# AI Pictures at a Mathematical Exhibition: How Applied Harmonic Analysis meets Machine Learning

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**Papers available online** at:
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# High-Level Overview

In this series of lectures, we discuss a few harmonic analysis techniques and problems applied to machine learning.

1. NN: Neural networks (NN) and their universal approximation property.
2. Lipschitz analysis: we provide rationals for studying Lipschitz properties of NNs, and then we perform a Lipschitz analysis of these networks. We focus on two aspects of this analysis: stochastic modelng of local vs. global analysis, and a scattering network inspired Lipschitz analysis of convolutive networks.
3. Invariance and Equivariance: We highlight the duality between invariance and covariance/equivariance, with focus on G-invariant representations.
4. Applications to data analysis and modeling: We present applications on a variety of problems: classification and regression on graphs; generative models for data sets; neural network based modeling of time-evolution of dynamical systems; discrete optimizatons.

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**Joint works with**:

Naveed Haghani (UMD,APL-JHU) Daniel Levy (UMD)

Maneesh Singh (Verisk, Comcast) Efstratos Tsoukanis (UMD)

R. Balan, N. Haghani, M. Singh, "Permutation Invariant Representations with Applications to Graph Deep Learning", arXiv preprint: 2203.07546 [math.FA], [cs.LG]

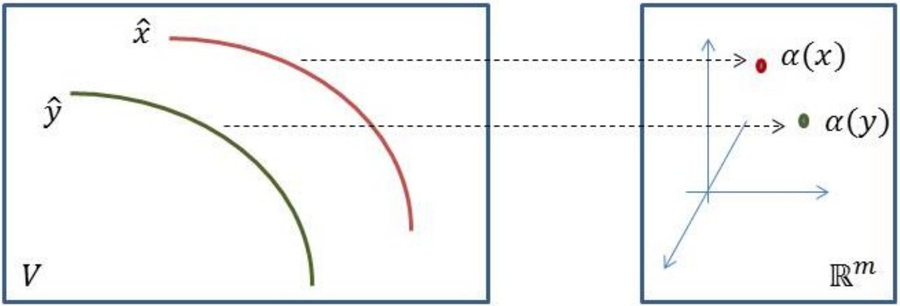
R. Balan, E. Tsoukanis, "Relationships between the Phase Retrieval Problem and Permutation Invariant Embeddings", arXiv preprint: 2306.13111 [math.FA]

# High-Level View

Two related problems with many variations:

Given a (discrete) group $G$ acting on a normed space $V$:

1. Construct a (bi)Lipschitz Euclidean embedding of the quotient space $V/G$, $\alpha : \hat{V} \to \mathbb{R}^m$. Classification of cosets.
2. Construct the projection onto cosets, $\pi : V \to [y] = \hat{y} = \{g.y \ , \ g \in G\}$.



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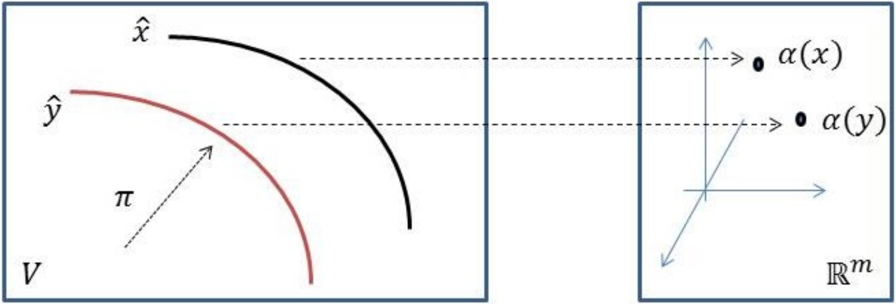
2. Construct projections onto cosets, $\pi : V \to [y] = \hat{y} = \{g.y \; , \; g \in G\}$. Optimizations within cosets.



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# A. Similarity of Matrices

Consider two symmetric matrices $A, B \in \mathrm{Sym}(n)$. When are they equivalent modulo an orthonormal change of coordinates?
Specificaly, is there an orthogonal matrix $U \in O(n)$ so that $B = UAU^T$ ?

An elementary derivation in linear algebra shows that $A \overset{O(n)}{\sim} B$ if and only if $A$ and $B$ have the same set of eigenvalues with exactly same multiplicities.

But what about other groups $G$? For instance what about the group of permutation matrices $\mathcal{S}_n$?
Find necessary and sufficient conditions so that $A \overset{\mathcal{S}_n}{\sim} B$.
Recall:

$$\mathcal{S}_n = \{P \in O(n) : P_{i,j} \in \{0,1\}\} = O(n) \cap \{W \in [0,1]^{n\times n} : W1 = 1, W^T 1 = 1\}$$

# A. The Graph Isomorphism Problem

Consider two graphs $G = (\mathcal{V}, \mathcal{E})$ and $\tilde{G} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ with $n$ nodes. The graph isomorphism problem is the computational problem of determining whether these graphs are identical after a relabeling of nodes.

If $A$ and $\tilde{A}$ denote their adjacency matrices, these graphs are isomorphic if and only if $\tilde{A} = \Pi A \Pi^T$ for some permutation matrix $\Pi \in \mathcal{S}_n$.

Current state-of-the-art (Wikipedia): Babai (2015,2017) presented a quasi-polynomial algorithm with running time $2^{O((\log n)^c)}$, for some fixed $c > 0$. Helfgott (2017) claims that one can take $c = 3$.

Similar problem can be stated for weighted graphs: $A, \tilde{A} \in \mathrm{Sym}(n)$ with nonnegative entries, isomorphic if and only if $\tilde{A} = \Pi A \Pi^T$ for some $\Pi \in \mathcal{S}_n$.

# B. Graph Alignment Problems

Consider two $n \times n$ symmetric matrices $A, B$. In the alignment problem for quadratic forms one seeks an orthogonal matrix $U \in O(n)$ that minimizes

$$\|UAU^T - B\|_F^2 := trace((UAU^T - B)^2) = \|A\|_F^2 + \|B\|_F^2 - 2trace(UAU^T B).$$

The solution is well-known and depends on the eigendecomposition of matrices $A, B$: if $A = U_1 D_1 U_1^T$, $B = U_2 D_2 U_2^T$ then

$$U_{opt} = U_2 U_1^T \quad , \quad \|U_{opt} A U_{opt}^T - B\|_F^2 = \sum_{k=1}^{n} |\lambda_k - \mu_k|^2,$$

where $D_1 = diag(\lambda_k)$ and $D_2 = diag(\mu_k)$ are diagonal matrices with eigenvalues ordered monotonically.

# B. Quadratic Assignment Problem

The challenging case is when $U$ is constrained to the permutation group as is the case in the *graph matching problem*. In this case, the optimization problem becomes

$$\min_{U \in \mathcal{S}_n} \|UAU^T - B\|_F$$

turns into a QAP:

$$\max_{U \in \mathcal{S}_n} trace(UAU^T B).$$

This is equivalent to computing the natural distance $\mathbf{d}(\hat{A}, \hat{B}) = \min_{P,Q \in \mathcal{S}_n} \|PAP^T - QBQ^T\|_F$ between the equivalence classes $\hat{A}, \hat{B} \in \widehat{\mathrm{Sym}(n)}$ induced by the group action $\mathcal{S}_n \times \mathrm{Sym}(n) \to \mathrm{Sym}(n)$, $(\Pi, A) \mapsto \Pi A \Pi^T$.

# C. Graph Learning Problems

Given a data graph (e.g., social network, transportation network, citation network, chemical network, protein network, biological networks):

- Graph adjacency or weight matrix, $A \in \mathbb{R}^{n \times n}$;
- Data matrix, $X \in \mathbb{R}^{n \times r}$, where each row corresponds to a feature vector per node.

Contruct a map $f : (A, X) \to f(A, X)$ that performs:

1. classification: $f(A, X) \in \{1, 2, \cdots, c\}$
2. regression/prediction: $f(A, X) \in \mathbb{R}$.

Key observation: The outcome should be invariant to vertex permutation: $f(PAP^T, PX) = f(A, X)$, for every $P \in \mathcal{S}_n$.

# Invariance vs. Equivariance

Graph learning problems are prime examples of the difference between *invariant* vs. *equivariant* representations.
If the machine learning task is **node** classification or regression:

$$f : (A, X) \mapsto f(A, X) \ \in \ \{1, 2, \cdots, c\}^n \ \text{ or } \ \mathbb{R}^n$$

where $f(A, X)$ is a graph signal, i.e., $f(A, X)_i$ is signal at node $i$, then the nonlinear map $f$ is *equivariant* and must satisfy $f(PAP^T, PX) = Pf(A, X)$, for all $P \in \mathcal{S}_n$.

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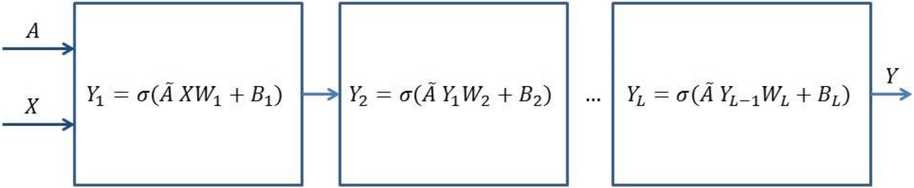
On the other hand, if the machine learning task is **graph** classification or regression,

$$f : (A, X) \mapsto f(A, X) \in \{1, 2, \cdots, c\} \text{ or } \mathbb{R}$$

where $f(A, X)$ is assigned for the entire graph, then the nonlinear map $f$ is *invariant* and must satisfy $f(PAP^T, PX) = f(A, X)$, for all $P \in \mathcal{S}_n$.

# C. Graph Convolution Networks (GCN), Graph Neural Networks (GNN)

General architecture of a GCN/GNN




GCN (Kipf and Welling ('16)) choses $\tilde{A} = I + A$; GNN (Scarselli et.al. ('08), Bronstein et.al. ('16)) choses $\tilde{A} = p_l(A)$, a polynomial in adjacency matrix. $L$-layer GNN has parameters $(p_1, W_1, B_1, \cdots, p_L, W_L, B_L)$.

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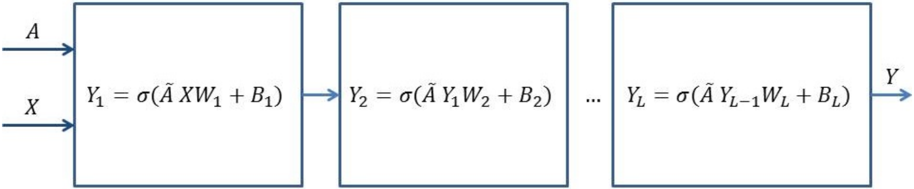


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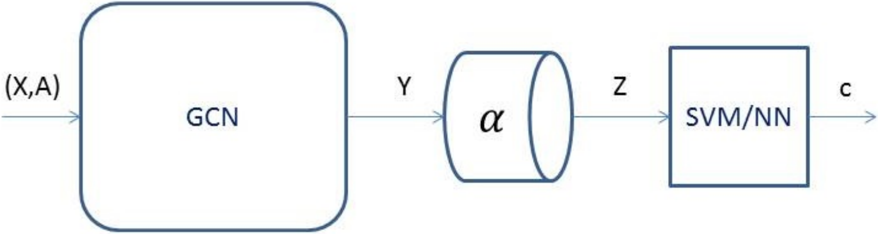
Note the *covariance (or, equivariance) property*: for any $P \in O(n)$ (including $\mathcal{S}_n$), if $(A, X) \mapsto (PAP^T, PX)$ and $B_i \mapsto PB_i$ then $Y \mapsto PY$.

# C. Deep Learning with GCN/GNN

The approach for the two learning tasks (classification or regression) is based on the following scheme (see also Maron et.al. ('19)):




where $\alpha$ is a permutation invariant map (embedding), and SVM/NN is a single-layer or a deep neural network (Support Vector Machine or a Fully Connected Neural Network) trained on invariant representations.

The purpose of this talk is to analyze the $\alpha$ component.

# The metric space $\hat{V}$ when $V = \mathbb{R}^{n\times d}$

Recall the equivalence relation $\sim$ on $V = \mathbb{R}^{n\times d}$ induced by the group of permutation matrices $\mathcal{S}_n$ acting on $V$ by left multiplication: for any $X, X' \in \mathbb{R}^{n\times d}$,

$$X \sim X' \quad \Leftrightarrow \quad X' = PX \ , \ \text{for some } P \in \mathcal{S}_n$$

Let $\widehat{\mathbb{R}^{n\times d}} = \mathbb{R}^{n\times d} / \sim$ be the quotient space endowed with the natural distance induced by Frobenius norm $\|\cdot\|_F$

$$\mathbf{d}(\hat{X}_1, \hat{X}_2) = \min_{P\in S_n} \|X_1 - PX_2\|_F \quad , \quad \hat{X}_1, \hat{X}_2 \in \widehat{\mathbb{R}^{n\times d}}.$$

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The computation of the minimum distance is performed by solving the Linear Assignment Problem (LAP) whose convex relaxation is exact:

$$\max_{P\in\mathcal{S}_n} trace(PX_2X_1^T) = \max_{W\in DS(n)} trace(WX_2X_1^T)$$

where $DS(n) = \{W \in [0,1]^{n\times n} : W1 = 1, W^T1 = 1\}$ is the convex set of doubly stochastic matrices.

# The embedding problem

Problem: Construct a bi-Lipschitz embedding $\hat{\alpha} : \widehat{\mathbb{R}^{n\times d}} \to \mathbb{R}^m$, i.e., an integer $m = m(n,d)$, a map $\alpha : \mathbb{R}^{n\times d} \to \mathbb{R}^m$ with constants $0 < a \leq b < \infty$ so that for any $X, X' \in \mathbb{R}^{n\times d}$,

1. If $X \sim X'$ then $\alpha(X) = \alpha(X')$.
2. If $\alpha(X) = \alpha(X')$ then $X \sim X'$.
3. $a \cdot \mathbf{d}(\hat{X}, \hat{X}') \leq \|\alpha(X) - \alpha(X')\|_2 \leq b \cdot \mathbf{d}(\hat{X}, \hat{X}')$.

where $\mathbf{d}(\hat{X}, \hat{X}') = \min_{P\in\mathcal{S}_n} \|X - PX'\|_F$.

# A Universal Embedding

Consider the map

$$\mu : \widehat{\mathbb{R}^{n\times d}} \to \mathcal{P}(\mathbb{R}^d) \quad , \quad \mu(X)(x) = \frac{1}{n}\sum_{k=1}^{n} \delta(x - x_k)$$

where $\mathcal{P}(\mathbb{R}^d)$ denotes the convex set of probability measures over $\mathbb{R}^d$, and $\delta$ denotes the Dirac measure. $x_k$ is the $k^{th}$ row of $X$.
Clearly $\mu(X') = \mu(X)$ iff $X' = PX$ for some $P \in \mathcal{S}_n$.
The Wasserstein-2 distance is isometrically equivalent to **d**:

$$W_2(\mu(X), \mu(Y))^2 := \inf_{q \in J(\mu(X),\mu(Y))} \mathbb{E}_q[\|x - y\|_2^2] = \min_{P \in \mathcal{S}_n} \|Y - PX\|^2$$

By Kantorovich-Rubinstein theorem, the Wasserstein-1 distance (the Earth moving distance) extends to a norm on the space of signed Borel measures.

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By Kantorovich-Rubinstein theorem, the Wasserstein-1 distance (the Earth moving distance) extends to a norm on the space of signed Borel measures.
Main drawback: $\mathcal{P}(\mathbb{R}^d)$ is infinite dimensional!

# Finite Dimensional Embeddings

Idea: "Project" the measure onto a finite dimensional space. This is accomplished by *kernel methods*:

Fix a family of functions $f_1, \cdots, f_m$ and consider:

$$\mu(X) \mapsto \int_{\mathbb{R}^d} f_j(x) d\mu(X) = \frac{1}{n} \sum_{k=1}^{n} f_j(x_k) \quad , \quad j \in [m]$$

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Possible choices:

1. Polynomial embeddings: $\mathbb{R}[X]^{\mathcal{S}_n}$, ring of invariant polynomials; [Lipman&al.],[Peyré&al.],[Sanay&al.],[Kemper book] ...
2. Gaussian kernels: $f_j(x) = exp(-\|x - a_j\|^2/\sigma_j^2)$ ; [Gilmer&al.],[Zaheer&al.], [Vinyals&al.],...
3. Fourier kernels (cmplx embd): $f_j(x) = exp(2\pi i \langle x, \omega_j \rangle)$; related to Prony method; [Li&Liao] for bi-Lipschitz estimates.

Main drawback: No global bi-Lipschitz embeddings [Cahill&al.'19]. Ok on (some) compacts.

# Polynomial Expansions - Quadratics

In the case $d = 1$ recall Vieta's formulas, Newton-Girard identities

$$P(X) = \prod_{k=1}^{N} (X - x_k) \leftrightarrow (\sum_k x_k, \sum_k x_k^2, ..., \sum_k x_k^n)$$

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For $d > 1$, consider the quadratic $d$-variate polynomial:

$$\begin{aligned} P(Z_1, \cdots, Z_d) &= \prod_{k=1}^{n} \left( (Z_1 - x_{k,1})^2 + \cdots + (Z_d - x_{k,d})^2 \right) \\ &= \sum_{p_1,\ldots,p_d=0}^{2n} a_{p_1,\ldots,p_d} Z_1^{p_1} \cdots Z_d^{p_d} \end{aligned}$$

Encoding complexity:

$$m = \binom{2n+d}{d} \sim (2n)^d.$$

# Polynomial Expansions - Quadratics (2)

A more careful analysis of $P(Z_1,...,Z_d)$ reveals a form:

$$P(Z_1,...,Z_d) = t^n + Q_1(Z_1,...,Z_d)t^{n-1} + \cdots + Q_{n-1}(Z_1,...,Z_d)t + Q_n(Z_1,...,Z_d)$$

where $t = Z_1^2 + \cdots + Z_d^2$ and each $Q_k(Z_1,...,Z_d) \in \mathbb{R}_k[Z_1,...,Z_d]$ is a (non-homogeneous) polynomial of degree $k$. Hence one needs to encode:

$$m = \begin{pmatrix} d+1 \\ 1 \end{pmatrix} + \begin{pmatrix} d+2 \\ 2 \end{pmatrix} + \cdots + \begin{pmatrix} d+n \\ n \end{pmatrix} = \begin{pmatrix} d+n+1 \\ n \end{pmatrix} - 1$$

number of coefficients.

A significant drawback: Inversion is numerically unstable and embedding is not Lipschitz.

# Readout Mapping Approach

## Polynomial Expansion - Linear Forms

A stable (Lipschitz, not bi-Lipschitz!) embedding can be constructed as follows (see also Gobels' algorithm (1996) or [Derksen, Kemper '02]). Consider the $n$ linear forms $\lambda_k(Z_1, ..., Z_d) = x_{k,1}Z_1 + \cdots x_{k,d}Z_d$. Construct the polynomial in variable $t$ with coefficients in $\mathbb{R}[Z_1, ..., Z_d]$:

$$P(t) = \prod_{k=1}^{n}(t-\lambda_k(Z_1, ..., Z_d)) = t^n - e_1(Z_1, .., Z_d)t^{n-1} + \cdots (-1)^n e_n(Z_1, ..., Z_d)$$

$$= t^n + \sum_{\substack{p_0, p_1, \cdots, p_d \geq 0 \\ p_0 + p_1 + \cdots + p_d = n \ , \ p_0 < n}} c_{p_0,p_1,\cdots,p_d} t^{p_0} Z_1^{p_1} \cdots Z_d^{p_d}$$

The elementary symmetric polynomials $(e_1, ..., e_n)$ are in 1-1 correspondence (Newton-Girard theorem) with the moments: $\mu_p = \sum_{k=1}^{n} \lambda_k^p(Z_1, ..., Z_d)$, $1 \leq p \leq n$.

# Polynomial Expansions - Linear Forms (2)

Each $\mu_p$ is a homogeneous polynomial of degree $p$ in $d$ variables. Hence to encode each of them one needs $\binom{d+p-1}{p}$ coefficients. Hence the embedding dimension is

$$m_0 = \binom{d}{1} + \binom{d+1}{2} + \cdots + \binom{d+n-1}{n} = \binom{d+n}{n} - 1$$

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The map $\alpha_0 : \mathbb{R}^{n\times d} \to \mathbb{R}^{m_0}$, $X \mapsto (c_{p_0,p_1,\cdots,p_d})_{p_0,p_1,\cdots,p_d}$ is injective modulo $\mathcal{S}_n$ but it is not Lipschitz. However a simple modification as suggested by [Cahill et.al.'19] makes it Lipschitz.

# Polynomial Lipschitz embedding

Denote by $L_0$ the Lipschitz constant of $\alpha_0$ when restricted to the closed unit ball $B_1(\mathbb{R}^{n\times d}) : \{X \in \mathbb{R}^{n\times d}\ ,\ \|X\| \leq 1\}$ of $\mathbb{R}^{n\times d}$, i.e. $\|\alpha_0(X) - \alpha_0(Y)\| \leq L_0\|X - Y\|$ for any $X, Y \in \mathbb{R}^{n\times d}$ with $\|X\|, \|Y\| \leq 1$. Let $\varphi_0 : \mathbb{R} \to [0,1]$, $\varphi_0(x) = min(1, \frac{1}{x})$ be a Lipschitz monotone decreasing function with Lipschitz constant 1.

**Theorem**

*The map:*

$$\alpha_1 : \mathbb{R}^{n\times d} \to \mathbb{R}^m \ ,\ \alpha_1(X) = \begin{pmatrix} \alpha_0\left(\varphi_0(\|X\|)X\right) \\ \|X\| \end{pmatrix},$$

*with* $m = \begin{pmatrix} n+d \\ d \end{pmatrix} = m_0 + 1$ *lifts to an injective and globally Lipschitz map* $\hat{\alpha}_1 : \widehat{\mathbb{R}^{n\times d}} \to \mathbb{R}^m$ *with Lipschitz constant* $Lip(\hat{\alpha}_1) \leq \sqrt{1 + L_0^2}$.

# Minimality

For $d = 1$, $m = n$ which is minimal.

For $d = 2$, $m = \frac{n^2+3n}{2}$. Is this minimal?

# Algebraic Embedding

## Encoding using Complex Roots

Idea: Consider the case $d = 2$. Then each $x_1, \cdots, x_n \in \mathbb{R}^2$ can be replaced by $n$ complex numbers $z_1, \cdots, z_n \in \mathbb{C}$, $z_k = x_{k,1} + i x_{k,2}$.
Consider the complex polynomial:

$$Q(z) = \prod_{k=1}^{n} (z - z_k) = z^n + \sum_{k=1}^{n} \sigma_k z^{n-k}$$

which requires $n$ complex numbers, or $2n$ real numbers.

# Algebraic Embedding

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Open problem: Can this construction be extended to $d \geq 3$?
Remark: A drawback of polynomial (algebraic) embeddings: [Cahill'19] showed that polynomial embeddings of translation invariant spaces cannot be bi-Lipschitz.

# The Max Pool approach

The idea is provided by the following observation.
Let $\downarrow: \mathbb{R}^n \to \mathbb{R}^n$ denote the *sorting map* $x \mapsto \downarrow x = \Pi x$, $\Pi \in \mathcal{S}_n$, so that

$$(\Pi x)_1 \geq (\Pi x)_2 \geq \cdots \geq (\Pi x)_n.$$

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## Lemma

$\downarrow : \widehat{\mathbb{R}^n} \to \mathbb{R}^n$ *is an isometry (hence bi-Lipschitz):*

$$\| \downarrow (x) - \downarrow (y) \| = \min_{P \in \mathcal{S}_n} \|x - Py\| \ , \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

Proof is based on the rearrangement inequality (see Wikipedia, or Hardy-Littlewood-Pólya "Inequalities" §10.2).

Our main goal is to extend this construction from $\mathbb{R}^n$ to $\mathbb{R}^{n \times d}$

# The Encoder $\beta_A$

## Notations

Recall the equivalence relation, for $X, Y \in \mathbb{R}^{n\times d}$,

$$X \sim Y \quad \Leftrightarrow \quad \exists \Pi \in \mathcal{S}_n \ , \ Y = \Pi X$$

that induces a quotient space $\widehat{\mathbb{R}^{n\times d}} = \mathbb{R}^{n\times d}/\sim$ and the natural distance

$$\mathbf{d} : \widehat{\mathbb{R}^{n\times d}} \times \widehat{\mathbb{R}^{n\times d}} \to \mathbb{R} \ \ , \ \ \mathbf{d}([X],[Y]) = \min_{\Pi\in\mathcal{S}_n} \|X - \Pi Y\|_F$$

In the following we construct an Euclidean embedding of the form

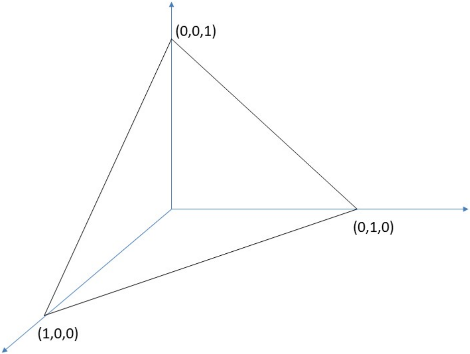
$$\beta_A : \mathbb{R}^{n\times d} \to \mathbb{R}^{n\times D} \ \ , \ \ \beta_A(X) = \downarrow (XA)$$

where $\downarrow(\cdot)$ sorts decreasingly each column of $\cdot$, independently.
The matrix $A \in \mathbb{R}^{d\times D}$ is called the *key* of encoder $\beta_A$.
The key is called *universal* if $\widehat{\beta_A} : \widehat{\mathbb{R}^{n\times d}} \to \mathbb{R}^{n\times D}$ is injective.

# Intuition behind universality of keys

Consider the case
$n = 2$ , $d = 3$

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix}$$



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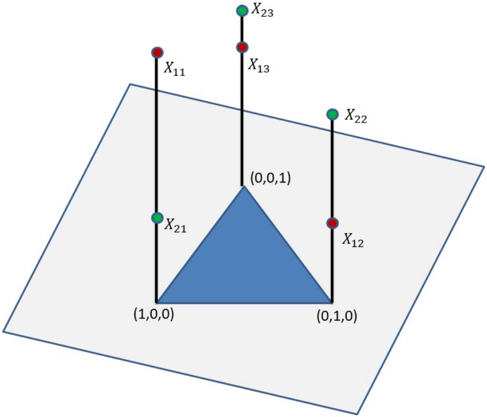


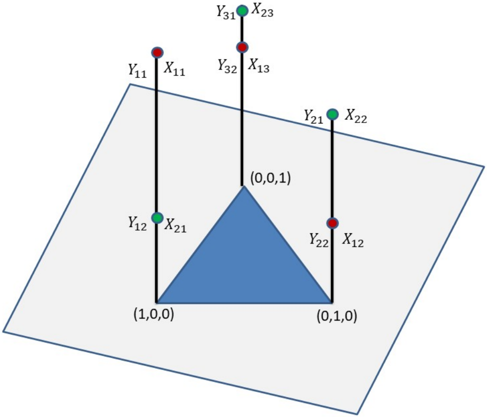
# Intuition behind universality of keys

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix}$$

$$Y = \downarrow X$$

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \end{bmatrix}$$

Information lost!



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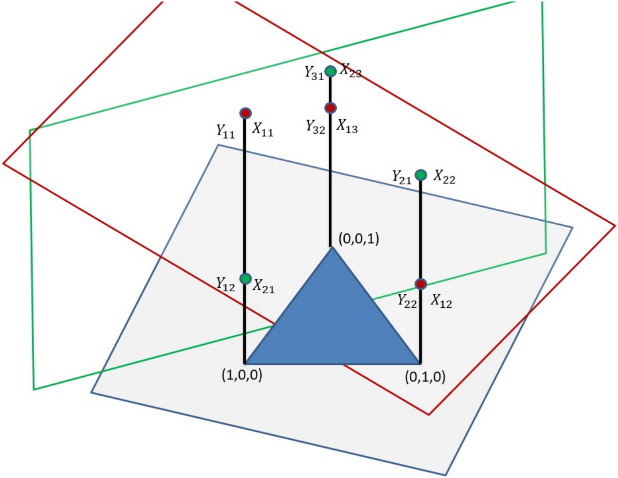
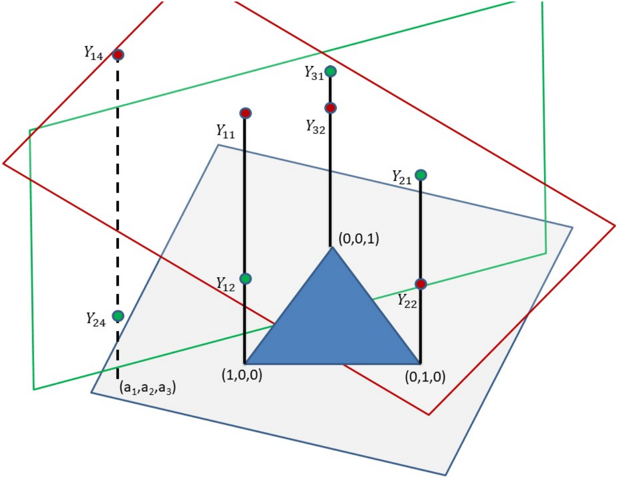


# Intuition for this encoder

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix}$$

$$Y = \downarrow \begin{bmatrix} X & Xa \end{bmatrix}$$

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \end{bmatrix}$$



# Three results (1)

## Existence of Universal Keys

### Theorem

*Consider the metric space* $(\widehat{\mathbb{R}^{n\times d}}, \mathbf{d})$. *Set* $D = 1 + (d-1)n!$ *and let* $A \in \mathbb{R}^{d\times D}$ *be a matrix whose columns form a full spark frame. Then the key* $A$ *is universal and the induced map* $\hat{\beta}_A : \widehat{\mathbb{R}^{n\times d}} \to \mathbb{R}^{n\times D}$, $\hat{\beta}_A([X]) = \downarrow(XA)$ *is injective. Furthermore,* $\hat{\beta}_A$ *is bi-Lipschitz with constants* $a_0 = \min_{J\subset[D],|J|=d} s_d(A[J])$ *and* $b_0 = s_1(A)$, *where* $s_1(A)$ *denotes the largest singular value of* $A$, $A[J]$ *denotes the submatrix of* $A$ *formed by columns indexed by* $J$, *and* $s_d(A[J])$ *denotes the* $d^{th}$ *singular value (in this case, the smallest) of* $A[J]$. *Specifically, for any* $X, Y \in \mathbb{R}^{n\times d}$,

$$a_0\mathbf{d}([X],[Y]) \le \|\beta_A(X) - \beta_A(Y)\| \le b_0\mathbf{d}([X],[Y]) \qquad (3.1)$$

*where all norms are Frobenius norms.*

# Three results (2)

## Bi-Lipschitz Property of Universal Keys

### Theorem

*Assume the key* $A \in \mathbb{R}^{d \times D}$ *is universal, i.e., the induced map* $\hat{\beta}_A : \widehat{\mathbb{R}^{n \times d}} \to \mathbb{R}^{n \times D}$, $[X] \mapsto \beta_A(X) = \downarrow(XA)$ *is injective. Then* $\hat{\beta}_A$ *is bi-Lipschitz, that is, there are constants* $a_0 > 0$ *and* $b_0 > 0$ *so that for all* $X, Y \in \mathbb{R}^{n \times d}$,

$$a_0\, \mathbf{d}([X],[Y]) \leq \|\beta_A(X) - \beta_A(Y)\| \leq b_0\, \mathbf{d}([X],[Y]) \qquad (3.2)$$

*where all are Frobenius norms. Furthermore, an estimate for* $b_0$ *is provided by the largest singular value of A,* $b_0 = s_1(A)$.

# Three results (3)

## Dimension Reduction

### Theorem

*Assume $A \in \mathbb{R}^{d \times D}$ is a universal key for $\widehat{\mathbb{R}^{n \times d}}$ with $D \geq 2d$. Then, for $m \geq 2nd$, a generic linear operator $B : \mathbb{R}^{n \times D} \to \mathbb{R}^m$ with respect to Zariski topology on $\mathbb{R}^{n \times D \times m}$, the map*

$$\hat{\beta}_{A,B} : \widehat{\mathbb{R}^{n \times d}} \to \mathbb{R}^{2nd} \; , \; \hat{\beta}_{A,B}(\hat{X}) = B\left(\hat{\beta}_A(\hat{X})\right) \tag{3.3}$$

*is bi-Lipschitz. In particular, almost every full-rank linear operator $B : \mathbb{R}^{n \times D} \to \mathbb{R}^{2nd}$ produces such a bi-Lipschitz map.*

This result is compatible with a Whitney embedding theorem with the important caveat that the Whitney embedding result applies to smooth manifolds, whereas $\widehat{\mathbb{R}^{n \times d}}$ is not a manifold.

# Highlights of proofs

## First result: Universal keys

The upper bound is imediate. For lower bound, fix $X, Y \in \mathbb{R}^{n \times d}$:

$$\|\beta_A(X) - \beta_A(Y)\|_2^2 = \sum_{k=1}^{D} \| \downarrow (Xa_k) - \downarrow (Ya_k)\|_2^2 = \sum_{k=1}^{D} \|P_k X a_k - Q_k Y a_k\|_2^2$$

$$\stackrel{\Pi_k := Q_k^T P_k}{=} \sum_{k=1}^{D} \|(\Pi_k X - Y) a_k\|_2^2$$

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$$\overset{\Pi_k := Q_k^T P_k}{=} \sum_{k=1}^{D} \|(\Pi_k X - Y) a_k\|_2^2 \geq \sum_{j=1}^{d} \|(\Pi_{k_j} X - Y) a_{k_j}\|_2^2$$

so that $\Pi_{k_1} = \cdots = \Pi_{k_d} = \Pi_0$ (pigeonhole principle: needs $D > (d-1)n!$).

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## First result: Universal keys

The upper bound is imediate. For lower bound, fix $X, Y \in \mathbb{R}^{n\times d}$:

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so that $\Pi_{k_1}=\cdots=\Pi_{k_d}=\Pi_0$ (pigeonhole principle: needs $D>(d-1)n!$). Then:

$$\|\beta_A(X)-\beta_A(Y)\|_2^2\geq\sum_{j=1}^{d}\|(\Pi_0X-Y)a_{k_j}\|_2^2\stackrel{full\ \ spark}{\geq}s_d(A[J])^2\|\Pi_0X-Y\|^2$$

$$\geq s_d(A[J])^2\min_{\Pi\in\mathcal{S}_n}\|\Pi X-Y\|^2=s_d(A[J])^2\mathbf{d}([X],[Y])^2$$

# Highlights of proofs

## Second result: Bi-Lipschitz Property

The proof resembles the treatment of phase retrieval problem:

1. Homogeneity and compactness reduce the problem to local analysis.
2. The encoder is "locally" linearized. The failure of local lower Lipschitz bound implies a certain behavior for a Quadratically Constrained Ratio of Quadratics (QCRQ).
3. QCRQ has a minimizer:inf $\Rightarrow$ min. [Teboulle&al.]
   This step took most of time and lots of (self)convincing !
4. Contradiction to injectivity assumption.

# More detailed proof of the bi-Lipschitz result (1)

1. Reduction to local lower Lipschitz bound.
Assume $\inf_{X \not\sim Y} \|\beta_A(X) - \beta_A(Y)\|_2 / \mathbf{d}([X],[Y]) = 0$. By *homogeneity* and *compactness*, extract/construct sequences $(X_j)_j$ and $(Y_j)_j$ so that: (i) $X_j \to Z$; (ii) $Y_j \to Z$; (iii) $\|Y_j\| \leq \|X_j\| = \|Z\| = 1$; (iv) $\mathbf{d}([X_j],[Z]) = \|X_j - Z\|$; (v) $\mathbf{d}([X_j],[Y_j]) = \|X_j - Y_j\|$; (vi) $\mathbf{d}([Y_j],[Z]) = \|Y_j - z\|$.

2. Local linearization.
Let $H = \{P \in \mathcal{S}_n \ ; \ PZ = Z\}$ denote the stabilizer of $Z$. Let $U_j = X_j - Z$ and $V_j = Y_j - Z$. Then:

$$\lim_{j \to \infty} \frac{\sum_{k=1}^{D} \min_{Q \in H} \|QU_j a_k - V_j a_k\|^2}{\|U_j - V_j\|^2} = 0\ , \quad \|U_j - V_j\| \leq \|U_j - PV_j\|, \ \forall P \in H.$$

# More detailed proof of the bi-Lipschitz result (2)

3. QCQP

Last limit implies:

$$\inf_{\substack{(u,v)\in\mathbb{R}^{n\times d}:\\ U\neq QV,\forall V\in H}} \max_{P\in H} \frac{\sum_{k=1}^{D}\|(U-\Pi_k V)a_k\|_2^2}{\|U-PV\|^2} = 0$$

where $\Pi_k$ achieves alignment between $U_j a_k$ and $V_j a_k$.

Since these groups are finite, we obtain that the infimum is achieved!

Hence:

4. Injectivity no-go

There are $U, V \in \mathbb{R}^{n\times d}$ so that $Z+U \not\sim Z+V$ and yet $(Z+U)a_k = \Pi_k(Z+V)a_k$ for all $k\in[D]$. This shows $\beta_A(Z+U) = \beta_A(Z+V)$ which contradicts injectivity!

Q.E.D.

# Highlights of proofs

## Third result: Dimension Reduction

The proof follows the approach in [Cahill&al.], [Dufresne]:

$$0 = B(\beta_A(X)) - B(\beta_A(Y)) \Rightarrow \beta_A(X) - \beta_A(Y) \in \ker(B)$$

Need to show: $\beta_A(X) - \beta_A(Y) = 0$, or, $Ran(\Delta) \cap \ker(B) = \{0\}$, where

$$\Delta : \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d} \to \mathbb{R}^{n\times D} \ , \ \Delta(X,Y) = \beta_A(X) - \beta_A(Y).$$

In the polynomial case, [Cahill&al.] exploit arguments from algebraic geometry. Here the problem is simpler since $Ran(\Delta)$ is included in a finite union of linear subspaces of dimension at most $2nd$.
By a dimension argument it follows that the target space for $B$ must be of dimension at least $2nd$ to obtain an injective embedding. In this case, generically, $Ran(\Delta)$ and $\ker(B)$ intersect transversally.

# Towards universal keys

The arXiv preprint provides necessary and sufficient conditions for a key to be universal.

Open Problem: Given $(n, d)$ find the smallest dimension $D$ so that there exists a universal key $A \in \mathbb{R}^{d \times D}$ for $\mathbb{R}^{n \times d}$.

So far we obtained (joint with Daniel Levy (UMD) ):

| n | d | D-d |
|---|---|---|
| 2 | 2 | 1 |
| 3 | 2 | 2 |
| 4 | 2 | 2 |
| 5 | 2 | 3 |
| 6 | 2 | $\geq 4$ |

Open Problem: If a universal key exists for a triple $(n, d, D)$ then is it true that universal keys are generic in $\mathbb{R}^{d \times D}$ ?

# Related results

A sequence of preprints came out almost simultaneously:

1. R. Balan, N. Haghani, M.Singh, Permutation Invariant Representations with Applications to Graph Deep Learning, arXiv:2203.07546 (2022)
2. N. Dym, S. J. Gortler, Low Dimensional Invariant Embeddings for Universal Geometric Learning, arXiv:2205.02956 (2022)
3. J. Cahill, J. W. Iverson, D. G. Mixon, D. Packer, Group-invariant max filtering, arXiv:2205.14039 (2022)

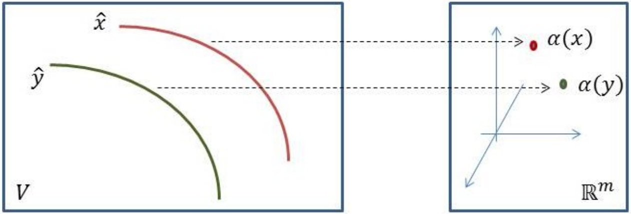
all of them based on sorting in one way or another. [Dym and Gortler] shows that the key size should be significantly smaller than $n!$. [Cahill et.al.'22] introduced the concept of *max filter* which is a special case of a more general G-invariant representation discussed next.

# Table of Contents

# High-Level View

Recall the framework for Euclidean embeddings of metric spaces induced by orthogonal representations of (finite) groups $G$ acting on a linear space $V$.

Metric space $(\hat{V}, \mathbf{d})$ where: $\hat{V} = V/G$ is the set of orbits, $[x] = \{U_g \ , \ g \in Gx\}$, for $x \in V$; and $\mathbf{d}(\hat{x}, \hat{y}) = \min_{u \in \hat{x}, v \in \hat{y}} \|u - v\|_V$.



# The Program

Given a (discrete) group $G$ acting unitarily on a normed space $V$, we formulate four general problems

1. Construct injective embeddings of the quotient space $V/G$, $\alpha : \hat{V} \to \mathbb{R}^m$. The injectivity problem.
2. Construct/Obtain bi-Lipschitz properties for the Euclidean embedding $\alpha : \hat{V} \to \mathbb{R}^m$. The stability problem.
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4. Analyze specific cases. Applications.

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Today we focus on the first two problems: injectivity and bi-Lipschitz stability.

# Invariant Representations

Let $V$ be a $d$-dimensional Hilbert space and $G$ a finite group of size $N = |G|$ acting unitarily on $V$, $\{U_g, g \in G\}$.
The quotient space $\widehat{V} = V/G$ is the set of orbits $[x] = \{U_g x\ ,\ g \in G\}$ induced by the group action, where for $x, y \in V$, $x \sim y$ iff $y = U_g x$ for some $g \in G$. $(\widehat{V}, \mathbf{d})$ becomes a metric space with the natural distance

$$\mathbf{d}([x],[y]) = \min_{g \in G} \|x - U_g y\|$$

How to construct an invariant representation?
The standard method in the computational invariant theory: Find generators of the ring of invariant polynomials in $d$ variables. This method goes back to Cayley, Hilbert, Noether .... However this approach has a drawback: it cannot produce bi-Lipschitz embeddings [1], unless special cases.

[1] J. Cahill, A. Contreras, A.C. Hip, Complete Set of translation Invariant Measurements with Lipschitz Bounds, ACHA 2020

# Sorting based Representations

Different approaches were considered recently [2],[3],[4] based on sorting. A unified framework for these approaches is presented here.

Fix a generator $w \in V$ (call it, window or template) and consider the nonlinear map induced by sorting its coorbit:

$$\phi_w : V \to \mathbb{R}^N \quad , \quad \phi_w(x) = \downarrow ((\langle x, U_g w \rangle)_{g \in G}) .$$

where $\downarrow(y) = (y_{\pi(i)})_{i \in [N]}$ is the non-increasing sorting operator: $y_{\pi(1)} \geq \cdots \geq y_{\pi(N)}$.

[2] R. Balan, N. Haghani, M.Singh, Permutation Invariant Representations with Applications to Graph Deep Learning, arXiv:2203.07546 (2022)

[3] N. Dym, S. J. Gortler, Low Dimensional Invariant Embeddings for Universal Geometric Learning, arXiv:2205.02956 (2022)

[4] J. Cahill, J. W. Iverson, D. G. Mixon, D. Packer, Group-invariant max filtering, arXiv:2205.14039 (2022)

# Representations based on sorting (2)

$$\phi_w : V \to \mathbb{R}^N \quad , \quad \phi_w(x) = \downarrow \left( (\langle x, U_g w \rangle)_{g \in G} \right).$$

Remarks:

1. $\phi_w(U_g x) = \phi_w(x)$ for every $g \in G$ and $x \in V$. Thus $\phi_w$ lifts to the quotient space $\widehat{V}$.
2. Invariant polynomials, and more generally, invariant functions obtained by the averaging operator (the Reynolds operator), can be obtained as:

$$K \mapsto F_K(x) = \frac{1}{|G|} \sum_{g \in G} K(\langle U_g x, w \rangle) = \frac{1}{|G|} \sum_{g \in G} K(\phi_w(x))$$

# Invariant Coorbit Representations

For a collection $\mathbf{w} = (w_1, \cdots, w_p) \in V^p$ let

$$\Phi_{\mathbf{w}} : V \to \mathbb{R}^{N \times p} \quad , \quad \Phi_{\mathbf{w}}(x) = \left[ \phi_{w_1}(x) | \cdots | \phi_{w_p}(x) \right] .$$

For a subset $S \subset [N] \times [p]$ of cardinal $m = |S|$, let

$$\Phi_{\mathbf{w},S} : V \to l^2(S) \sim \mathbb{R}^m \ , \ \Phi_{\mathbf{w},S}(x) = (\Phi_{\mathbf{w}}(x))|_S$$

be the restriction of $\Phi_{\mathbf{w}}$ to $S$. For a linear operator $\mathcal{L} : l^2(S) \to \mathbb{R}^m$, let

$$\Psi_{\mathbf{w},S,\mathcal{L}} : V \to \mathbb{R}^m \quad , \quad \Psi_{\mathbf{w},\mathcal{L}}(x) = \mathcal{L}(\Phi_{\mathbf{w},S}(x))$$

be the "projection" of $\Phi_{\mathbf{w},S}$ through $\mathcal{L}$ into $\mathbb{R}^m$.
Problems: Construct $(\mathbf{w}, S)$ so that $\Phi_{\mathbf{w},S}$ is a bi-Lipschitz embedding of $\widehat{V}$. Construct $(\mathbf{w}, S, \mathcal{L})$ so that $\Psi_{\mathbf{w},S,\mathcal{L}}$ is bi-Lipschitz.

# Invariant Coorbit Representations

Special cases:
1. If $G = S_n$ and $V = \mathbb{R}^{n \times d}$ with action $(P, X) \mapsto PX$, then [5] introduced the embedding $\beta_A(X) = \downarrow(XA)$, for *key* $A \in \mathbb{R}^{d \times D}$ and sorting operator acting independently in each column.
Equivalent recasting: Let $w_1 = \delta_1 \cdot a_1^T, \ldots, w_D = \delta_1 \cdot a_D^T$, where $\delta_1 = (1, 0, \cdots, 0)^T$ and $A = [a_1 | \cdots | a_D]$. Then note $\phi_{w_1}(X) = \downarrow(Xa_1) \otimes 1_{(n-1)!}$. Thus $\Phi_{\mathbf{w}}(X) = \beta_A(X) \otimes 1_{(n-1)!}$. Thus $\beta_A(X) = \Phi_{\mathbf{w},S}(X)$ for an appropriate subset $S \subset [n!] \times [D]$ of size $nD$.
2. The *max filter* introduced in [6] for some template $w \in V$ is defined by $\langle\langle \cdot, w \rangle\rangle : V \to \mathbb{R}$, $\langle\langle x, w \rangle\rangle = \max_{g \in G} \langle x, U_g w \rangle$. Equivalent recasting: $\langle\langle x, w \rangle\rangle = \Phi_{w,S}(X)$, for $S = \{1\}$.

[5] R. Balan, N. Haghani, M.Singh, Permutation Invariant Representations with Applications to Graph Deep Learning, arXiv:2203.07546 (2022)
[6] J. Cahill, J. W. Iverson, D. G. Mixon, D. Packer, Group-invariant max filtering, arXiv:2205.14039 (2022)

# Sufficient conditions for an injective embedding

## Theorem

*Consider $G$ finite group of size $N$ acting unitarily on the d-dimensional $V$. Let $\mathbf{w} \in V^p$, $S \subset [N] \times [p]$, $S_k$ the $k^{th}$ slice, and linear map $\mathcal{L} : l^2(S) \to \mathbb{R}^m$. Denote $\gamma_2 = \min_{g \in G, g \neq 1} \min_{\lambda \in \mathbb{R}} rank(\lambda I_d - U_g)$, $\gamma_3 = \max_{g \in G, g \neq 1} \min_{\lambda \in \mathbb{R}} rank(\lambda I_d - U_g)$. Then for almost every $\mathbf{w}$ and $\mathcal{L}$ the maps $\Phi_{\mathbf{w},S}$ or $\Psi_{\mathbf{w},S,\mathcal{L}}$ are injective on $\widehat{V}$ in any of the following cases:*

1. *(Max filter, Cahill et.al. 2022) If $p \geq 2d$ and $S_{max} = \{(1,1), \cdots, (1,p)\}$ then the max filterbank $\Phi_{\mathbf{w},S_{max}}$ is injective for a.e. $\mathbf{w} \in V^p$.*
2. *(variation of previous result) If $p \geq 2d$ and $|S_k| \geq 1$ for all $k \in [p]$ then $\Phi_{\mathbf{w},S}$ is injective for a.e. $\mathbf{w} \in V^p$.*
3. [a] *If $G$ is a reflection group and $p \geq d$ then the max filterbank $\Phi_{\mathbf{w},S_{max}}$ is injective for a.e. $\mathbf{w} \in V^p$.*

---

[a] D. Mixon, Y. Qaddura, Injectivity, stability, and positive definiteness of max filtering, arXiv:2212.11156

# Sufficient conditions for injective embedding (cont)

**Theorem**

4. *If* $p \geq 2d - \gamma_2$, $|S| \geq 2d$, *and for each* $k$, $|S_k| \in \{1,2\}$ *then* $\Phi_{\mathbf{w},S}$ *is injective for a.e.* $\mathbf{w} \in V^p$.
5. *If* $2d - \gamma_3 \leq p \leq 2d$, $|S_1| = \cdots = |S_{2d-p}| = N$, *and* $|S_{2d-p+1}| = \cdots = |S_p| = 1$ *then* $\Phi_{\mathbf{w},S}$ *is injective for a.e.* $\mathbf{w} \in V^p$.
6. *If* $\Phi_{\mathbf{w},S}$ *is injective and* $m \geq 2d$ *then the map* $\Psi_{\mathbf{w},S,\mathcal{L}}$ *is injective for a.e. linear map* $\mathcal{L} : l^2(S) \to \mathbb{R}^m$.

Remark:
This result can be extended to the case when $S$ has an irregular structure. However this requires some involved spectral conditions.

# Injectivity – sketch of proof

The proof provides a semi-algebraic characterization of the set of "bad" windows, i.e., windows $\mathbf{w}$ that fail to separate, say $\mathcal{F}$.

$$\mathcal{F} \subset \bigcup_{\mathbf{g},\mathbf{h} \in G^{p*}} \mathcal{F}_{\mathbf{g},\mathbf{h}} \ , \quad G^{p*} = \{(g_i^k)_{(i,k)\in S} \ , \ \forall k, (g_i^k)_{i\in S_k} \in G^{|S_k|} \text{are distinct}\}$$

$$\mathcal{F}_{\mathbf{g},\mathbf{h}} = \bigcup_{(x,y)\in\Gamma} \otimes_{k=1}^{p} \{U_{g_1^k}x - U_{h_1^k}y, \ldots, U_{g_{m_k}^k}x - U_{h_{m_k}^k}y\}^{\perp}$$

where $\Gamma = \{(x,y) \in V^2 \ : \ x \not\sim y, \|x\|^2 + \|y\|^2 = 1\}$, $m_k = |S_k|$. Using the "lift-and-project" technique, we realize each $\mathcal{F}_{\mathbf{g},\mathbf{h}}$ as finite unions of projection onto second term of total manifolds of certain real-analytic vector bundles. The vector bundles have as base manifolds subsets of $\Gamma$ where dimension of the orthogonal complement of constant. In turn those subsets are controled by spectral properties of $U_g$'s.

# Injectivity – sketch of proof

The base manifolds of these vector bundles are themselves total spaces of a different vector bundles living over Grassmanian manifolds. For instance, for $|S_k| = m_k = 2$, First construct the bundle $(Gr(1,\mathbb{R}^2),\pi,E)$ over $Gr(1,\mathbb{R}^2) = \mathbb{RP}^1 \sim [0,\pi)$ with total space

$$E = \{(\theta,x,y) \in [0,\pi)\times V^2 \; ; \; cos(\theta)(U_{g_1}x - U_{h_1}y) + sin(\theta)(U_{g_2}x - U_{h_2}y) = 0\}$$

Most of fibers are $d$-dimensional except what $tan(\theta)$ is an eigenvalue of some unitary $U_g$. Those two cases induce a disjoint partition $\Gamma = (\Gamma \setminus \Pi_2(E)) \cup (\Gamma \cap \Pi_2(E))$ so that

$$\begin{array}{rcl} (x,y) \in \Gamma_1 := \Gamma \setminus \Pi_2(E) & \rightarrow & \dim\{U_{g_1}x - U_{h_1}y), (U_{g_2}x - U_{h_2}y\}^{\perp} = d-2 \\ (x,y) \in \Gamma_2 := \Gamma \cap \Pi_2(E) & \rightarrow & \dim\{U_{g_1}x - U_{h_1}y), (U_{g_2}x - U_{h_2}y\}^{\perp} = d-1 \end{array}$$

from where the dimension estimates arise.

# Main Result

## Theorem

*Consider $G$ finite group of size $N$ acting unitarily on the $d$-dimensional $V$. Let $\mathbf{w} \in V^p$, $S \subset [N] \times [p]$ and $\mathcal{L} : l^2(S) \to \mathbb{R}^m$. Let*

$$B = \max_{\substack{\sigma_1, \cdots, \sigma_p \subset G, \\ |\sigma_k| = |S_k|, \forall k}} \lambda_{max}\left(\sum_{k=1}^{p} \sum_{g \in \sigma_k} U_g w_k w_k^T U_g^T\right)$$

*where $S_k = \{i \in [N], (i,k) \in S\}$ for each $k \in [p]$.*

1. $\Phi_{\mathbf{w},S} : (\widehat{V}, \mathbf{d}) \to l^2(S)$ *is Lipschitz with constant upper bounded by* $\sqrt{B}$.
2. *If* $S = [N] \times [p]$ *and* $\Phi_{\mathbf{w},S} : (\widehat{V}, \mathbf{d}) \to (\mathbb{R}^m, \|\cdot\|_2)$ *is injective then it is also bi-Lipschitz;*
3. *If* $S = [N] \times [p]$ *and* $\Phi_{\mathbf{w},S} : (\widehat{V}, \mathbf{d}) \to (\mathbb{R}^m, \|\cdot\|_2)$ *is injective then for a generic* $\mathcal{L}$ *with* $m \geq 2d$, *the map* $\Psi_{\mathbf{w},S,\mathcal{L}} : (\widehat{V}, \mathbf{d}) \to (\mathbb{R}^m, \|\cdot\|_2)$ *is injective and bi-Lipschitz.*

# Sketch of Proof

1. The upper Lipschitz bound is not too hard. A quick way to obtain it is by the Fundamental Theorem of Calculus: Fix $x, y \in V$ and choose them so that $\mathbf{d}([x],[y]) = \|x-y\|$. The function $f:[0,1] \to l^2(S)$, $f(t) = \Phi_{\mathbf{w},S}((1-t)x+ty)$ is Lipschitz because the sorting operator $\downarrow$ is Lipschitz. The upper Lipschitz constant is computable from FTC and Lebesgue's differentiation theorem:

$$\|f(1)-f(0)\|_2 = \| \int_0^1 (Jf)|_{(1-t)x+ty}(y-x)dt\| \leq sup_z \|J\Phi_{\mathbf{w},S}(z)\|_\infty \mathbf{d}([x],[y])$$

But wherever $\Phi$ is differentiable, $J\Phi_{\mathbf{w},S}(z) = \left[ \left( U_{g(\pi_k(i)} w_k \right)^T \right]_{(i,k)\in S}$ where $\pi_k$ is the permutation that sorts $\phi_{w_k}(z)$. From here one obtains the upper bound.

The same goes for $\Psi_{\mathbf{w},S,\mathcal{L}}$.

# Sketch of Proof (2)

2. The lower Lipschitz bound is more challanging. The proof follows the recipe from (Balan et.al. 2022) and is by contradiction.

Assume the lower bound is $A = 0$.

Step 1. A compactness argument together with the homogeneity of map $\Phi$ implies the local lower Lipschitz constant must vanish: there is $z \in S_1(V)$:

$$\lim_{r \downarrow 0} \inf_{\substack{x \not\sim y \\ \mathbf{d}([x],[z])<r, \mathbf{d}([y],[z])<r}} \frac{\|\Phi_{\mathbf{w},S}(x) - \Phi_{\mathbf{w},S}(y)\|_2}{\mathbf{d}([x],[y])} = 0.$$

Step 2. Construct sequences $(x_n)_n$ and $(y_n)_n$ so that: (i) $\|x_n\| = 1$, (ii) $\|y_n\| \leq 1$; (iii) $\mathbf{d}([x_n],[y_n]) = \|x_n - y_n\|$; (iv) $\mathbf{d}([x_n],[z]) = \|x_n - z\|$; and $x_n \to z$, $[y_n] \to [z]$, and $y_n \to y_\infty$.

Step 3. Let $H = \{g \in G : U_g z = z\}$ denote the stabilizer of $z$. Let $\Delta_0 = \min_{g \in G \setminus H} \|z - U_g z\| > 0$. Assume $n$ large enough so that $u_n = x_n - z$, $v_n = y_n - z$ satisfy $\|u_n\|, \|v_n\| < \frac{1}{4}\Delta_0$. This forces $y_\infty = z$.

# Sketch of Proof (2)

Step 4. By finiteness of $G$, we extract subsequences so that $(\Phi_{\mathbf{w},S}(x_n))_{i,k} = \langle x_n, U_{g(1,i,k)} w_k \rangle$ and $(\Phi_{\mathbf{w},S}(y_n))_{i,k} = \langle y_n, U_{g(2,i,k)} w_k \rangle$ (note the group elements are independent on n !). It follows:

$$\lim_{n\to\infty} \frac{1}{\|u_n - v_n\|^2} \sum_{(i,k)\in S} |\langle w_k, U^T_{g(1,i,k)} u_n - U^T_{g(2,i,k)} v_n \rangle|^2 = 0$$

Step 5. Using an argument about ratios of quadratics, it follows that one is able to produce $u, v$ so that $u \not\sim v$ and $\langle w_k, U^T_{g(1,i,k)} u - U^T_{g(2,i,k)} v \rangle = 0$ for all $(i,k) \in S$. Then for $s > 0$ small enough, $x = z + su$ and $y = z + sv$ we have $\mathbf{d}([x],[y]) > 0$ and yet $\Phi_{\mathbf{w},S}(x) = \Phi_{\mathbf{w},S}(y)$. Contradiction!

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Thank you!

Questions?