+(H)$, the "natural" metric lifts to

$$D_p : Sym^+(H) \times Sym^+(H) \to \mathbb{R} \ , \ \ D_p(X,Y) = \min_{\substack{VV^* = X \\ WW^* = Y}} \|V - WU\|_p.$$

Metric Structures on $Sym(H)$

Natural metric vs. Bures/Helinger

Let $X, Y \in Sym^+(H)$. For the natural distance we choose $p = 2$:

$$D_{natural}(X, Y) = \min_{\substack{VV^* = X \\ WW^* = Y}} \|V - W\|_F$$

Fact:

$$D_{natural}(X, Y) = \min_{U \in U(n)} \|X^{1/2} - Y^{1/2}U\|_F = \sqrt{\mathrm{tr}(X) + tr(Y) - 2\|X^{1/2}Y^{1/2}\|_1}$$

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Another distance: Bures/Helinger distance:

$$D_{Bures}(X,Y) = \|X^{1/2} - Y^{1/2}\|_F = d_2(X^{1/2}, Y^{1/2})$$

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A consequence of the Arithmetic-Geometric Mean Inequality [BK00]:

$$\frac{1}{2} D^2_{Bures}(X, Y) \le D^2_{natural}(X, Y) \le D^2_{Bures}(X, Y).$$

Metric Structures

p-dependency

Lemma (BZ16)

1. $(d_p)_{1\leq p\leq\infty}$ *are equivalent metrics and the identity map* $i:(\hat{H},d_p)\rightarrow(\hat{H},d_q)$, $i(x)=x$ *has Lipschitz constant* $Lip^d_{p,q,n}=\max(1,2^{\frac{1}{q}-\frac{1}{p}})$.
2. *The metric space* $(\hat{H},d_p)$ *is isometrically isomorphic to* $\mathcal{S}^{1,0}$ *endowed with the p-norm via* $\kappa_\beta:\hat{H}\rightarrow\mathcal{S}^{1,0}\quad,\quad x\mapsto\kappa_\beta(x)=xx^*$.

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2. *The metric space* $(\hat{H},D_2)$ *is Lipschitz isomorphic to* $\mathcal{S}^{1,0}$ *endowed with the 2-norm via* $\kappa_\alpha:\hat{H}\rightarrow\mathcal{S}^{1,0}\quad,\quad x\mapsto\kappa_\alpha(x)=\frac{1}{\|x\|}xx^*$.

Metric Structures
Distinct Structures

Two different structures: topologically equivalent, BUT the metrics are NOT equivalent:

Lemma (BZ16)

The identity map $i : (\hat{H}, D_p) \rightarrow (\hat{H}, d_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Likewise, the identity map $i : (\hat{H}, d_p) \rightarrow (\hat{H}, D_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Hence the induced topologies on $(\hat{H}, D_p)$ and $(\hat{H}, d_p)$ are the same, but the corresponding metrics are not Lipschitz equivalent.

Obviously, the same result holds for $(Sym^+(H), D_{natural})$ and $(Sym^+(H), d_2)$.

Lipschitz Stability in Phase Retrieval

Lipschitz inversion: α

Theorem (BZ16)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in H$:*

$$A_0 \min_{\varphi} \|x - e^{i\varphi}y\|_2^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k\rangle| - |\langle y, f_k\rangle| \right|^2 \leq B_0 \min_{\varphi} \|x - e^{i\varphi}y\|_2^2.$$

2. *There is a Lipschitz map $\omega : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\omega) \leq \frac{4+3\sqrt{2}}{\sqrt{A_0}} = \frac{8.24}{\sqrt{A_0}}$.*

Lipschitz Stability in Phase Retrieval

Lipschitz inversion: β

Theorem (BZ16)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\beta : (\hat{H}, d_1) \rightarrow (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in H$:*

$$a_0 \|xx^* - yy^*\|_1^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k \rangle|^2 - |\langle y, f_k \rangle|^2 \right|^2 \leq b_0 \|xx^* - yy^*\|_1^2.$$

2. *There is a Lipschitz map $\psi : (\mathbb{R}^m, \|\cdot\|_2) \rightarrow (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\psi) \leq \frac{4+3\sqrt{2}}{\sqrt{a_0}} = \frac{8.24}{\sqrt{a_0}}$.*

Stability Results in Quantum Tomography

Bi-Lipschitz properties of α and β on Quantum States

Fix a closed subset $S \subset Sym^+(H)$. For instance $S = St(H)$, or $St^r(H)$, or $\mathcal{S}^{r,0}$.

Theorem

Assume $\mathcal{F} = \{F_1, \cdots, F_m\} \subset Sym^+(H)$ so that $\alpha|_S$ and $\beta|_S$ are injective. Then there are constants $a_0, A_0, b_0, B_0 > 0$ so that for every $X, Y \in S$,

$$A_0 D^2_{natural}(X,Y) \leq \sum_{k=1}^{m} \left| \sqrt{\langle X, F_k \rangle} - \sqrt{\langle Y, F_k \rangle} \right|^2 \leq B_0 D^2_{natural}(X,Y)$$

$$a_0 \|X - Y\|_F^2 \leq \sum_{k=1}^{m} |\langle X, F_k \rangle - \langle Y, F_k \rangle|^2 \leq b_0 \|X - Y\|_F^2.$$

Next Results

Lipschitz inversion of α and β on Quantum States

Consider the measurement map

$$\beta : (St^r(H), d_1) \rightarrow (\mathbb{R}^m, \|\cdot\|_2) \ \ , \ \ \beta(T) = (tr(TF_k))_{1 \leq k \leq m}$$

where $St^r(H) = \{T = T^* \geq 0 \ , \ tr(T) = 1 \ , \ rank(T) \leq r\}$.
If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.
Conjecture: If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract.

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If conjecture is true, it follows that even if β is injective on rank r quantum states, it cannot admit a Lipschitz (or even continuous) left inverse defined globally on $\mathbb{R}^m$.

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If conjecture is true, it follows that even if β is injective on rank r quantum states, it cannot admit a Lipschitz (or even continuous) left inverse defined globally on $\mathbb{R}^m$.
A similar result should hold true for

$$\alpha : (St^r(H), D_2) \to (\mathbb{R}^m, \|\cdot\|_2) \quad , \quad \alpha(T) = (\sqrt{tr(TF_k)})_{1 \leq k \leq m}.$$

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