Geometric and Analytic Properties of Positive Semi-Definite Matrices

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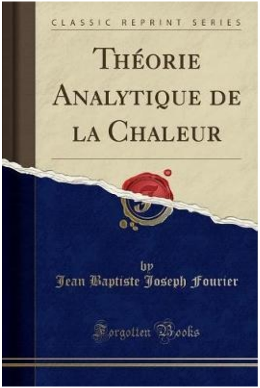
Dedicated to Hans Feichtinger for his 70th Birthday

Norbert Wiener Center for Harmonic Analysis and Applications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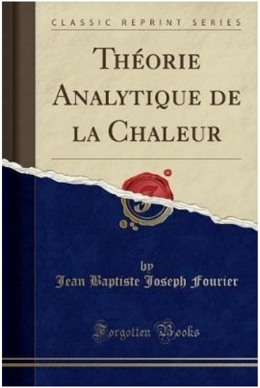
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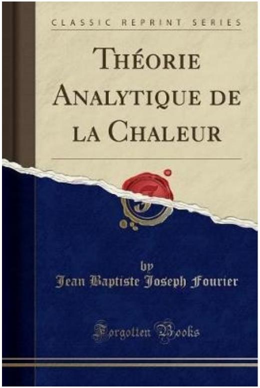
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Collaborators:
Kasso Okoudjou (Tufts), Anirudha Poria(Bar-Ilan U.), Michael Rawson (UMD), Yang Wang (HKUST), Rui Zhang (HKUST)

Works:

1. R. Balan, K.A. Okoudjou, M. Rawson, Y. Wang, R. Zhang, *Optimal l1 Rank One Matrix Decomposition*, in "Harmonic Analysis and Applications", Rassias M., Ed. Springer (2021)
2. R. Balan, K. Okoudjou, A. Poria, *On a Feichtinger Problem*, Operators and Matrices vol. 12(3), 881-891 (2018) http://dx.doi.org/10.7153/oam-2018-12-53

Problem Formulation

Let $Sym^+(\mathbb{C}^n) = \{A \in \mathbb{C}^{n\times n}\ ,\ A^* = A \geq 0\}$. For $A \in Sym^+(\mathbb{C}^n)$,

$$\gamma_+(A) := \inf_{A=\sum_{k\geq 1} x_x x_k^*} \sum_k \|x_k\|_1^2$$

The *matrix conjecture*: There is a universal constant C_0 such that, for every $n \geq 1$ and $A \in Sym^+(\mathbb{C}^n)$,

$$\gamma_+(A) \leq C_0 \|A\|_1 := C_0 \sum_{k,l=1}^{n} |A_{k,l}|$$

Motivation

A Feichtinger Problem

At a 2004 Oberwolfach meeting, H.F. asked the following question: (Q1) Given a positive semi-definite trace-class operator $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$, $Tf(x) = \int K(x,y)f(y)dy$, with $K \in M^1(\mathbb{R}^d \times \mathbb{R}^d)$, and its spectral factorization, $T = \sum_k \langle \cdot, h_k \rangle h_k$, must it be $\sum_k \|h_k\|_{M^1}^2 < \infty$?

A modified version of the question is: (Q2) Given T as before ($T = T^* \geq 0$, $K \in M^1(\mathbb{R}^d \times \mathbb{R}^d)$), is there a factorization $T = \sum_k \langle \cdot, g_k \rangle g_k$ such that $\sum_k \|g_k\|_{M^1}^2 < \infty$?

Using (Heil,Larson '08) and some functional analysis arguments:

Proposition

If (Q2) is answered affirmatively, then the matrix conjecture must be true.

Current Status of the Matrix Conjecture

The infimum is achieved:

$$\gamma_+(A) := \inf_{A=\sum_{k\geq 1} x_x x_k^*} \sum_k \|x_k\|_1^2 = \min_{A=\sum_{k=1}^{n^2} x_x x_k^*} \sum_k \|x_k\|_1^2.$$

Upper bounds:

$$\gamma_+(A) \leq n\, trace(A) \leq n\|A\|_1 := n\sum_{k,j} |A_{k,j}|$$

Lower bounds:

$$\|A\|_1 = \min_{A=\sum_{k\geq 1} x_x y_k^*} \sum_k \|x_k\|_1 \|y_k\|_1 \leq \gamma_+(A)$$

Convexity: for $A, B \in Sym^+(\mathbb{C}^n)$ and $t \geq 0$,

$$\gamma_+(A+B) \leq \gamma_+(A) + \gamma_+(B) \quad , \quad \gamma_+(tA) = t\gamma_+(A)$$

Current Status of the Matrix Conjecture

Lower bound is achieved:

1. If $A = xx^*$ is of rank one, then $\gamma_+(A) = \|x\|_1^2 = \|A\|_1$.
2. If $A \geq 0$ is diagonally dominant matrix, then $\gamma_+(A) = \|A\|_1$.

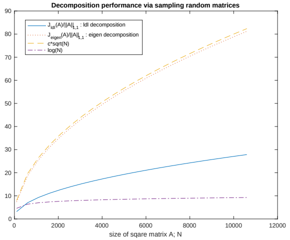
Continuity:

1. Let $Sym^{++}(\mathbb{C}^n) = \{A = A^* > 0\}$. Then $\gamma_+|_{Sym^{++}} : Sym^{++}(\mathbb{C}^n) \to \mathbb{R}$ is continuous.
2. If $A, B \in Sym^+(\mathbb{C}^n)$, $trace(A), trace(B) \leq 1$ and $A, B \geq \delta I$ then

$$|\gamma_+(A) - \gamma_+(B)| \leq \left(\frac{n}{\delta^2} + n^2\right) \|A - B\|_{Op}$$

hence Lipschitz continuous.

Maximum of $\sum_k \|x_k\|_1^2 / \|A\|_1$ over 30 random noise realizations, where $x_k's$ are obtained from the eigendecomposition, or the LDL factorization.



First New Result: Measure Optimization

Let $S_1 = \{x \in \mathbb{C}^n \ , \ \|x\|_1 = 1\}$ denote the compact unit sphere with respect to the l^1 norm, and let $\mathcal{B}(S_1)$ denote the set of Borel measures over S_1. For $A \in \mathrm{Sym}(\mathbb{C}^n)^+(\mathbb{C}^n)$ consider the optimization problem:

$$(p^*, \mu^*) = inf_{\mu \in \mathcal{B}(S_1) : \int_{S_1} xx^* d\mu(x) = A} \mu(S_1) \quad (M)$$

Theorem (Optimal Measure)

For any $A \in Sym^+(\mathbb{C}^n)$ the optimization problem (M) is convex and its global minimum is achieved by

$$p^* = \gamma_+(A) \quad , \quad \mu^*(x) = \sum_{k=1}^{m} \lambda_k \delta(x - g_k)$$

where $A = \sum_{k=1}^m (\sqrt{\lambda_k} g_k)(\sqrt{\lambda_k} g_k)^*$ is an optimal decomposition that achieves $\gamma_+(A) = \sum_{k=1}^m \lambda_k$.

Super-resolution and Convex Optimizations

$$\gamma_+(A) = \min_{x_1,\dots,x_m \,:\, A=\sum_k x_k x_k^*} \sum_{k=1}^{m} \|x_k\|_1^2 \;,\;\; m=n^2 \quad (P)$$

$$p^* = \inf_{\mu\in\mathcal{B}(S_1) \,:\, A=\int_{S_1} xx^* d\mu(x)} \int_{S_1} d\mu(x) \quad (M)$$

Remarks

1. *The optimization problem (P) is non-convex, but finite-dimensional. The optimization problem (M) is convex, but infinite-dimensional.*
2. *If $g_1,\dots,g_m \in S_1$ in the support of μ^* are known so that $\mu^* = \sum_{k=1}^m \lambda_k \delta(x-g_k)$, then the optimal $\lambda_1,\dots,\lambda_m \geq 0$ are determined by a linear program. More general, (M) is an infinite-dimensional linear program.*
3. *Finding the support of μ^* is an example of a super-resolution problem. One possible approach is to choose a redundant dictionary (frame) that includes the support of μ^*, and then solve the induced linear program.*

Second New Result: The Continuity Property

Theorem (The Continuity Property)

The map $\gamma_+ : (Sym^+(\mathbb{C}^n), \|\cdot\|) \to \mathbb{R}$ *is continuous.*

Remarks

1. This statement extends the continuity result from $Sym^{++}(\mathbb{C}^n) = \{A = A^* > 0\}$ to $Sym^+(\mathbb{C}^n) = \{A = A^* \geq 0\}$.
2. The proof is based on (possibly new) operator comparison results.

Proof: The Optimal Measure Result

Recall: we want to show the following problems admit same solution:

$$\gamma_+(A) = \min_{x_1,\ldots,x_m \,:\, A=\sum_k x_k x_k^*} \sum_{k=1}^{m} \|x_k\|_1^2 \;,\;\; m = n^2 \quad (P)$$

$$p^* = \inf_{\mu\in\mathcal{B}(S_1) \,:\, A=\int_{S_1} xx^*\, d\mu(x)} \int_{S_1} d\mu(x) \quad (M)$$

a. Assume $A = \sum_{k=1}^{m} x_k x_k^*$ is a global minimum for (P). Then $\mu(x) = \sum_{k=1}^{m} \|x_k\|_1^2 \delta(x - \frac{x_k}{\|x_k\|_1})$ is a feasible solution for (M). This shows $p^* \leq \gamma_+(A)$.

b. For reverse: Let μ^* be an optimal measure in (M). Fix $\varepsilon > 0$. Construct a disjoint partition $(U_l)_{1\leq l\leq L}$ of S_1 so that each U_l is included in some ball $B_\varepsilon(z_l)$ of radius ε with $\|z_l\|_1 = 1$. Thus $U_l \subset B_\varepsilon(z_l) \cap S_1$.

For each l, compute $x_l = \frac{1}{\mu^*(U_l)} \int_{U_l} x\, d\mu^*(x) \in B_\varepsilon(z_l)$. Let $g_l = \sqrt{\mu^*(U_l)}x_l$.

Proof: The Optimal Measure Result (cont)

Key inequality:

$$0 \leq R_l := \int_{U_l} (x - x_l)(x - x_l)^* \, d\mu^*(x) = \int_{U_l} xx^* \, d\mu^*(x) \ - \mu^*(U_l) x_l x_l^*$$

Sum over l and with $R = \sum_{l=1}^{L} R_l$ get

$$A = \int_{S_1} xx^* \, d\mu^*(x) \leq \sum_{l=1}^{L} g_l g_l^* + R$$

By sub-additivity and homogeneity:

$$\gamma_+(A) \leq \sum_{l=1}^{L} \|g_l\|_1^2 + \gamma_+(R) \leq \sum_{l=1}^{L} \mu^*(U_l) \|x_l\|_1^2 + n \, trace(R)$$

But $\|x_l - z_l\|_1 \leq \varepsilon$ and $\|x - x_l\|_1 \leq 2\varepsilon$ for every $x \in U_l$. Hence $\|x_l\|_1 \leq 1 + \varepsilon$ and $trace(R_l) \leq 4\mu^*(U_l)\varepsilon^2$.

Proof: The Optimal Measure Result (end)

Thus:

$$\gamma_+(A) \leq \mu^*(S_1) + (2\varepsilon + \varepsilon^2 + 4n\varepsilon^2)\mu^*(S_1)$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$\gamma_+(A) \leq \mu^*(S_1) = p^*$$

This ends the proof of the measure result. □

The Continuity Property

The proof is based on the following two lemmas:

Lemma (L1)

Let $A \in Sym^+(\mathbb{C}^n)$ of rank $r > 0$. Let $\lambda_r > 0$ denote the r^{th} eigenvalue of A, and let $P_{A,r}$ denote the orthogonal projection onto the range of A. For any $0 < \varepsilon < 1$ and $B \in Sym^+(\mathbb{C}^n)$ such that $\|A - B\|_{Op} \leq \frac{\varepsilon\lambda_r}{1-\varepsilon}$, the following holds true:

$$A - (1-\varepsilon)P_{A,r}BP_{A,r} \geq 0 \qquad (1)$$

Lemma (L2)

Let $B \in Sym^+(\mathbb{C}^n)$ of rank $r > 0$. Let $\lambda_r > 0$ denote the r^{th} eigenvalue of B. For any $0 < \varepsilon < \frac{1}{2}$ and $A \in Sym^+(\mathbb{C}^n)$ such that $\|A - B\|_{Op} \leq \varepsilon\lambda_r$, the following holds true:

$$A - (1-\varepsilon)P_{A,r}BP_{A,r} \geq 0 \qquad (2)$$

where $P_{A,r}$ denotes the orthogonal projection onto the top r eigenspace of A.

Proof of Continuity of γ_+

Fix $A \in Sym^+(\mathbb{C}^n)$. Let $(B_j)_{j\geq 1}$, $B_j \in Sym^+(\mathbb{C}^n)$, be a convergent sequence to A. We need to show $\gamma_+(B_j) \to \gamma_+(A)$.

Let $A = \sum_{k=1}^{n^2} x_k x_k^*$ be the optimal decomposition of A such that $\gamma_+(A) = \sum_{k=1}^{n^2} \|x_k\|_1^2$.

If $A = 0$ then $\gamma_+(A) = 0$ and

$$0 \leq \gamma_+(B_j) \leq n\, trace(B_j) \leq n^2 \|B_j\|_{Op}.$$

Hence $\lim_j \gamma_+(B_j) = 0$.

Assume $rank(A) = r > 0$ and let $\lambda_r > 0$ denote the smallest strictly positive eigenvalue of A. Let $\varepsilon \in (0, \frac{1}{2})$ be arbitrary. Let $J = J(\varepsilon)$ be so that $\|A - B_j\|_{Op} < \varepsilon \lambda_r$ for all $j > J$. Let $B_j = \sum_{k=1}^{n^2} y_{j,k} y_{j,k}^*$ be the optimal decomposition of B_j such that $\gamma_+(B_j) = \sum_{k=1}^{n^2} \|y_{j,k}\|_1^2$.

Let $\Delta_j = A - (1-\varepsilon) P_{A,r} B_j P_{A,r}$. By Lemma L1, for any $j > J$,

$$\gamma_+(A) \leq (1-\varepsilon)\gamma_+(P_{A,r} B_j P_{A,r}) + \gamma_+(\Delta_j) \leq (1-\varepsilon) \sum_{k=1}^{n^2} \|P_{A,r} y_{j,k}\|_1^2 + n\, trace(\Delta_j)$$

Proof of Continuity of γ_+ (cont)

Pass to a subsequence j' of j so that $y_{j',k} \to y_k$, for every $k \in [n^2]$, and $\gamma_+(B_{j'}) \to \liminf_j \gamma_+(B_j)$. Then $\lim_{j'} P_{A,r} y_{j',k} = P_{A,r} y_k = y_k$ and

$$\lim_{j'} \sum_{k=1}^{n^2} \|P_{A,r} y_{j',k}\|_1^2 = \lim_{j'} \sum_{k=1}^{n^2} \|y_{j',k}\|_1^2 = \liminf_j \gamma_+(B_j)$$

On the other hand, $\lim_j trace(\Delta_j) = \varepsilon\, trace(A)$. Hence:

$$\gamma_+(A) \leq (1-\varepsilon) \liminf_j \gamma_+(B_j) + \varepsilon\, trace(A)$$

Since $\varepsilon > 0$ is arbitrary, it follows $\gamma_+(A) \leq \liminf_j \gamma_+(B_j)$.
The inequality $\limsup_j \gamma_+(B_j) \leq \gamma_+(A)$ follows from Lemma L2 similarly.
This ends the proof of continuity. $\square$

Thank you!

Thank you for listening!
HAPPY BIRTHDAY HANS!