Lipschitz Extensions in Inverse Problems

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Based on joint works with: Yang Wang (HKST), Dongmian Zou (IMA), David Bekkerman and Wenbo Li (UMD).

Happy Birthday Akram!

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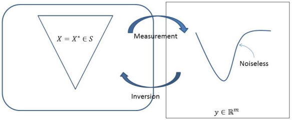
High-Level Problem Formulation

Given: A nonlinear map (analysis) $\alpha : \mathcal{S} \to \mathbb{R}^m$ from a metric space $(\mathcal{S}, d)$ to the Euclidean space $(\mathbb{R}^m, \|\cdot\|_2)$.
Wanted: A left inverse $\omega : \mathbb{R}^m \to \mathcal{S}$ that is globally Lipschitz.



Today problems: The case when $\mathcal{S} \subset Sym^+(\mathbb{C}^n)$ is a class of psd matrices, or $\mathcal{S} \subset \mathbb{R}^n$ is the class of sparse signals.

Quantum Tomography

Setup

A quantum system is characterized by the density matrix $M \in \mathbb{C}^{n\times n}$. Given a set of observables $Y_1, \cdots, Y_m$ that can be measured simultaneously, the problem is to estimate (compute) the density matrix $M = M^* \geq 0$ from noisy measurements:

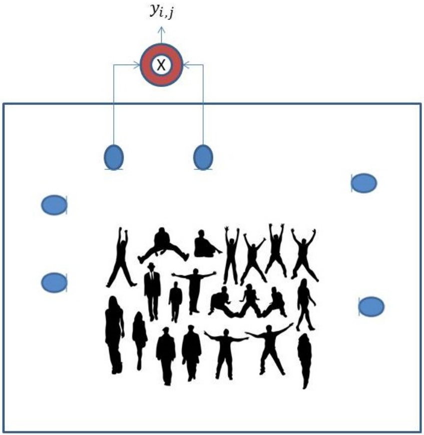
$$y_k = trace(MY_k) + \nu_k.$$

Constraints: (1) $trace(M) = 1$ (2) weakly mixed system, i.e. M has low rank, $rank(M) \leq d$:

$$\mathcal{S} = St^d(\mathbb{C}^n) = \{X = X^* \geq 0 \ , \ \ trace(X) = 1 \ , \ rank(X) \leq d\}.$$

Scene Understanding from Power Measurements

Setup



Mixing model: d decorrelated sources (acoustic, RF, etc) monitored by n sensors. A subset S of all possible ordered pairs $\{(i,j)\ ;\ 1 \leq i \leq j \leq n\}$ of sensors determines signal covariance, i.e. the measurements are:

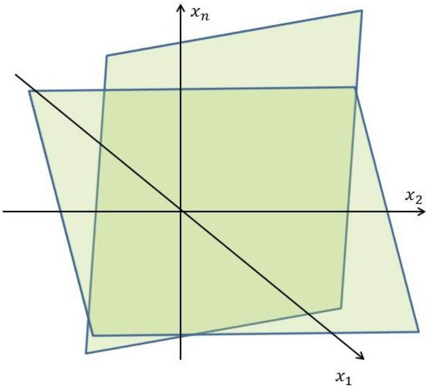
$$y_\alpha = \mathbb{E}[x_i \overline{x_j}] + \nu_\alpha = R_{i,j} + \nu_\alpha .$$

for $\alpha = (i,j) \in S$ and $R = \mathbb{E}[xx^*]$ is the $n \times n$ cov. matrix of rank d.

The problem is to estimate R from $\{y_\alpha\ ,\ \alpha \in S\}$ ($|S| = m$).
Here: $\mathcal{S} = \mathbb{S}^{d,0} = \{X = X^* \geq 0\ \ ,\ \ rank(X) \leq d\}$.

Compressive Sampling Scenario

Setup



Signal Model: x: d-sparse $\mathbb{R}^n$-vector.
Measurement Model:

$$y = Ax + \nu \in \mathbb{R}^m.$$

Here:

$$\mathcal{S} = \mathbb{R}^n_d = \{x \in \mathbb{R}^n \ , \ \|x\|_0 \leq d\}.$$

Notations

$H = \mathbb{F}^n$ a finite dimensional Euclidean space, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$,.

- $Sym(H) = \{T \in H^{n \times n}\ ,\ T = T^*\}$
- Convex cone of PSD: $Sym^+(H) = \{T \in Sym(H)\ ,\ T = T^* \geq 0\}$
- Quantum states: $St(H) = \{T \in Sym^+(H)\ ,\ trace(T) = 1\}$
- Low-rank quantum states
  $St^r(H) = \{T \in Sym^+(H)\ ,\ trace(T) = 1\ ,\ rank(T) \leq r\}$
- Cone of low-rank mixed signature matrices:

$$\mathbb{S}^{p,q} = \{T \in Sym(H)\ ,\ T \text{ has at most } p \text{ positive and } q \text{ negative eigenvalues}\}$$

In particular $\mathbb{S}^{1,0} = \{xx^*\ ,\ x \in H\}$, set of rank (at most) one PSDs.

- Cone of sparse signals:

$$H_d = \mathbb{R}^n_d = \{x \in H = \mathbb{R}^n\ ,\ \|x\|_0 \leq d\}.$$

Problem Formulation
Models

Forward maps:

$$\alpha : Sym^+(H) \rightarrow \mathbb{R}^m \quad , \quad (\alpha(X))_k = \sqrt{trace(XF_k)} = \sqrt{\langle X, F_k \rangle}$$

$$\beta : Sym^+(H) \rightarrow \mathbb{R}^m \quad , \quad (\beta(X))_k = trace(XF_k) =: \langle X, F_k \rangle$$

where $F_1, \cdots, F_m \in Sym^+(H)$ are fixed PSD matrices.

$$\gamma : H_d \rightarrow \mathbb{R}^m \quad , \quad \gamma(x) = Ax$$

where $A \in \mathbb{R}^{m \times n}$ is a "fat" measurement matrix $(n > m \geq 2d)$.

Problem Formulation
Models

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Spaces:

- Phase Retrieval: $\mathcal{S} = \mathbb{S}^{1,0} = \{xx^* \ , \ x \in H\}$ or $\mathcal{S} = \hat{H} = H/T^1$.
- Quantum Tomography:
  $\mathcal{S} = St^r(H) = \{X = X^* \geq 0 \ , \ trace(X) = 1 \ , \ rank(X) \leq r\}$.
- Covariance Matrix Estimation: $\mathcal{S} = \mathbb{S}^{d,0}$.
- Sparse Signal Estimation: $\mathcal{S} = \mathbb{R}^n_d$.

Problem Formulation

The phase retrieval problem

Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad (\alpha(x))_k = |\langle x, f_k \rangle| = \sqrt{\langle xx^*, f_k f_k^* \rangle}.$$

$$\beta : \hat{H} \to \mathbb{R}^m \quad , \quad (\beta(x))_k = |\langle x, f_k \rangle|^2 = \langle xx^*, f_k f_k^* \rangle.$$

Assume α, β are injective, the problem is to construct global Lipschitz inverses and to study their Lipschitz constants.

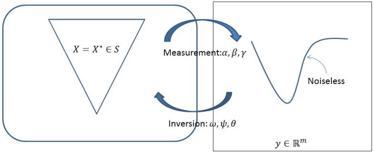
Problem Formulation

Lipschitz reconstruction: the general case

Assume the maps $\alpha, \beta, \gamma : \mathcal{S} \to \mathbb{R}^m$ are injective, where

$$(\alpha(X))_k = \sqrt{trace(XF_k)} \quad , \quad (\beta(X))_k = trace(XF_k) \quad , \quad \gamma(x) = Ax.$$

Our Problem Today:



Construct Lipschitz maps $\omega, \psi, \theta : \mathbb{R}^m \to \mathbb{S}$ so that $\omega \circ \alpha = 1_X$, $\psi \circ \beta = 1_X$, $\theta \circ \gamma = 1_{\mathcal{S}}$. Determine $Lip(\omega)$, $Lip(\psi)$ and $Lip(\theta)$.

Metric Structures on $\hat{H}$ and $Sym(H)$

Norm Induced Metric

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance* on $Sym(H)$:

$$d_p : Sym(H) \times Sym(H) \to \mathbb{R} \ , \ d_p(X,Y) = \|X-Y\|_p,$$

the p-norm of singular values (nuclear $p=1$, Frobenius $p=2$, operator $p=\infty$).

On $\hat{H} = H/T^1$ it induces the metric

$$\mathbf{d}_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \mathbf{d}_p(\hat{x},\hat{y}) = \|xx^* - yy^*\|_p$$

so that $\mathbf{d}_p(\hat{x},\hat{y}) = d_p(xx^*, yy^*)$. In the case $p=2$ we obtain

$$d_2(X,Y) = \|X-Y\|_F \quad , \quad \mathbf{d}_2(x,y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x,y\rangle|^2}$$

Metric Structures on $\hat{H}$ and $Sym(H)$

The Natural Metric

The *natural metric*

$$\mathbf{D}_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ \mathbf{D}_p(\hat{x}, \hat{y}) = \min_{\varphi} \|x - e^{i\varphi} y\|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

$$\mathbf{D}_2(\hat{x}, \hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y\rangle|}$$

On $Sym^+(H)$, the "natural" metric lifts to

$$D_p : Sym^+(H) \times Sym^+(H) \to \mathbb{R} \ , \ \ D_p(X, Y) = \min_{\substack{VV^* = X \\ WW^* = Y}} \|V - W\|_p.$$

Metric Structures on $Sym(H)$

Natural metric vs. Bures/Helinger

Let $X, Y \in Sym^+(H)$. For the natural distance we choose $p = 2$:

$$D_{natural}(X,Y) = \min_{\substack{VV^* = X \\ WW^* = Y}} \|V - W\|_F$$

Fact:

$$D_{natural}(X,Y) = \min_{U \in U(n)} \|X^{1/2} - Y^{1/2}U\|_F = \sqrt{\mathrm{tr}(X) + tr(Y) - 2\|X^{1/2}Y^{1/2}\|_1}$$

Metric Structures on $Sym(H)$

Natural metric vs. Bures/Helinger

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Another distance: Bures/Helinger distance:

$$D_{Bures}(X,Y) = \|X^{1/2} - Y^{1/2}\|_F = d_2(X^{1/2}, Y^{1/2})$$

A consequence of the Arithmetic-Geometric Mean Inequality [BhatiaKittaneh00]:

$$\frac{1}{2}\|X^{\frac{1}{2}} - Y^{\frac{1}{2}}\|_F^2 \le \min_{U \in U(n)} \|X^{\frac{1}{2}} - Y^{\frac{1}{2}}U\|_F^2 \le \|X^{\frac{1}{2}} - Y^{\frac{1}{2}}\|_F^2$$

Stability Results for the forward maps

Bi-Lipschitz properties of α and β

Fix a closed subset $\mathcal{S} \subset Sym^+(H)$. For instance $\mathcal{S} = St(H)$, or $\mathcal{S} = \mathbb{S}^{r,0}$, or $\mathcal{S} = St^r(H) = St(H) \cap \mathbb{S}^{r,0}$.

Theorem

Assume $\mathcal{F} = \{F_1, \cdots, F_m\} \subset Sym^+(H)$ so that $\alpha|_{\mathcal{S}}$ and $\beta|_{\mathcal{S}}$ are injective. Then there are constants $a_0, A_0, b_0, B_0 > 0$ so that for every $X, Y \in \mathcal{S}$,

$$A_0 \|X^{1/2} - Y^{1/2}\|_F^2 \leq \sum_{k=1}^m \left| \sqrt{\langle X, F_k \rangle} - \sqrt{\langle Y, F_k \rangle} \right|^2 \leq B_0 \|X^{1/2} - Y^{1/2}\|_F^2$$

$$a_0 \|X - Y\|_F^2 \leq \sum_{k=1}^m |\langle X, F_k \rangle - \langle Y, F_k \rangle|^2 \leq b_0 \|X - Y\|_F^2.$$

Stability Results for the inverse map

Lipschitz inversion of α and β on Quantum States

Consider the measurement maps

$$\alpha,\beta : (St^r(H),d_1) \to (\mathbb{R}^m, \|\cdot\|_2)\ ,\ (\alpha(T))_k = \sqrt{tr(TF_k)}\ ,\ (\beta(T))_k = tr(TF_k)$$

where $St^r(H) = \{T = T^* \geq 0\ ,\ tr(T) = 1\ ,\ rank(T) \leq r\}$.
If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.
If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract ($St^1(H) = P(H)$).

Stability Results for the inverse map

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If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.
If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract ($St^1(H) = P(H)$). Consequence:

Theorem

Fix $1 \leq r < n$. For any set of matrices $F_1, \cdots, F_m \in Sym^+(H)$ thre are no continuous maps $\omega : \mathbb{R}^m \to St^r(H)$ or $\psi : \mathbb{R}^m \to St^r(H)$ so that $\omega(\alpha(T) = T$ for every $T \in Sym^+(H)$, or $\psi(\beta(T)) = T$ for every $T \in Sym^+(H)$.

Lipschitz inversion of α on $\mathbb{S}^{r,0}$

Theorem

Assume the map

$$\alpha : (\mathbb{S}^{r,0}(H), D_{Bures}) \to (\mathbb{R}^m, \|\cdot\|_2) \quad , \quad (\alpha(T))_k = \sqrt{trace(TF_k)}$$

is injective, where $\mathbb{S}^{r,0}(H) = \{T = T^* \geq 0 \ , \ rank(T) \leq r\}$. *Then there exists a Lipschitz map* $\omega : \mathbb{R}^m \to \mathbb{S}$ *so that* $\omega(\alpha(T)) = T$ *for every* $T \in \mathbb{S}^{r,0}$, *and*

$$Lip(\omega) = \sup_{c \neq d \in \mathbb{R}^m} \frac{\|(\omega(c))^{1/2} - (\omega(d))^{1/2}\|_F}{\|c - d\|_2} \leq \frac{\sqrt{r+1}}{\sqrt{A_0}}.$$

Lipschitz inversion of β on $\mathbb{S}^{r,0}$

Theorem

Assume the map

$$\beta : (\mathbb{S}^{r,0}(H), \|\cdot\|_F) \to (\mathbb{R}^m, \|\cdot\|_2) \quad , \quad (\beta(T))_k = trace(TF_k)$$

is injective, where $\mathbb{S}^{r,0}(H) = \{T = T^* \geq 0 \ , \ rank(T) \leq r\}$. *Then there exists a Lipschitz map* $\psi : \mathbb{R}^m \to \mathbb{S}$ *so that* $\psi(\beta(T)) = T$ *for every* $T \in \mathbb{S}^{r,0}$, *and*

$$Lip(\psi) = \sup_{c \neq d \in \mathbb{R}^m} \frac{\|\psi(c) - \psi(d)\|_F}{\|c - d\|_2} \leq \frac{\sqrt{r+1}}{\sqrt{a_0}}.$$

Phase Retrieval: Lipschitz inversion of α

Theorem (B.Li18,B.Zou15,BWang15,BCMN14)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\alpha : (\hat{\mathbb{C}}^n, \mathbf{D}_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in \mathbb{C}^n$:*

$$A_0 \min_{\varphi} \|x - e^{i\varphi}y\|_2^2 \leq \sum_{k=1}^{m} ||\langle x, f_k\rangle| - |\langle y, f_k\rangle||^2 \leq B_0 \min_{\varphi} \|x - e^{i\varphi}y\|_2^2.$$

2. *$B_0 = B$, the frame upper bound.*
3. *In the real case: $A_0 = \min_{I \subset [m]} A[I] + A[I^c]$.*
4. *There is a Lipschitz map $\omega : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{\mathbb{C}}^n$, and (ii) its Lipschitz constant is $Lip(\omega) \leq \frac{2}{\sqrt{A_0}}$.*

Phase Retrieval: Lipschitz inversion of β

Theorem (B.Li18,B.Zou15,BWang15,BCMN14)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map* $\beta : (\hat{\mathbb{C}}^n, \mathbf{d}_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ *is bi-Lipschitz. Let* $\sqrt{a_0}, \sqrt{b_0}$ *denote its Lipschitz constants: for every* $x, y \in \mathbb{C}^n$*:*

$$a_0 \|xx^* - yy^*\|_1^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k\rangle|^2 - |\langle y, f_k\rangle|^2 \right|^2 \leq b_0 \|xx^* - yy^*\|_1^2.$$

2. $b_0 = \max_{\|x\|=1} \|Fx\|_4^4$.
3. *There is a Lipschitz map* $\psi : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, d_1)$ *so that: (i)* $\psi(\beta(x)) = x$ *for every* $x \in \hat{\mathbb{C}}^n$*, and (ii) its Lipschitz constant is* $Lip(\psi) \leq \frac{2}{\sqrt{a_0}}$.

Global Lipschitz inversion in Compressive Sampling

Theorem

Assume that every 2d columns of the $m \times n$ matrix A are linearly independent. Let $c_0 = \min_{|I|=2d} \sigma_{2d}(A[I])$ (square root of the smallest lower Riesz bound among all possible combinations of 2d columns). Let $\gamma : \mathbb{R}^n_d \to \mathbb{R}^m$, $\gamma(x) = Ax$, where $\mathbb{R}^n_d$ denotes the space of d-sparse signals in $\mathbb{R}^n$. Then

1. *For every $x, y \in \mathbb{R}^n_d$, $\|\gamma(x) - \gamma(y)\|_0 \geq c_0 \|x - y\|_2$.*
2. *There is a Lipschitz maps $\theta : \mathbb{R}^m \to \mathbb{R}^n_d$ so that: (i) $\theta(\gamma(x)) = x$ for all $x \in \mathbb{R}^n_d$; (ii) $Lip(\theta) \leq \frac{\sqrt{d+1}}{c_0}$.*

Note: Same bounds for $\mathbb{C}^n_d$.

Lipschitz Inversion
Overview

The extension mechanism involves three steps:

1. Embed the metric space $(\mathcal{S}, d)$ into a Hilbert space K ($Sym(H)$ or H);
2. Use Kirszbraun's theorem to obtain an isometric extension;
3. Construct and apply a Lipschitz projection in K onto the image of $(\mathcal{S}, d)$.

We exemplify this mechanism on the phase retrieval (PR) problem. The Low-Rank PSD Case: Similar to the PR case; different Lipschitz retraction for $\mathbb{S}^{r,0}(H)$. Same for the compressive sampling problem.
Note: The same mechanism works in the Johnson-Lindenstrauss theorem.

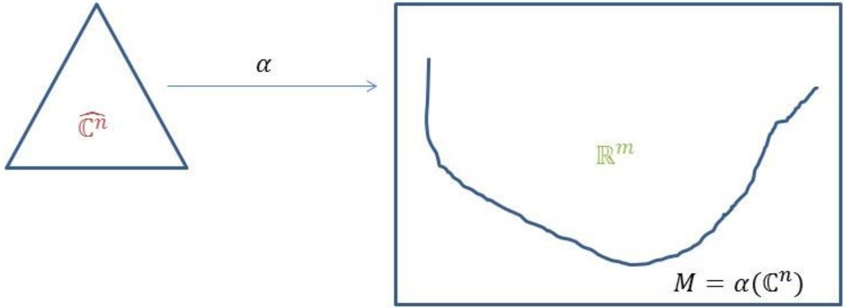
PR Inversion

Extension of the inverse for α

We know $\alpha : (\hat{H}, \mathbf{D}_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

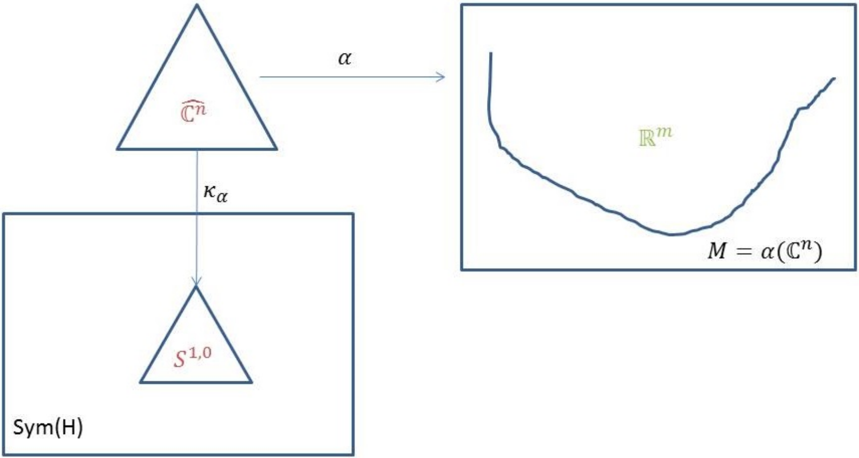
$$A_0 \mathbf{D}_2(x,y)^2 \leq \|\alpha(x) - \alpha(y)\|^2 \leq b_0 \mathbf{D}_2(x,y)^2$$

Let $M = \alpha(\hat{H}) \subset \mathbb{R}^m$.



PR Inversion
Extension of the inverse for α: Step 1

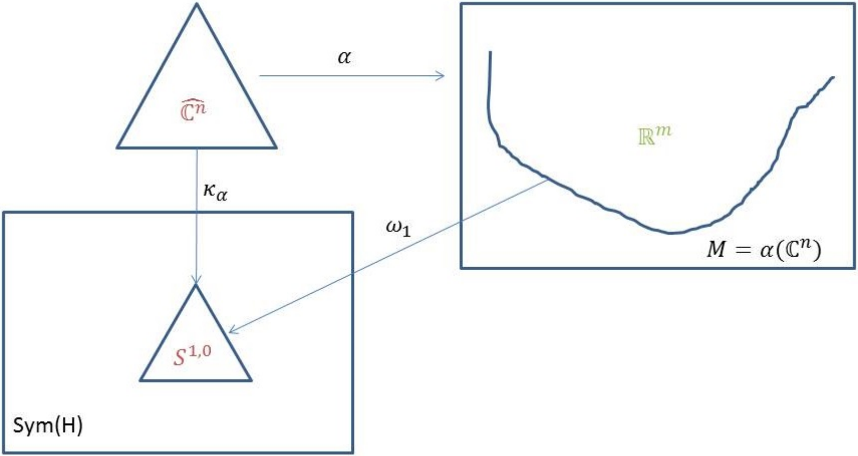
First identify (=embed) $\hat{H}$ with $\mathbb{S}^{1,0}(H)$.



PR Inversion

Extension of the inverse for α: Step 1

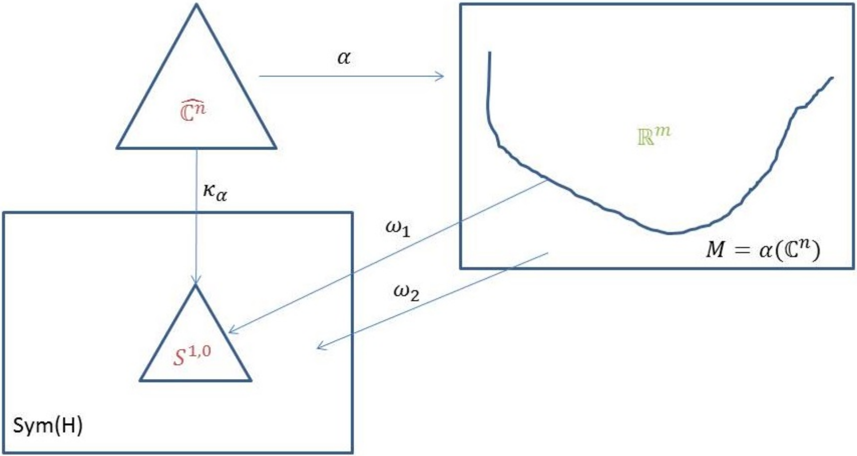
Then construct the local left inverse $\omega_1 : M \rightarrow \hat{H}$ with $Lip(\omega_1) = \frac{1}{\sqrt{A_0}}$.



PR Inversion

Extension of the inverse for α: Step 2

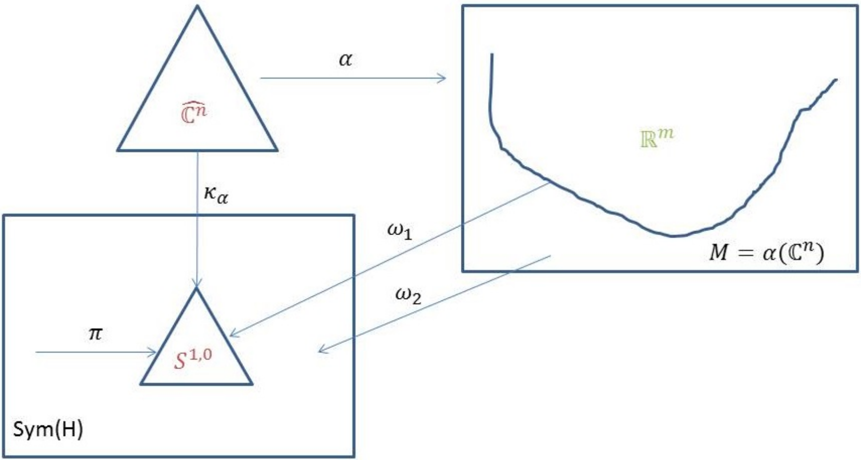
Use Kirszbraun's theorem to extend isometrically $\omega_2 : \mathbb{R}^m \to Sym(H)$.



PR Inversion

Extension of the inverse for α: Step 3

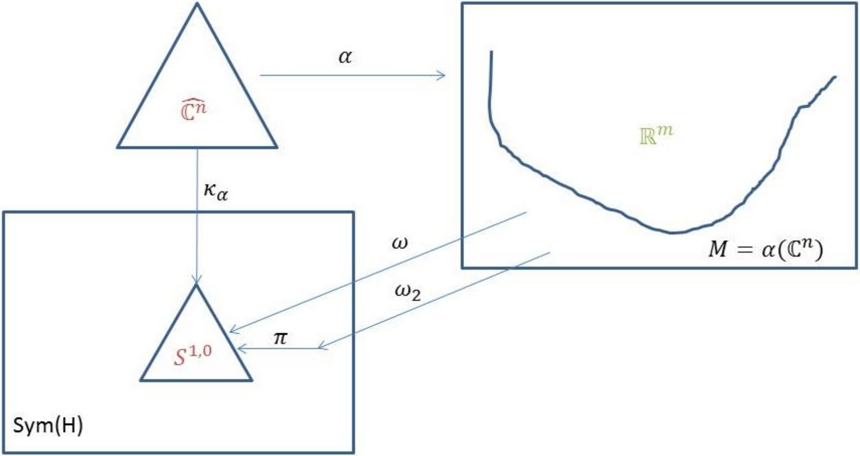
Construct a Lipschitz "projection" $\pi : Sym(H) \rightarrow \mathbb{S}^{1,0}(H)$.



PR Inversion

Extension of the inverse for α: Final process

Compose the two maps to get $\omega : \mathbb{R}^m \to \mathbb{S}^{1,0}$, $\omega = \pi \circ \omega_2$.



Part 2: $\mathbb{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the spectral decomposition of the self-adjoint operator A in $Sym(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : Sym(H) \to \mathbb{S}^{1,0}(H) \quad , \quad \pi(A) = (\lambda_1 - \lambda_2)P_1$$

satisfies the following two properties:

1. $\pi : (Sym(H), \|\cdot\|_F) \to (\mathbb{S}^{1,0}(H), \|\cdot\|_F)$ *is Lipschitz with* $Lip(\pi) = \sqrt{2}$.
2. $\pi(A) = A$ *for all* $A \in \mathbb{S}^{1,0}(H)$.

In [B.Zou'15] paper we proved, for $\pi : (Sym(H), d_p) \to (\mathbb{S}^{1,0}(H), d_p)$, $Lip(\pi) \leq 3 + 2^{1+\frac{1}{p}}$.
Recently [March 2018], Wenbo Li [AMSC/UMD] proved $Lip(\pi) = 2$ for $p = \infty$.

$\mathbb{S}^{r,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the nonlinear projector P_+ onto the cone of PSD matrices $Sym^+(H)$. Then the map

$$\pi_r : Sym(H) \to \mathbb{S}^{1,0}(H) \quad , \quad \pi(A) = P_+(A - \lambda_{r+1}(A)I)$$

satisfies the following two properties:

1. $\pi_r : (Sym(H), \|\cdot\|_F) \to (\mathbb{S}^{r,0}(H), \|\cdot\|_F)$ *is Lipschitz with* $Lip(\pi_r) = \sqrt{r+1}$.
2. $\pi_r(A) = A$ *for all* $A \in \mathbb{S}^{r,0}(H)$.

$H_d = \mathbb{R}^n_d$ as Lipschitz retract in $H = \mathbb{R}^n$

Lemma

Consider the nonlinear soft thresholding operator $\tau_\theta(t) = sign(t)[|t| - \theta]_+$. *Consider the map*

$$P_d : \mathbb{R}^n \to \mathbb{R}^n_d \quad , \quad (P_d(x))_k = \tau_\theta(x_k) \ , \ \theta = |\tilde{x}_{d+1}|$$

where $\tilde{x}_{d+1}$ *is the* $d+1^{st}$ *largest entry in magnitude. Then* P_d *satisfies the following two properties:*

1. $P_d : (H, \|\cdot\|_2) \to (H_d, \|\cdot\|_2)$ *is Lipschitz with* $Lip(P_d) = \sqrt{d+1}$.
2. $P_d(x) = x$ *for all* $x \in H_d$.

THANK YOU!!

Questions ?

References

[BCE06] R. Balan, P. Casazza, D. Edidin, On signal reconstruction without phase, Appl.Comput.Harmon.Anal. **20** (2006), 345–356.

[BBCE07] R. Balan, B. Bodmann, P. Casazza, D. Edidin, Painless reconstruction from Magnitudes of Frame Coefficients, J.Fourier Anal.Applic., **15** (4) (2009), 488–501.

[B12] R. Balan, Reconstruction of Signals from Magnitudes of Frame Representations, arXiv submission arXiv:1207.1134

[B13] R. Balan, Reconstruction of Signals from Magnitudes of Redundant Representations: The Complex Case, available online arXiv:1304.1839v1, Found.Comput.Math. 2015, http://dx.doi.org/10.1007/s10208-015-9261-0

[BW14] R. Balan and Y. Wang, Invertibility and Robustness of Phaseless Reconstruction, available online arXiv:1308.4718v1, Appl. Comp. Harm. Anal., 38 (2015), 469–488.

[BZ14] R. Balan, D. Zou, On Lipschitz inversion of nonlinear redundant representations, to appear in Contemporary Mathematics 2015.

[BZ15] R. Balan, D. Zou, On Lipschitz Analysis and Lipschitz Synthesis for the Phase Retrieval Problem, available online arXiv 1506.02092v1 [mathFA], 6 June 2015; Lin. Alg. and Appl. 2016.

[Bal15] R. Balan, The Fisher Information Matrix and the Cramer-Rao Lower Bound in a Non-Additive White Gaussian Noise Model for the Phase Retrieval Problem, proceedings of SampTA 2015.

[BCMN14] A. S. Bandeira, J. Cahill, D. Mixon, A. A. Nelson, Saving phase: Injectivity and Stability for phase retrieval, arXiv submission , arXiv: 1302.4618, Appl. Comp. Harm. Anal. 37 (1) (2014), 106–125.

[BK00] R. Bhatia, F. Kittaneh, *Notes on matrix arithmetic-geometric mean inequalities*, Lin.Alg.Appl. **308** (2000), 203–211.

[EM12] Y. C. Eldar, S. Mendelson, *Phase retrieval: Stability and recovery guarantees*, available online: arXiv:1211.0872, Appl. Comp. Harm. Anal. (2014), .

[HG13] M.J. Hirn, E. Le Gruyer, *A general theorem of existence of quasi absolutely minimal Lipschitz extensions*, arXiv:1211.5700v2 [math.FA], 8 Aug 2013.

[ZB06] L. Zwald, G. Blanchard, *On the convergence of eigenspaces in kernel Principal Component Analysis*, Proc. NIPS 05, vol. 18, 1649-1656, MIT Press, 2006.